Obstacle Problems and Lattice Growth Models

Lionel Levine (MIT)

June 4, 2009

Joint work with Yuval Peres

Talk Outline

- Three growth models
  - Internal DLA
  - Divisible Sandpile
  - Rotor-router model
- Discrete potential theory and the obstacle problem.
- Scaling limit and quadrature domains.
- The abelian sandpile as a growth model
- Conjectures about pattern formation:
  - Scale invariance
  - Dimensional reduction

Internal DLA with Multiple Sources

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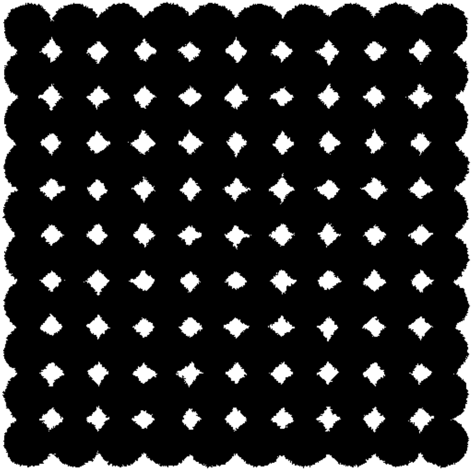
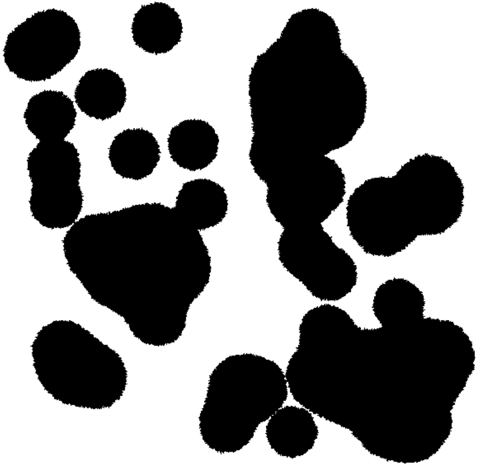
- Each particle performs **simple random walk** in $\mathbb{Z}^d$ until reaching an unoccupied site.
- Get a **random set** of km occupied sites in $\mathbb{Z}^d$.
- The distribution of this random set does not depend on the order of the walks (**Diaconis-Fulton '91**).



100 point sources arranged on a 10×10 grid in $\mathbb{Z}^2$.

Sources are at the points $(50i, 50j)$ for $0 \le i, j \le 9$.
Each source started with 2200 particles.



50 point sources arranged at random in a box in $\mathbb{Z}^2$.

The sources are iid uniform in the box $[0, 500]^2$.
Each source started with 3000 particles.

Questions

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- Not clear how to define dynamics in $\mathbb{R}^d$.

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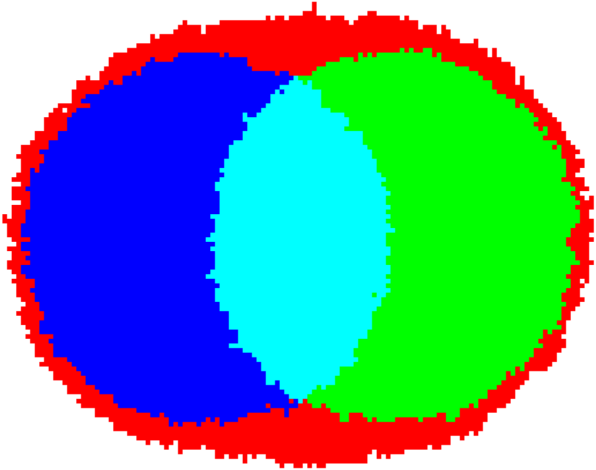
- **Hard part**: How does the shape change when the particles in the overlaps continue walking until they reach unoccupied sites?



Two-source internal DLA cluster built from overlapping single-source clusters.

