The Scaling Limit of Diaconis-Fulton Addition

Lionel Levine

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Joint work with Yuval Peres
Diaconis-Fulton Addition

- Finite sets $A, B \subset \mathbb{Z}^d$. 
Diaconis-Fulton Addition

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- $A \cap B = \{x_1, \ldots, x_k\}$. 
Diaconis-Fulton Addition

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- $A \cap B = \{x_1, \ldots, x_k\}$.
- To form $A + B$, let $C_0 = A \cup B$ and

$$C_j = C_{j-1} \cup \{y_j\}$$

where $y_j$ is the endpoint of a random walk started at $x_j$ and stopped on exiting $C_{j-1}$. 
Diaconis-Fulton Addition

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  where $y_j$ is the endpoint of a random walk started at $x_j$ and stopped on exiting $C_{j-1}$.
- Define $A + B = C_k$.
- Abeilan property: the law of $A + B$ does not depend on the ordering of $x_1, \ldots, x_k$. 

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The Scaling Limit of Diaconis-Fulton Addition
Internal DLA

$A_1 = \{ o \}, \ A_n = A_{n-1} + \{ o \}.$
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- More precisely, for any \( \varepsilon > 0 \), with probability one we have

\[
B_r(1-\varepsilon) \subset A_{\lfloor \omega_d r^d \rfloor} \subset B_r(1+\varepsilon)
\]

for all sufficiently large \( r \).
Internal DLA

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- Lawler, Bramson and Griffeath (1992) proved that the limiting shape is a ball.
- More precisely, for any $\varepsilon > 0$, with probability one we have
  \[ B_{r(1-\varepsilon)} \subset A_{\lfloor \omega_d r^d \rfloor} \subset B_{r(1+\varepsilon)} \]
  for all sufficiently large $r$.
- Here $B_r = \{x \in \mathbb{Z}^d : |x| < r\}$, and $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. 
The Rotor-Router Model

- Deterministic analogue of random walk.
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- Each site $x \in \mathbb{Z}^2$ has a rotor pointing North, South, East or West.
  (Start all rotors pointing North, say.)
The Rotor-Router Model

- Deterministic analogue of random walk.
- Each site $x \in \mathbb{Z}^2$ has a rotor pointing North, South, East or West.
  (Start all rotors pointing North, say.)
- A particle starts at the origin. At each site it comes to, it
  1. Turns the rotor clockwise by 90 degrees;
  2. Takes a step in direction of the rotor.
Rotor-Router Aggregation

Sequence of lattice regions

\[ A_1 = \{o\} \]

\[ A_n = A_{n-1} \cup \{x_n\}, \]
Rotor-Router Aggregation

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where

- \( x_n \in \mathbb{Z}^2 \) is the site at which rotor walk first leaves the region \( A_{n-1} \).
Rotor-Router Aggregation

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where
- \( x_n \in \mathbb{Z}^2 \) is the site at which rotor walk first leaves the region \( A_{n-1} \).

- Makes sense in \( \mathbb{Z}^d \) for any \( d \).
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The Scaling Limit of Diaconis-Fulton Addition
Spherical Asymptotics

**Theorem** (L.-Peres) Let $A_n$ be the region of $n$ particles formed by rotor-router aggregation in $\mathbb{Z}^d$.

$B_r - c \log r \subset A_n \subset B_r(1 + c' r^{-1/d} \log r)$,

where $B_\rho$ is the ball of radius $\rho$ centered at the origin.

$n = \omega d r^d$, where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$.

$c, c'$ depend only on $d$.

**Corollary**: Inradius/Outradius $\to 1$ as $n \to \infty$.

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**Spherical Asymptotics**

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The Scaling Limit of Diaconis-Fulton Addition
The Abelian Property

 Choices of which particles to route in what order don’t affect the final shape generated or the final rotor directions.
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- **Abelian sandpile, or chip-firing** model:
  - When 4 or more grains of sand accumulate at a site in \( \mathbb{Z}^2 \), it *topples*, sending one grain to each neighbor.
The Abelian Property

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Abelian sandpile, or chip-firing model:

- When 4 or more grains of sand accumulate at a site in $\mathbb{Z}^2$, it topples, sending one grain to each neighbor.
- Choices of which sites to topple in what order don’t affect the final sandpile shape.
The Abelian Property

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  - When 4 or more grains of sand accumulate at a site in $\mathbb{Z}^2$, it **topples**, sending one grain to each neighbor.
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- **Equivalent models**:
  - Start with $n$ particles at the origin.
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  - Choices of which sites to topple in what order don’t affect the final sandpile shape.

