Logarithmic Fluctuations From Circularity

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Joint work with David Jerison and Scott Sheffield

From random walk to growth model

Internal DLA

- Start with n particles at the origin in the square grid $\mathbb{Z}^2$.
- Each particle in turn performs a simple random walk until it finds an unoccupied site, stays there.
- $A(n)$: the resulting random set of n sites in $\mathbb{Z}^2$.

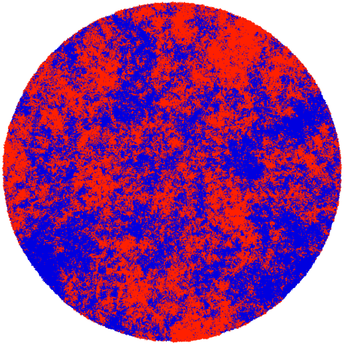
Growth rule:

- Let $A(1) = \{o\}$, and

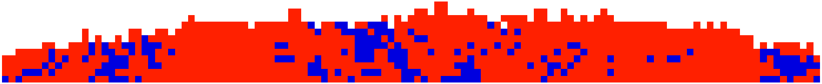
$$A(n+1) = A(n) \cup \{X^n(\tau^n)\}$$

where $X^1, X^2, \dots$ are independent random walks, and

$$\tau^n = \min\{t \mid X^n(t) \notin A(n)\}.$$



Internal DLA cluster in $\mathbb{Z}^2$.



Closeup of the boundary.

Questions

- Limiting shape
- Fluctuations

Meakin & Deutch, J. Chem. Phys. 1986

- "It is also of some fundamental significance to know just how smooth a surface formed by diffusion limited processes may be."

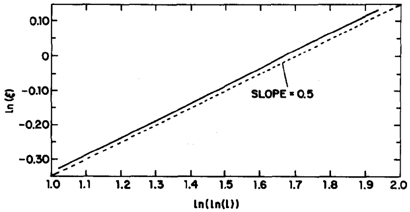


FIG. 2. Dependence of the variance of the surface height (ξ) on the strip width l for two-dimensional (square lattice) diffusion limited annihilation in the long time ($\bar{h} \gg l$) limit.

- "Initially, we plotted $\ln(\xi)$ vs $\ln(\ell)$ but the resulting plots were quite noticably curved. Figure 2 shows the dependence of $\ln(\xi)$ on $\ln[\ln(\ell)]$."

History of the Problem

- **Diaconis-Fulton 1991**: Addition operation on subsets of $\mathbb{Z}^d$.
- **Lawler-Bramson-Griffeath 1992**: w.p.1,

$$B_{(1-\varepsilon)r} \subset A(\pi r^2) \subset B_{(1+\varepsilon)r} \quad \text{eventually.}$$

- **Lawler 1995**: w.p.1,

$$\mathbf{B}_{r-r^{1/3}\log^2 r} \subset A(\pi r^2) \subset \mathbf{B}_{r+r^{1/3}\log^4 r} \quad \text{eventually.}$$

"A more interesting question... is whether the errors are $o(n^\alpha)$ for some $\alpha < 1/3$."

Logarithmic Fluctuations Theorem

Jerison - L. - Sheffield 2010: with probability 1,

$$\mathbf{B}_{r-C\log r} \subset A(\pi r^2) \subset \mathbf{B}_{r+C\log r} \quad \text{eventually.}$$

Asselah - Gaudillière 2010 independently obtained

$$\mathbf{B}_{r-C\log r} \subset A(\pi r^2) \subset \mathbf{B}_{r+C\log^2 r} \quad \text{eventually.}$$

Logarithmic Fluctuations in Higher Dimensions

In dimension $d \geq 3$, let ω_d be the volume of the unit ball in $\mathbb{R}^d$. Then with probability 1,

$$\mathbf{B}_{r-C\sqrt{\log r}} \subset A(\omega_d r^d) \subset \mathbf{B}_{r+C\sqrt{\log r}} \quad \text{eventually}$$

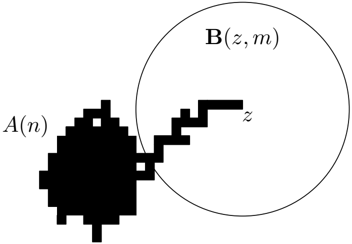
for a constant C depending only on d.