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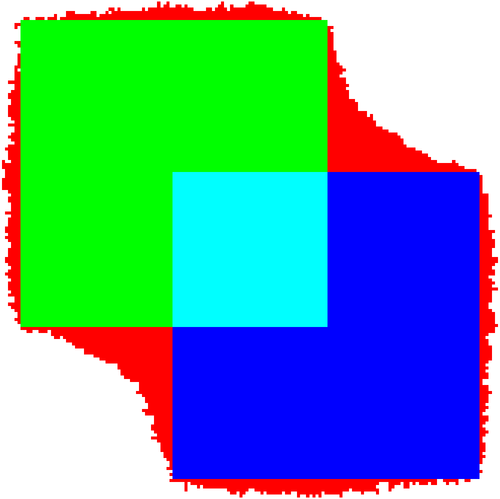
- **Abeilan property**: the law of $A + B$ does not depend on the ordering of $y_1, \ldots, y_k$.



Diaconis-Fulton sum of two squares in $\mathbb{Z}^2$ overlapping in a smaller square.

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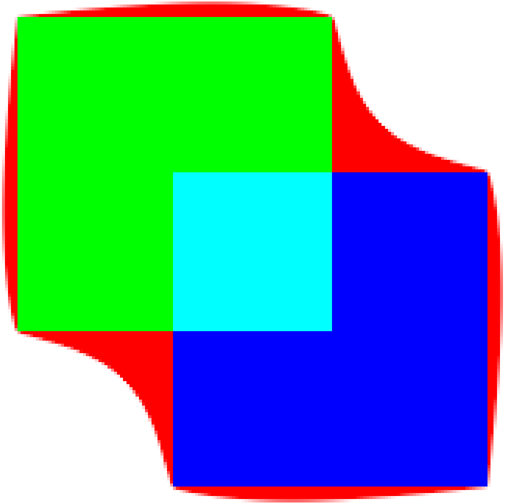
- Abelian property: $A \oplus B$ does not depend on the choices.



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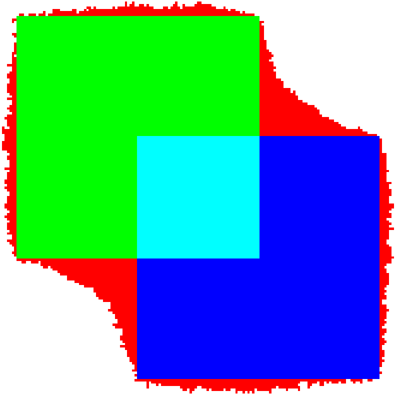

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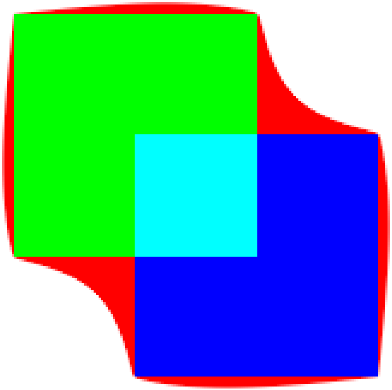

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- Need additional information to determine the domain $A \oplus B$.

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- Unknown function u, unknown domain $D = \{u > 0\}$.

$$u \geq 0$$
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\end{aligned}
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Least Superharmonic Majorant

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- Then the odometer function is $\boxed{u = s - \gamma.}$

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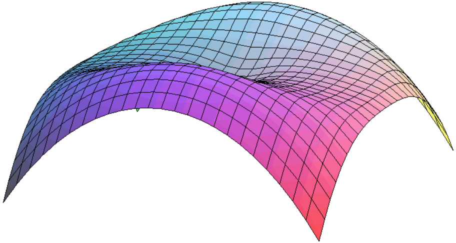
is the least superharmonic majorant of γ.