- **Equivalent models:**
  - Start with $n$ particles at the origin.
  - If there are $m$ particles at a site, send $\lfloor m/4 \rfloor$ to each neighbor.
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  - When 4 or more grains of sand accumulate at a site in $\mathbb{Z}^2$, it *topples*, sending one grain to each neighbor.
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  - **Sandpile**: Leave the extra particles where they are.
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- **Equivalent models**:
  - Start with $n$ particles at the origin.
  - If there are $m$ particles at a site, send $\lfloor m/4 \rfloor$ to each neighbor.
  - **Sandpile**: Leave the extra particles where they are.
  - **Rotor**: Send extra particles according to the usual rotor rule.
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The Scaling Limit of Diaconis-Fulton Addition
Bounds for the Abelian Sandpile

- **Theorem** (L.-Peres) Let $S_n$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from $n$ particles at the origin.

- Improves the bounds of Le Borgne and Rossin.
Bounds for the Abelian Sandpile

**Theorem** (L.-Peres) Let $S_n$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from $n$ particles at the origin. Then

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\left(\text{Ball of volume } \frac{n-o(n)}{2d-1}\right) \subset S_n \subset \left(\text{Ball of volume } \frac{n+o(n)}{d}\right).
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Bounds for the Abelian Sandpile

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Improves the bounds of Le Borgne and Rossin.
(Disk of area $n/3) \subset S_n \subset (Disk of area n/2)$
Divisible Sandpile

- Start with mass $m$ at the origin.
Divisible Sandpile

- Start with mass \( m \) at the origin.
- Each site keeps mass 1, divides excess mass equally among its neighbors.

Theorem (L.-Peres): There are constants \( c \) and \( c' \) depending only on \( d \), such that

\[
B_{r-c} \subset A_m \subset B_{r+c'}
\]

where \( m = \omega d r d \).
Divisible Sandpile

- Start with mass $m$ at the origin.
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- As $t \to \infty$, get a limiting region $A_m$ of mass 1, fractional mass on $\partial A_m$, and zero outside.

\textbf{Theorem} (L.-Peres): There are constants $c$ and $c'$ depending only on $d$, such that $B_r - c \subset A_m \subset B_r + c'$ where $m = \omega d r^d$. 

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Divisible Sandpile

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- **Theorem** (L.-Peres): There are constants $c$ and $c'$ depending only on $d$, such that
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Questions

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- Is it the same for all three models?
Questions

- As the lattice spacing goes to zero, is there a scaling limit?
- If so, can we describe the limiting shape?
- Is it the same for all three models?
- Not clear how to define dynamics in $\mathbb{R}^d$. 
Odometer Function

$u(x) = \text{total mass emitted from } x.$
Odometer Function

- \( u(x) = \) total mass emitted from \( x \).
- Discrete Laplacian:

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\Delta u(x) = \frac{1}{2d} \sum_{y \sim x} u(y) - u(x)
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= \text{mass received} - \text{mass emitted}
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Odometer Function

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\]

= mass received − mass emitted

\[
= \begin{cases} 
  -1 & x \in A \cap B \\
  0 & x \in A \cup B - A \cap B \\
  1 & x \in A \oplus B - A \cup B.
\end{cases}
\]
Least Superharmonic Majorant

Let

$$\gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y),$$

where $g$ is the Green’s function for SRW in $\mathbb{Z}^d$, $d \geq 3$. 

Claim: odometer = $\gamma - \gamma$. 

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The Scaling Limit of Diaconis-Fulton Addition
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where $g$ is the Green’s function for SRW in $\mathbb{Z}^d$, $d \geq 3$.

In dimension two, we use the negative of the potential kernel in place of $g$.

Claim: odometer = $s(x) - \gamma(x)$. 

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The Scaling Limit of Diaconis-Fulton Addition
Least Superharmonic Majorant

Let

\[ \gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y), \]

where \( g \) is the Green’s function for SRW in \( \mathbb{Z}^d, d \geq 3 \).