(Jerison - L. - Sheffield 2010; Asselah - Gaudillière 2010)

Elements of the proof

- Thin tentacles are unlikely.
- Martingales to detect fluctuations from circularity.
- "Self-improvement"

Thin tentacles are unlikely



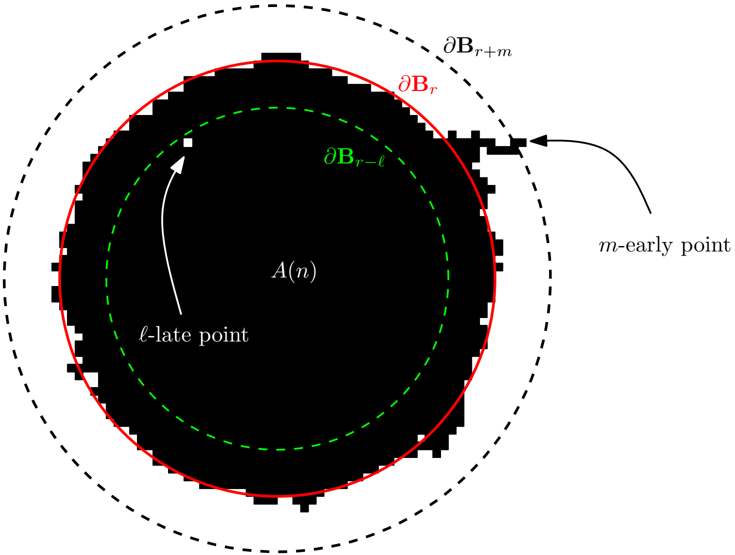
A thin tentacle.

Lemma. If $0 \notin \mathbf{B}(z,m)$, then

$$\mathbb{P}\left\{z \in A(n),\ \#(A(n) \cap \mathbf{B}(z,m)) \leq bm^d\right\} \leq \begin{cases} Ce^{-cm^2/\log m}, & d=2 \\ Ce^{-cm^2}, & d \geq 3 \end{cases}$$

for constants $b, c, C > 0$ depending only on the dimension d.

Early and late points in $A(n)$, for $n = \pi r^2$



Early and late points

Definition 1. z is an m-early point if:

$$z \in A(n), \quad n < \pi(|z| - m)^2$$

Definition 2. z is an ℓ-late point if:

$$z \notin A(n), \quad n > \pi(|z| + \ell)^2$$

$\mathcal{E}_m[n]$ = event that some point in $A(n)$ is m-early

$\mathcal{L}_\ell[n]$ = event that some point in $\mathbf{B}_{\sqrt{n/\pi}-\ell}$ is ℓ-late

Structure of the argument: Self-improvement

LEMMA 1. No ℓ-late points implies no m-early points:
If $m \geq C\ell$, then

$$\mathbb{P}(\mathcal{E}_m[n] \cap \mathcal{L}_\ell[n]^c) < n^{-10}.$$

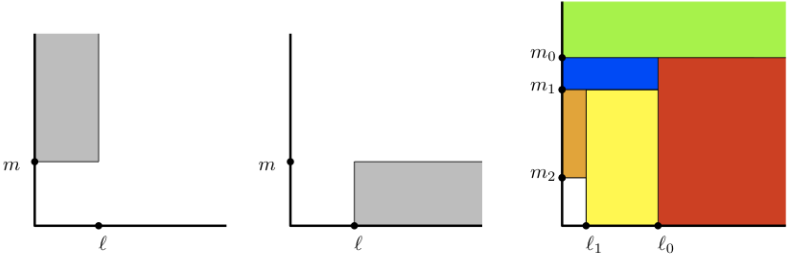
LEMMA 2. No m-early points implies no ℓ-late points:
If $\ell \geq \sqrt{C(\log n)m}$, then

$$\mathbb{P}(\mathcal{L}_\ell[n] \cap \mathcal{E}_m[n]^c) < n^{-10}.$$

Iterate, $\ell \mapsto \sqrt{C(\log n)C\ell}$, which is decreasing until

$$\ell = C^2 \log n.$$

Iterating Lemmas 1 and 2



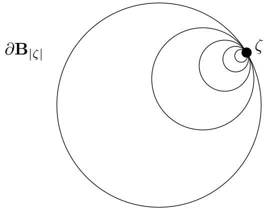
- Fix n and let ℓ, m be the maximal lateness and earliness occurring by time n. Iterate starting from $m_0 = n$:
- (ℓ, m) unlikely to belong to a vertical rectangle by Lemma 1.
- (ℓ, m) unlikely to belong to a horizontal rectangle by Lemma 2.

Early and late point detector

To detect early points near $\zeta \in \mathbb{Z}^2$, we use the martingale

$$M_\zeta(n) = \sum_{z \in \widetilde{A}(n)} (H_\zeta(z) - H_\zeta(0))$$

where H_ζ is a discrete harmonic function approximating $\mathrm{Re}\left(\frac{\zeta/|\zeta|}{\zeta - z}\right)$.



The fine print:

- Discrete harmonicity fails at three points $z = \zeta, \zeta+1, \zeta+1+i$.
- Modified growth process $\widetilde{A}(n)$ stops at $\partial B_{|\zeta|}(0)$.