- Obstacle for two overlapping disks A and B:



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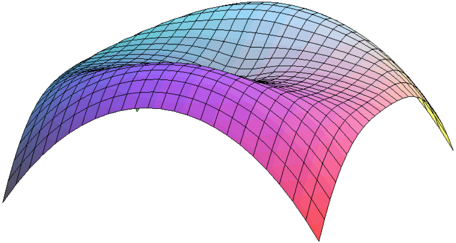

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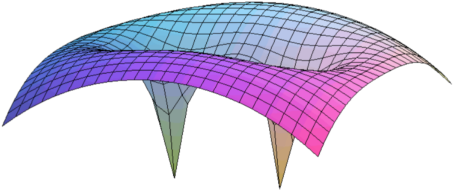
- Obstacle for two point sources x_1 and x_2:



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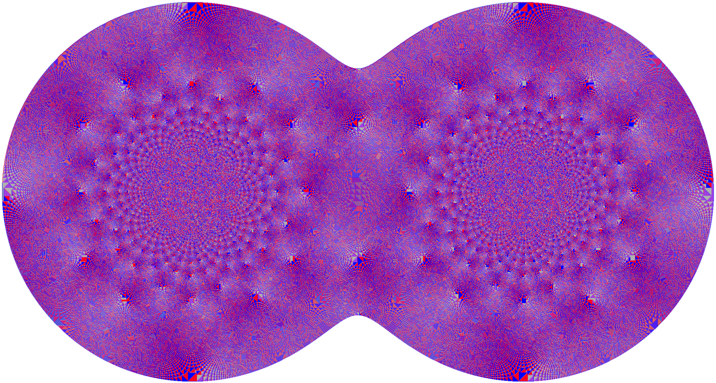

The smash sum

$$A \oplus B = A \cup B \cup \{s > \gamma\}$$

of two overlapping disks $A, B \subset \mathbb{R}^2$.

Properties of the Smash Sum

- Associativity: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.
  - Analogous to the abelian property of the divisible sandpile.
- Volume conservation: $\mathrm{vol}(A \oplus B) = \mathrm{vol}(A) + \mathrm{vol}(B)$.
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- **Quadrature identity**: If h is an integrable superharmonic function on $A \oplus B$, then

$$\int_{A \oplus B} h(x)dx \leq \int_A h(x)dx + \int_B h(x)dx.$$

  - One can also take this as the defining property of the smash sum (**Gustafsson '88**).

Two Physical Interpretations of the Smash Sum

- Hele-Shaw "stamping" problem:
  - Blob of incompressible fluid in the narrow gap between two plates.
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- Electrostatic interpretation (S. Sheffield):
  - Positively charged solid in $A \cap B$ (charge density $+1$).
  - Neutral solid in $A \cup B - A \cap B$.
  - Negatively charged fluid (charge density -1) outside $A \cup B$.
  - Total energy is minimized when the fluid occupies $A \oplus B - A \cup B$.

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  - Convergence is in the sense of ε-neighborhoods: for all $\varepsilon > 0$

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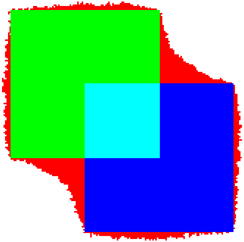
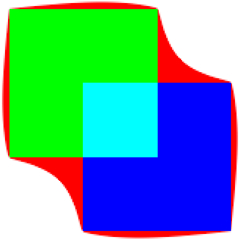
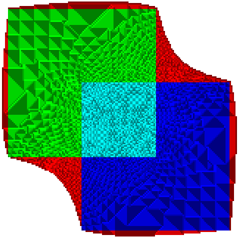


Internal DLA



Divisible Sandpile



Rotor-Router Model

Steps of the Proof

convergence of densities

$$\Downarrow$$

convergence of obstacles

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  - D is the smash sum of balls $B(x_i, r_i)$, where $\lambda_i = \omega_d r_i^d$.