In dimension two, we use the negative of the potential kernel in place of \( g \).

Let \( s(x) = \inf\{\phi(x) \mid \phi \text{ superharmonic, } \phi \geq \gamma\} \).
Least Superharmonic Majorant

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  \[ \gamma(x) = -|x|^2 - \sum_{y \in A} g(x, y) - \sum_{y \in B} g(x, y), \]

  where \( g \) is the Green’s function for SRW in \( \mathbb{Z}^d, d \geq 3 \).
  - In dimension two, we use the negative of the potential kernel in place of \( g \).

- Let \( s(x) = \inf \{ \phi(x) \mid \phi \text{ superharmonic, } \phi \geq \gamma \} \).

- **Claim:** odometer = \( s - \gamma \).
Proof of the claim

Let $m(x)$ = amount of mass present at $x$ in the final state.
Proof of the claim

Let \( m(x) \) = amount of mass present at \( x \) in the final state. Then

\[
\Delta u = m - 1_A - 1_B
\]
Proof of the claim

Let $m(x)$ = amount of mass present at $x$ in the final state. Then

$$\Delta u = m - 1_A - 1_B \leq 1 - 1_A - 1_B.$$
Proof of the claim

Let \( m(x) = \) amount of mass present at \( x \) in the final state. Then

\[
\Delta u = m - 1_A - 1_B
\]

\leq 1 - 1_A - 1_B.

Since

\[
\Delta \gamma = 1_A + 1_B - 1
\]

the sum \( u + \gamma \) is superharmonic, so \( u + \gamma \geq s \).
Proof of the claim

Let $m(x) =$ amount of mass present at $x$ in the final state.

Then

$$
\Delta u = m - 1_A - 1_B \\
\leq 1 - 1_A - 1_B.
$$

Since

$$
\Delta \gamma = 1_A + 1_B - 1
$$

the sum $u + \gamma$ is superharmonic, so $u + \gamma \geq s$.

Reverse inequality: $s - \gamma - u$ is superharmonic on $A \oplus B$ and nonnegative outside $A \oplus B$, hence nonnegative inside as well.
Defining the Scaling Limit

- \( A, B \subset \mathbb{R}^d \) bounded open sets such that \( \partial A, \partial B \) have measure zero.
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- Let

$$D = A \cup B \cup \{s > \gamma\}$$
Defining the Scaling Limit

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- Let

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where

\[ \gamma(x) = -|x|^2 - \int_A g(x, y)dy - \int_B g(x, y)dy \]
Defining the Scaling Limit

- $A, B \subset \mathbb{R}^d$ bounded open sets such that $\partial A, \partial B$ have measure zero
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  where
  \[ \gamma(x) = -|x|^2 - \int_A g(x, y)dy - \int_B g(x, y)dy \]
  and
  \[ s(x) = \inf\{ \phi(x) | \phi \text{ is continuous, superharmonic, and } \phi \geq \gamma \} \]
  is the least superharmonic majorant of $\gamma$. 

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Defining the Scaling Limit

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- Let

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is the least superharmonic majorant of $\gamma$.

- Odometer: $u = s - \gamma$. 

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The Scaling Limit of Diaconis-Fulton Addition
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.

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Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- Write $A^{\cdot} = A \cap \delta_n \mathbb{Z}^d$.
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- Write $A:: = A \cap \delta_n \mathbb{Z}^d$.
- **Theorem** (L.-Peres) For any $\varepsilon > 0$, with probability one

$$D_{\varepsilon}:: \subset D_n, R_n, I_n \subset D_{\varepsilon}::$$

for all sufficiently large $n$,
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
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- **Theorem** (L.-Peres) For any $\varepsilon > 0$, with probability one

$$D^\cdot_{\varepsilon} \subset D_n, R_n, I_n \subset D^\varepsilon$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the Diaconis-Fulton sums of $A^\cdot$ and $B^\cdot$ in the lattice $\delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.
Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets with $\partial A, \partial B$ having measure zero.
- Lattice spacing $\delta_n \downarrow 0$.
- Write $A^{\vdash} = A \cap \delta_n \mathbb{Z}^d$.
- **Theorem** (L.-Peres) For any $\varepsilon > 0$, with probability one

$$D^{\varepsilon \vdash} \subset D_n, R_n, I_n \subset D^{\varepsilon \vdash}$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the Diaconis-Fulton sums of $A^{\vdash}$ and $B^{\vdash}$ in the lattice $\delta_n \mathbb{Z}^d$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.
- $D = A \cup B \cup \{s > \gamma\}$. 