Time change of Brownian motion

- To get a *continuous time* martingale, we use Brownian motions on the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ instead of random walks.
- Then there is a standard Brownian motion B_ζ such that

$$M_\zeta(t) = B_\zeta(s_\zeta(t))$$

where

$$s_\zeta(t) = \lim \sum_{i=1}^{N} (M(t_i) - M(t_{i-1}))^2$$

is the quadratic variation of M_ζ.

LEMMA 1. No ℓ-late implies no $m = C\ell$-early

Event $Q[z,k]$:

- $z \in A(k) \setminus A(k-1)$.
- z is m-early ($z \in A(\pi r^2)$ for $r = |z| - m$).
- $\mathcal{E}_m[k-1]^c$: No previous point is m-early.
- $\mathcal{L}_\ell[n]^c$: No point is ℓ-late.

We will use M_ζ for $\zeta = (1+4m/r)z$ to show for $0 < k \leq n$,

$$\mathbb{P}(Q[z,k]) < n^{-20}.$$

Main idea: Early but no late would be a large deviation!

- Recall there is a Brownian motion B_ζ such that

$$M_\zeta(n) = B_\zeta(s_\zeta(n)).$$

- On the event $Q[z,k]$

$$\mathbb{P}\left(M_\zeta(k) > c_0 m\right) > 1 - n^{-20} \tag{1}$$

and

$$\mathbb{P}\left(s_\zeta(k) < 100 \log n\right) > 1 - n^{-20}. \tag{2}$$

- On the other hand, $(s = 100 \log n)$

$$\mathbb{P}\left(\sup_{s' \in [0,s]} B_\zeta(s') \geq s \right) \leq e^{-s/2} = n^{-50}.$$

Proof of (1)

On the event $Q[z,k]$

$$\mathbb{P}(M_\zeta(k) > c_0 m) > 1 - n^{-20}.$$

- Since $z \in A(k)$ and thin tentacles are unlikely, we have with high probability,

$$\#(A(k) \cap B(z,m)) \geq bm^2.$$

- For each of these bm^2 points, the value of H_ζ is order $1/m$, so their total contribution to $M_\zeta(k)$ is order m.
- No ℓ-late points means that points elsewhere cannot compensate.

Proof of (2): Controlling the Quadratic Variation

On the event $Q[z,k]$

$$\mathbb{P}(s_\zeta(k) < 100 \log n) > 1 - n^{-20}.$$

- Lemma: There are independent standard Brownian motions $B^1, B^2, \ldots$ such that

$$s_\zeta(i+1) - s_\zeta(i) \le \tau_i$$

where τ_i is the first exit time of B^i from the interval (a_i, b_i).

$$a_i = \min_{z \in \partial \tilde{A}(i)} H_\zeta(z) - H_\zeta(0)$$

$$b_i = \max_{z \in \partial \tilde{A}(i)} H_\zeta(z) - H_\zeta(0).$$

Proof of (2): Controlling the Quadratic Variation

On the event $Q[z,k]$

$$\mathbb{P}(s_\zeta(k) < 100 \log n) > 1 - n^{-20}.$$

- By independence of the τ_i,

$$\mathbb{E}e^{s_\zeta(k)} \leq \mathbb{E}e^{(\tau_1+\cdots+\tau_k)} = (\mathbb{E}e^{\tau_1})\cdots(\mathbb{E}e^{\tau_k}).$$

- By large deviations for Brownian exit times,

$$\mathbb{E}e^{\tau(-a,b)} \leq 1 + 10ab.$$

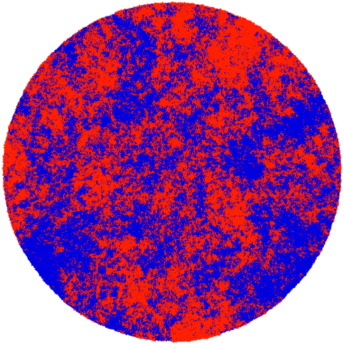
- Easy to estimate a_i, and use the fact that no previous point is m-early to bound b_i. Conclude that

$$\mathbb{E}\left[e^{s_\zeta(k)} 1_Q\right] \leq n^{50}.$$

What changes in higher dimensions?

- In dimension $d \geq 3$ the quadratic variation $s_\zeta(n)$ is constant order instead of $\log n$.
- So the fluctuations are instead dominated by thin tentacles, which can grow to length $\sqrt{\log n}$.
- Still open: prove matching lower bounds on the fluctuations of order $\log n$ in dimension 2 and $\sqrt{\log n}$ in dimensions $d \geq 3$.

Thank You!



Reference:

- D. Jerison, L. Levine and S. Sheffield, Logarithmic fluctuations for internal DLA. arXiv:1010.2483