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  $$D_\varepsilon^{::} \subset D_n, R_n, I_n \subset D^{\varepsilon ::}$$

  for all sufficiently large n, where
  - D_n, R_n, I_n are the domains of occupied sites $\delta_n \mathbb{Z}^d$, if $\lfloor \lambda_i \delta_n^{-d} \rfloor$ particles start at each site $x_i^{::}$ and perform divisible sandpile, rotor-router, and Diaconis-Fulton dynamics, respectively.
  - D is the smash sum of balls $B(x_i, r_i)$, where $\lambda_i = \omega_d r_i^d$.
- Follows from the main result and the case of a single point source.

Quadrature Domains

- Given $x_1, \dots x_k \in \mathbb{R}^d$ and $\lambda_1, \dots, \lambda_k > 0$.
- A domain $D \subset \mathbb{R}^d$ satisfying

$$\int_D h(x)dx \le \sum_{i=1}^{k} \lambda_i h(x_i).$$

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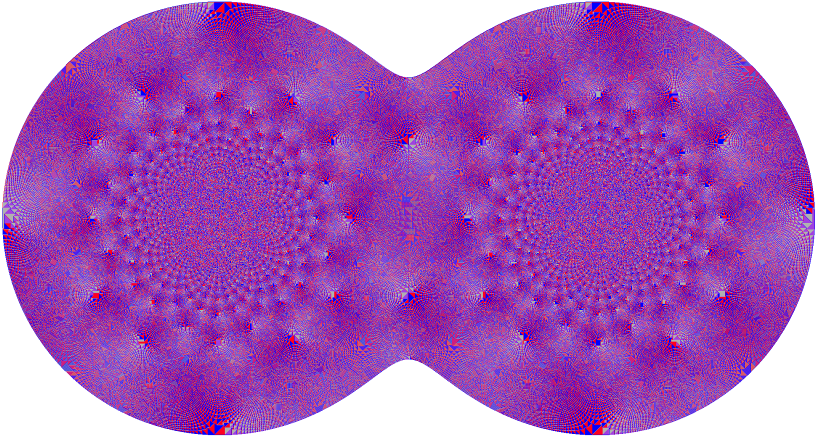
- In dimension two, the boundary of $B_1 \oplus \ldots \oplus B_k$ lies on an algebraic curve of degree $2k$.



$$\frac{1}{\pi r^2}\iint_D h(x,y)\,dx\,dy \le h(-1,0)+h(1,0)$$

The Abelian Sandpile as a Growth Model

- Start with n chips at the origin in $\mathbb{Z}^d$.
- If a site has at least $2d$ chips, it topples by sending one chip to each of the $2d$ neighboring sites.

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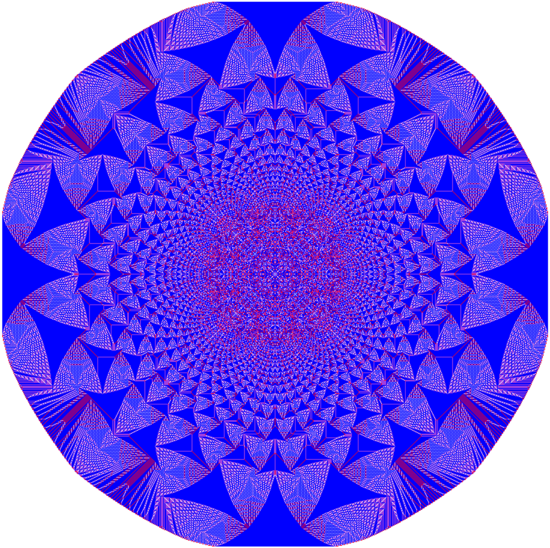
- **Bak-Tang-Wiesenfeld '87, Dhar '90, ...**

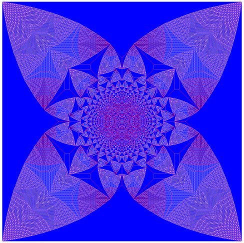
Sandpile of $1,000,000$ chips in $\mathbb{Z}^2$



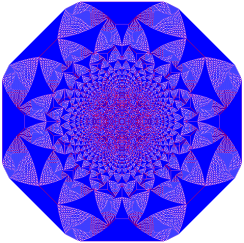
Growth on a General Background

- Let each site $x \in \mathbb{Z}^d$ start with $\sigma(x)$ chips. $(\sigma(x) \leq 2d-1)$
- We call σ the background configuration.
- Place n additional chips at the origin.
- Let $S_{n,\sigma}$ be the set of sites that topple.