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Main Result

- Let \( A, B \subset \mathbb{R}^d \) be bounded open sets with \( \partial A, \partial B \) having measure zero.
- Lattice spacing \( \delta_n \downarrow 0 \).
- Write \( A^\updownarrow = A \cap \delta_n \mathbb{Z}^d \).
- **Theorem (L.-Peres)** For any \( \varepsilon > 0 \), with probability one

\[
D^\updownarrow \subset D_n, R_n, I_n \subset D^\varepsilon
\]

for all sufficiently large \( n \), where

- \( D_n, R_n, I_n \) are the Diaconis-Fulton sums of \( A^\updownarrow \) and \( B^\updownarrow \) in the lattice \( \delta_n \mathbb{Z}^d \), computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.
- \( D = A \cup B \cup \{s > \gamma\} \).
- \( D_\varepsilon, D^\varepsilon \) are the inner and outer \( \varepsilon \)-neighborhoods of \( D \).
Multiple Point Sources

Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$. 

Theorem (L.-Peres) For any $\varepsilon > 0$, with probability one $D :\!:\!\varepsilon \subset D_n, R_n, I_n \subset D :\!\!\varepsilon$ for all sufficiently large $n$, where $D$ is the continuum Diaconis-Fulton sum of the balls $B(x_i, r_i)$, where $\lambda_i = \omega_d r_i d_i$. 

Follows from the main result and the case of a single point source. 

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The Scaling Limit of Diaconis-Fulton Addition
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.

- Theorem (L.-Peres) For any $\varepsilon > 0$, with probability one

$$D_\varepsilon \subset D_n, R_n, I_n \subset D_\varepsilon$$

for all sufficiently large $n$,.
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.

- **Theorem** (L.-Peres) For any $\varepsilon > 0$, with probability one

\[ D_{\varepsilon} \subset D_n, R_n, I_n \subset D^{\varepsilon} \]

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $\lfloor \lambda_i \delta_n^{-d} \rfloor$ particles start at each site $x_i$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.
Fix centers \( x_1, \ldots, x_k \in \mathbb{R}^d \) and \( \lambda_1, \ldots, \lambda_k > 0 \).

**Theorem** (L.-Peres) For any \( \varepsilon > 0 \), with probability one

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D_{\varepsilon} \subset D_n, R_n, I_n \subset D^\varepsilon
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for all sufficiently large \( n \), where

- \( D_n, R_n, I_n \) are the domains of occupied sites \( \delta_n \mathbb{Z}^d \), if \( \lfloor \lambda_i \delta_n \rfloor \) particles start at each site \( x_i \), computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.

- \( D \) is the continuum Diaconis-Fulton sum of the balls \( B(x_i, r_i) \), where \( \lambda_i = \omega_d r_i^d \).
Multiple Point Sources

- Fix centers $x_1, \ldots, x_k \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_k > 0$.

- **Theorem** (L.-Peres) For any $\varepsilon > 0$, with probability one

  $$D_\varepsilon \subset D_n, R_n, I_n \subset D_\varepsilon$$

for all sufficiently large $n$, where

- $D_n, R_n, I_n$ are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $[\lambda_i \delta_n^{-d}]$ particles start at each site $x_i$, computed using divisible sandpile, rotor-router, and internal DLA dynamics, respectively.

- $D$ is the continuum Diaconis-Fulton sum of the balls $B(x_i, r_i)$, where $\lambda_i = \omega_d r_i^d$.