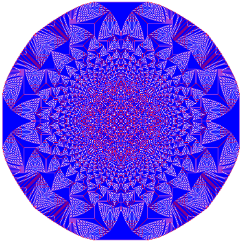
Constant Background $\sigma \equiv h$



$h = 2$

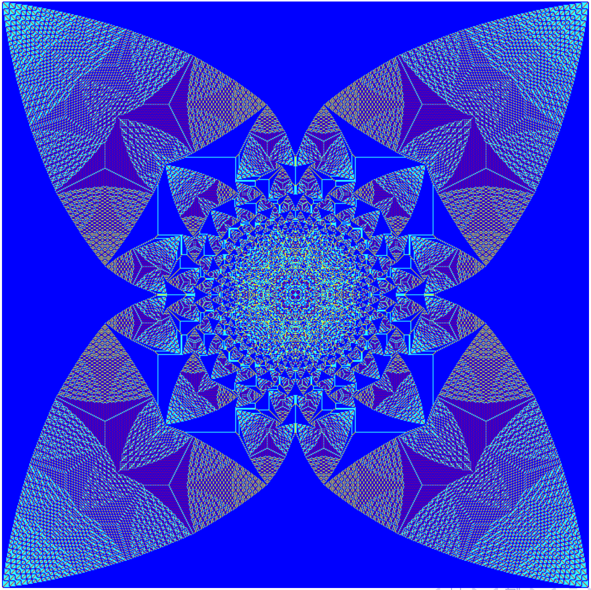


$h = 1$

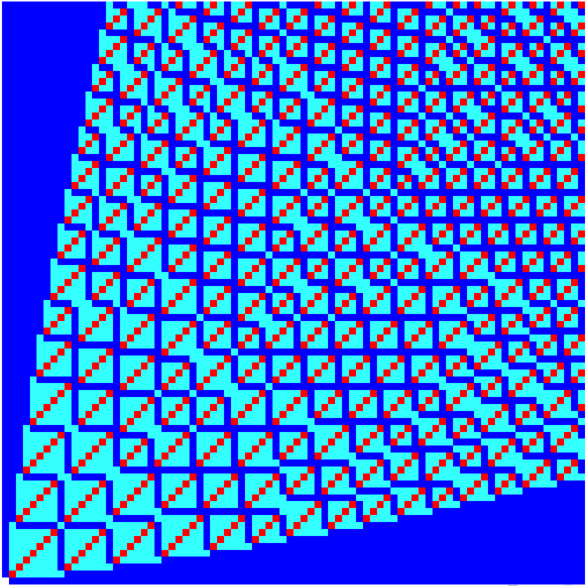


$h = 0$

The Square Sandpile: $d = h = 2$



Closeup of the Lower Left Corner



Odometer Function

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- Discrete Laplacian:

$$\begin{aligned}\Delta u(x) &= \sum_{y \sim x} u(y) - 2d\, u(x) \\ &= \text{chips received} - \text{chips emitted} \\ &= \tau^{\circ}(x) - \tau(x)\end{aligned}$$

where τ is the initial unstable chip configuration
and τ° is the final stable configuration.

Stabilizing Functions

- Given a chip configuration τ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \to \mathbb{Z}$, call u_1 stabilizing for τ if
$$\tau + \Delta u_1 \leq 2d - 1.$$

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- If u_1 and u_2 are stabilizing for τ, then

$$\begin{aligned} \tau + \Delta \min(u_1, u_2) &\le \tau + \max(\Delta u_1, \Delta u_2) \\ &= \max(\tau + \Delta u_1, \tau + \Delta u_2) \\ &\le 2d - 1 \end{aligned}$$

so $\min(u_1, u_2)$ is also stabilizing for τ.

Least Action Principle

- Let τ be a chip configuration on $\mathbb{Z}^d$ that stabilizes after finitely many topplings, and let u be its odometer function.
- Least Action Principle:

$$\text{If } u_1 : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0} \text{ is stabilizing for } \tau \text{, then } u \leq u_1 .$$

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- Least Action Principle:

  If $u_1 : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for τ, then $u \leq u_1$.

- So the odometer is minimal among all nonnegative stabilizing functions:

$$u(x) = \min\{u_1(x) \,|\, u_1 \geq 0 \text{ is stabilizing for } \tau\}.$$

- Interpretation: "Sandpiles are lazy."

Proof of LAP

- Odometer function u, stabilizing function u_1. Want $u \leq u_1$.
- Perform legal topplings in any order, without allowing any site x to topple more than $u_1(x)$ times, until no such toppling is possible.

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- Otherwise, τ' has some unstable site y, and $u'(y) = u_1(y)$.
- Further topplings according to $u_1 - u'$ can only increase the number of chips at y.
- But y is stable in $\tau + \Delta u_1$. $\Rightarrow\Leftarrow$

Obstacle Problem with an Integrality Condition

- **Lemma**. The abelian sandpile odometer function is given by

$$u = s - \gamma$$

where

$$s(x) = \min \left\{ f(x) \;\middle|\; \begin{array}{c} f : \mathbb{Z}^d \to \mathbb{R} \text{ is superharmonic} \\ \text{and } f - \gamma \text{ is } \mathbb{Z}_{\geq 0}\text{-valued} \end{array} \right\}.$$

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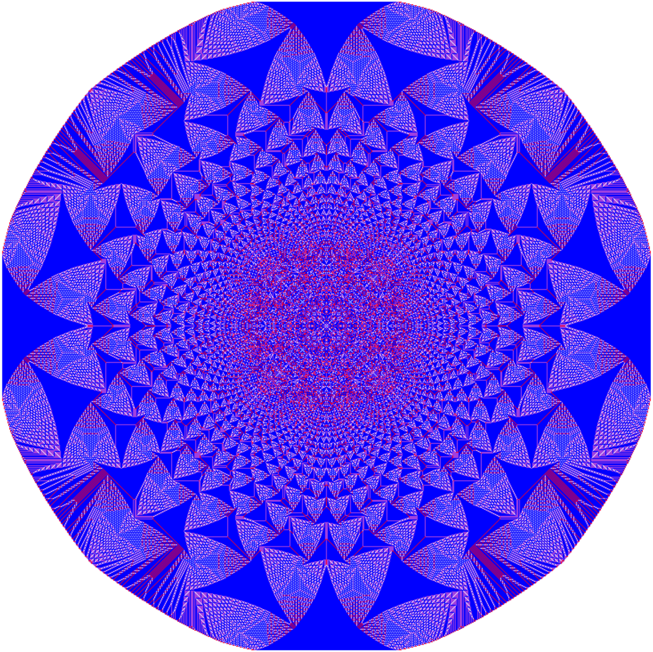
- The obstacle γ is given by

$$\gamma(x) = -\frac{(2d-1)|x|^2 + n \cdot g(o,x)}{2d}$$

where g is the Green's function for simple random walk in $\mathbb{Z}^d$

$$g(o,x) = \mathbb{E}_o \#\{k | X_k = x\}.$$



Bootstrapping From Small Values of h

- **Theorem** (L.-Peres) Let $S_{n,h}$ be the set of sites visited by the abelian sandpile in $\mathbb{Z}^d$, starting from n chips at the origin and constant background $h \leq d-1$.