- Follows from the main result and the case of a single point source.
Steps of the Proof

convergence of densities

⇓

convergence of obstacles
Steps of the Proof

convergence of densities
\[\downarrow\]
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convergence of odometer functions
Steps of the Proof

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convergence of domains.
Lower Bound for Internal DLA

- Inspired by the Lawler-Bramson-Griffeath argument for a single point source.
Lower Bound for Internal DLA

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- Inspired by the Lawler-Bramson-Griffeath argument for a single point source.
- After all particles have aggregated (stage 1), let them resume walking until they exit $D_\epsilon$ (stage 2).
- Fix $z \in D_\epsilon$, and let
  - $M =$ number of particles that visit $z$ during stages 1 and 2.
Lower Bound for Internal DLA

- Inspired by the Lawler-Bramson-Griffeath argument for a single point source.
- After all particles have aggregated (stage 1), let them resume walking until they exit $D^\varepsilon$ (stage 2).
- Fix $z \in D^\varepsilon$, and let $M =$ number of particles that visit $z$ during stages 1 and 2. $L =$ number of particles that visit $z$ during stage 2.
Lower Bound for Internal DLA

- Inspired by the Lawler-Bramson-Griffeath argument for a single point source.
- After all particles have aggregated (stage 1), let them resume walking until they exit $D$ (stage 2).
- Fix $z \in D$, and let
  - $M =$ number of particles that visit $z$ during stages 1 and 2.
  - $L =$ number of particles that visit $z$ during stage 2.
- $\mathbb{P}(z \notin I_n) = \mathbb{P}(L = M)$. 
Independent Indicators

Stage 2': Instead of starting particles where they have aggregated, start one particle at each point $y \in (D - A \cup B)^\circ$. 

$\tilde{L}$ = number of particles that visit $z$ during stage 2'. 

Since $\tilde{L} \geq L$ we have $P(L = M) \leq P(\tilde{L} \geq M)$. 

Strategy: show $E\tilde{L} < EM$ and use concentration of measure.
Independent Indicators

Stage 2': Instead of starting particles where they have aggregated, start one particle at each point \( y \in (D - (A \cup B))' \).

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Independent Indicators

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  $$\mathbb{P}(L = M) \leq \mathbb{P}(\tilde{L} \geq M).$$

- Strategy: show $\mathbb{E}\tilde{L} < \mathbb{E}M$ and use concentration of measure.
Let
\[ f(z) = g_n(z, z) \mathbb{E} (M - \tilde{L}) \]

where \( g_n \) is the Green’s function for SRW stopped on exiting \( D \).
Let
\[ f(z) = g_n(z, z) \mathbb{E} (M - \tilde{L}) \]
\[ = \sum_{y \in (A \cap B):} g_n(y, z) - \sum_{y \in (D - A \cup B):} g_n(y, z), \]
where \( g_n \) is the Green’s function for SRW stopped on exiting \( D \).

Then \( \Delta f = 1 - \frac{1}{A} - \frac{1}{B} \), on \( D \).

The divisible sandpile odometer satisfies \( \Delta u_n = 1 - \frac{1}{A} - \frac{1}{B} \), on \( D_n \).
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Then
\[ \Delta f = 1 - 1_A - 1_B, \quad \text{on } D \]
\[ f = 0, \quad \text{on } \partial D. \]
Dirichlet Problem

- Let

\[ f(z) = g_n(z, z) \mathbb{E} (M - \tilde{L}) = \sum_{y \in (A \cap B) :} g_n(y, z) - \sum_{y \in (D - A \cup B) :} g_n(y, z), \]

where \( g_n \) is the Green’s function for SRW stopped on exiting \( D \).

- Then

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- The divisible sandpile odometer satisfies

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\[ u_n = 0, \quad \text{on } \partial D_n. \]
Concentration of Measure

- Using the fact that $D_n \to D$, $u_n \to u$, and the positivity of $u$, can show that
  
  $$f > c_\varepsilon \delta_n^{-2} \quad \text{on } D_\varepsilon.$$

Lionel Levine (joint work with Yuval Peres)
Concentration of Measure

- Using the fact that $D_n \rightarrow D$, $u_n \rightarrow u$, and the positivity of $u$, can show that
  \[ f > c_\varepsilon \delta_n^{-2} \quad \text{on } D_\varepsilon. \]

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Large deviations:

$$P\left(\tilde{L} > (1 + \lambda) \mathbb{E}\tilde{L}\right) < 2e^{-c_\lambda \mathbb{E}\tilde{L}},$$

$$P\left(M < (1 - \lambda) \mathbb{E}M\right) < 2e^{-c_\lambda \mathbb{E}M}.$$
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Large deviations:

$$\mathbb{P} (\tilde{L} > (1 + \lambda) \mathbb{E} \tilde{L}) < 2e^{-c_{\lambda} \mathbb{E} \tilde{L}};$$

$$\mathbb{P} (M < (1 - \lambda) \mathbb{E} M) < 2e^{-c_{\lambda} \mathbb{E} M}.$$ 

Conclude that $\mathbb{P}(\tilde{L} \geq M) < 4e^{-c_{\epsilon} \delta_n^{-2}}$. 