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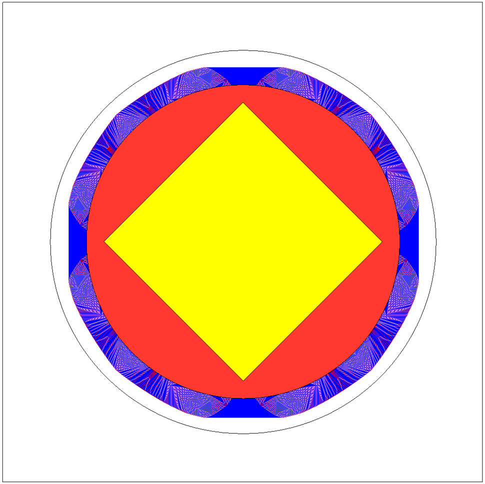
- Improves earlier bounds of Le Borgne and Rossin, Fey and Redig.

Bounds for the Abelian Sandpile Shape



$$(\text{Disk of area } n/3) \subset S_n \subset (\text{Disk of area } n/2)$$

Growth Rate of The Square Sandpile

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$$S_{n,2} \subset Q_r$$

for all sufficiently large n, where

$$r = \left(\frac{2}{\sqrt{\pi}} + \varepsilon\right)\sqrt{n}$$

and

$$Q_r = \{x \in \mathbb{Z}^2 : |x_1|, |x_2| \leq r\}.$$

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- Similar bound with $r = \Theta(n^{1/d})$ in d dimensions, for any constant background $h \leq 2d-2$.

A Mystery: Scale Invariance

- Big sandpiles look like scaled up small sandpiles!
- Let $\sigma_n(x)$ be the final number of chips at x in the sandpile of n particles on $\mathbb{Z}^d$.

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- Let $\sigma_n(x)$ be the final number of chips at x in the sandpile of n particles on $\mathbb{Z}^d$.
- Squint your eyes: for $x \in \mathbb{R}^d$ let

$$f_n(x) = \frac{1}{a_n^2} \sum_{\substack{y \in \mathbb{Z}^d \\ ||y - \sqrt{n}x|| \leq a_n}} \sigma_n(y).$$

where a_n is a sequence of integers such that

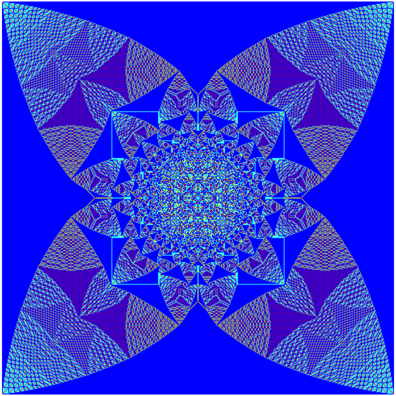
$$a_n \uparrow \infty \quad \text{and} \quad \frac{a_n}{\sqrt{n}} \downarrow 0.$$

Scale Invariance Conjecture

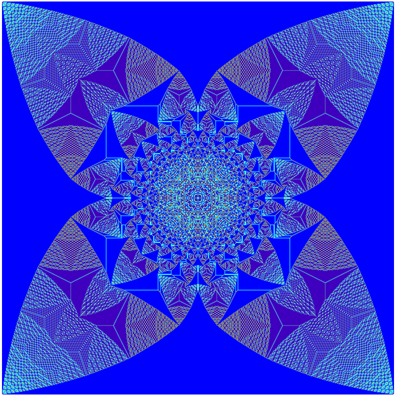
- **Conjecture**: There is a sequence a_n and a function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which is locally constant almost everywhere, such that $f_n \to f$ at all continuity points of f.

Two Sandpiles of Different Sizes

$n = 100,000$

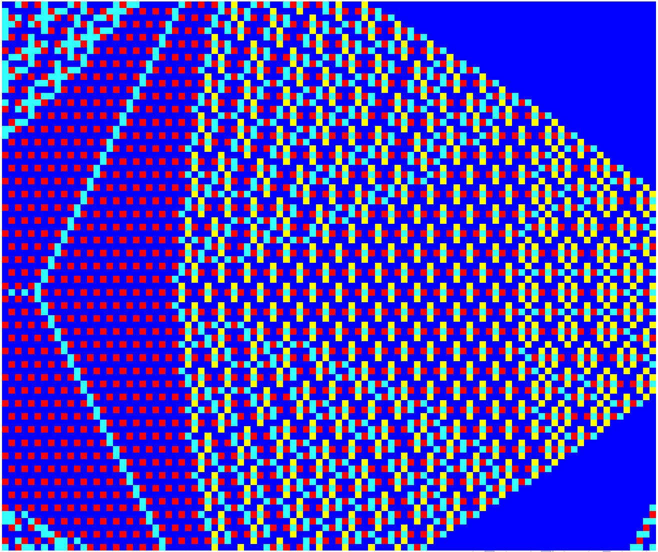


$n = 200,000$



(scaled down by $\sqrt{2}$)

Locally constant "steps" of f correspond to periodic patterns:



A Mystery: Dimensional Reduction

- Our argument used simple properties of one-dimensional sandpiles to bound the diameter of higher-dimensional sandpiles.
- Deepak Dhar pointed out that there seems to be a deeper relationship between sandpiles in d and $d-1$ dimensions...

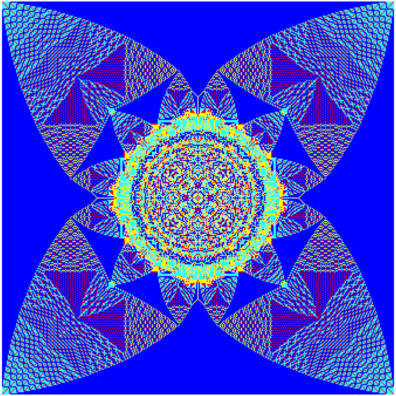
Dimensional Reduction Conjecture

- $\sigma_{n,d}$: sandpile of n chips on background $h = 2d - 2$ in $\mathbb{Z}^d$.
- **Conjecture**: For any n there exists m such that

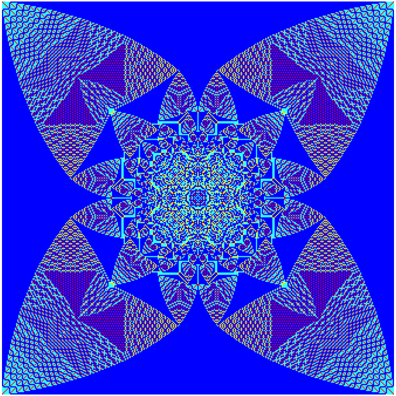
$$\sigma_{n,d}(x_1, \ldots, x_{d-1}, 0) = 2 + \sigma_{m,d-1}(x_1, \ldots, x_{d-1})$$

for almost all x sufficiently far from the origin.

A Two-Dimensional Slice of A Three-Dimensional Sandpile

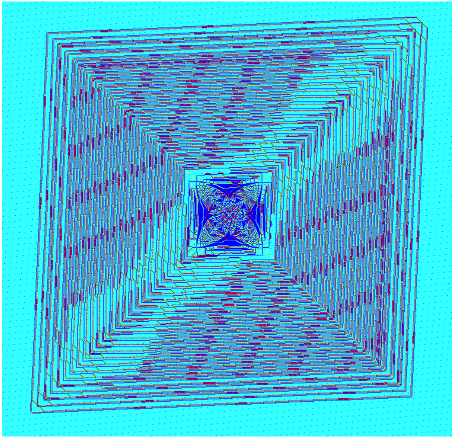


$d = 3$ (slice through origin)
$h = 4$
$n = 5,000,000$



$d = 2$
$h = 2$
$m = 46,490$

Thank You!



arXiv:0712.3378
arXiv:0901.3805