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Finishing Up

- Summing over $z \in D_\varepsilon$ and over $n$, by Borel-Cantelli only finitely many of the events $\{z \notin I_n\}$ occur, a.s.
- Hence $D_\varepsilon \subset I_n$ for sufficiently large $n$. 

Lionel Levine (joint work with Yuval Peres)

The Scaling Limit of Diaconis-Fulton Addition
Circularity for the Divisible Sandpile

▶ Dirichlet problem for the odometer function

\[ \Delta u = 1 \quad \text{on } A_m - \{o\} \]
\[ \Delta u(o) = 1 - m \]
\[ u = 0 \quad \text{on } \partial A_m. \]

▶ Idea: Compare \( u \) to the function

\[ \gamma(x) = |x|^2 - ma(x). \]

where \( a \) is the potential kernel

\[ a(x) = \lim_{n \to \infty} (G_n(o) - G_n(x)) \]

and \( G_n(x) \) is the expected number of visits to \( x \) by SRW before time \( n \).

▶ \( a(x) \) is harmonic off \( o \), and \( \Delta a(o) = 1 \).

▶ \( \Delta |x|^2 = 1 \)
Taylor expansion

- Standard estimate:
  \[ a(x) = \frac{2}{\pi} \log |x| + k + O(|x|^{-2}) \]
  gives
  \[ \gamma(x) = |x|^2 - \frac{2m}{\pi} \log |x| + km + O(m|x|^{-2}). \]

- Get a constant \( K = K(m) \) such that
  - If \( r \leq |x| < r + 1 \), then \( \gamma(x) = K + O(1) \).
  - \( \gamma(x) \geq K + (r - |x|)^2 + O \left( \frac{r^2}{|x|^2} \right) \).
Inner Radius

- $u - \gamma$ is superharmonic in $B_r$
- $u - \gamma \geq -K + O(1)$ on the boundary, hence on all of $B_r$.
- $\gamma$ grows quadratically as we move away from the boundary
- $\therefore u > 0$ on $B_{r-c}$.
Outer Radius

- $u - \gamma$ is harmonic in $A_m$
- $u - \gamma \leq -K + O(1)$ on the boundary, hence on all of $A_m$.
- If $x \in A_m$ with $r \leq |x| < r + 1$, then $u(x) \leq c'$.
- **Lemma:** If $y \in A_m - \{o\}$ there exists $z \sim y$ with $u(z) \geq u(y) + 1$.
- **Proof.** For some neighbor $z$,

$$u(z) \geq \frac{1}{4} \sum_{w \sim y} u(w) = u(y) + 1.$$ 

- $\therefore A_m \subset B_{r+c'}$. 

Lionel Levine (joint work with Yuval Peres)

The Scaling Limit of Diaconis-Fulton Addition
Adapting the Proof for Rotors

- Rotor-router odometer:
  \[ u(x) = \text{total number of particles emitted from } x. \]

- Instead of \( \Delta u = 1 \), we only know \(-2 \leq \Delta u \leq 4\).

- Repeating the argument only gives
  \[ B_{cr} \subset A_n \subset B_{c'r}. \]
Smoothing

To do better, let

\[ v(x) = \frac{1}{4k^2} \sum_{y \in S_k(x)} u(y) \]

where \( S_k(x) \) is a box of side length 2\( k \) centered at \( x \).

Using \( \Delta = \text{div grad} \), we get

\[ \Delta v(x) = \frac{1}{4k^2} \sum_{(y,z) \in \partial S_k(x)} \frac{u(z) - u(y)}{4} \]

\[ = 1 + O \left( \frac{1}{k} \right) \]

if \( o \notin S_k(x) \) and all sites in \( S_k(x) \) are occupied.
Fancier Smoothing

Let $T$ be the first exit time of $B_r$, and

$$v(x) = \mathbb{E}_x u(X_T) - \mathbb{E}_x T + n \mathbb{E} \#\{j < T | X_j = o\}.$$  

Boundary value problem:

$$\Delta v = 1 \quad \text{on } A_n \cap B_r - \{o\}$$

$$\Delta v(o) = 1 - n$$

$$v = 0 \quad \text{on } \partial A_n.$$  

Want to show $u \approx v$.  

Lionel Levine (joint work with Yuval Peres)  

The Scaling Limit of Diaconis-Fulton Addition
Green’s Function

End up getting

$$u(x) \geq v(x) - \sum_{y \in B_r} \sum_{z \sim y} |G_{B_r}(x, y) - G_{B_r}(x, z)|.$$ 

Error gets smaller as $x$ approaches the boundary, and we can show $B_{r - C\log r} \subset A_n$. 

But for the outer radius, the error is

$$\sum_{y \in A_n} \sum_{z \sim y} |G_{A_n}(x, y) - G_{A_n}(x, z)|.$$ 

Can’t control this, so we need another approach.
Spreading Out

- Spherical shells

\[ S_k = \{ x \in \mathbb{Z}^d : k \leq |x| < k + 1 \} . \]

- Lawler, Bramson, and Griffeath (1992): If \( j < k, x \in S_j, \ y \in S_k \), then

\[ \mathbb{P}_x(X_{T_k} = y) \leq \frac{C}{(k - j)^{d-1}} . \]

- Want to show the same holds for rotor-router walk, with frequency replacing probability.
Holroyd-Propp Bound

- recurrent graph $G$
- $Y \subset Z$ sets of vertices
- $s(x)$ particles start at $x$
- Stop walks when they hit $Z$; how many land in $Y$?
- Let $H(x) = \mathbb{P}_x(X_T \in Y)$. Then

$$|RR(s, Y) - RW(s, Y)| \leq \sum_{u \in G} \sum_{v \sim u} |H(u) - H(v)|$$

independent of $s$ and the initial rotor positions!
Outer Radius

- $N_j = \# \text{ particles that ever reach shell } S_j$.
- If $r < j < k$ with $N_k > N_j/2$, then
  \[
  \frac{CN_j}{(k-j)^{d-1}} \#(S_k \cap A_n) \geq \frac{N_j}{2}
  \]
  hence
  \[
  \sum_{i=j}^{k} \#(S_i \cap A_n) \geq C(k-j)^d.
  \]
- Since $B_{r-C\log r}$ is fully occupied,
  \[
  k \leq j + C(r^{d-1} \log r)^{1/d}
  \]
  which gives
  \[
  A_n \subset B_r(1+C r^{-1/d}(\log r)^{1+1/d}).
  \]
Further Directions and Open Problems: Rotor-Router

- How fast does $R(n) = \max_{k \leq n} (\text{outrad}(A_k) - \text{inrad}(A_k))$ really grow?

- Is the occupied region simply connected?

- Understand the patterns in the picture of rotor directions.

- Identify the limiting shape of the "broken rotor" models.

Lionel Levine (joint work with Yuval Peres)
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Lionel Levine (joint work with Yuval Peres)
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Lionel Levine (joint work with Yuval Peres)
The Scaling Limit of Diaconis-Fulton Addition
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Lionel Levine (joint work with Yuval Peres)
Lionel Levine (joint work with Yuval Peres)

The Scaling Limit of Diaconis-Fulton Addition
$z \mapsto 1/z^2$
Further Directions and Open Problems: Sandpile

- Fix an integer $h \in (-\infty, 2]$.
- Start every site in $\mathbb{Z}^2$ at height $h$.

Conjecture: As $n \to \infty$, the limiting shape $S_{n, h}$ is a (regular?) $(12 - 4h)$-gon.

Fey and Redig (2007) Case $h = 2$: The limiting shape of $S_{n, 2}$ is a square.

In all other cases, even the existence of a limiting shape is open.

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Lionel Levine (joint work with Yuval Peres)

The Scaling Limit of Diaconis-Fulton Addition

$h = 2$

$h = 1$

$h = 0$