Chip-Firing and A Devil's Staircase

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Two Ideas From Physics

- Mode locking: "Weakly coupled oscillators tend to synchronize their motion, i.e. their modes of oscillation acquire $\mathbb{Z}$-linear dependencies."
  - J. C. Lagarias, 1991.
- Self-organized criticality: "Dynamical systems with extended spatial degrees of freedom naturally evolve into self-organized critical structures of states which are barely stable."
  - Bak, Tang and Wiesenfeld, 1987.
- Our goal: A simple combinatorial model.

Chip-Firing on K_n

- At time t, each vertex $v \in [n]$ has $\sigma_t(v)$ chips
- If $\sigma_t(v) \geq n$, the vertex v is unstable, and can fire by sending one chip to every other vertex.

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- **Parallel update rule**: At each time step, all unstable vertices fire simultaneously:

$$\sigma_{t+1}(v) = \begin{cases} \sigma_t(v) + u_t, & \text{if } \sigma_t(v) \leq n-1 \\ \sigma_t(v) - n + u_t, & \text{if } \sigma_t(v) \geq n \end{cases}$$

where

$$u_t = \#\{v | \sigma_t(v) \geq n\}$$

is the number of unstable vertices at time t.

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  - $\Rightarrow$ The system may never reach a stable configuration.
  - $\Rightarrow$ Order of firings matters!

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  where

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  is the total number of firings before time t.
- Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$.

An Example on K_{10}

- $$\sigma_0 = (\quad 6 \quad 6 \quad 7 \quad 7 \quad 8 \quad 8 \quad 9 \quad 9 \quad 10 \quad 10)$$

An Example on K_{10}

- $$\begin{array}{lcccccccccc} \sigma_0 = (& 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10) \\ \sigma_1 = (& 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11 & 2 & 2) \end{array}$$

An Example on K_{10}

- $$\begin{array}{llllllllllll} \sigma_0 = (& 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10) \\ \sigma_1 = (& 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11 & 2 & 2) \\ \sigma_2 = (& 12 & 12 & 13 & 13 & 4 & 4 & 5 & 5 & 6 & 6) \end{array}$$

An Example on K_{10}

- $$\begin{aligned}
\sigma_0 &= (& 6 && 6 && 7 && 7 && 8 && 8 && 9 && 9 && 10 && 10) \\
\sigma_1 &= (& 8 && 8 && 9 && 9 && 10 && 10 && 11 && 11 && 2 && 2) \\
\sigma_2 &= (& 12 && 12 && 13 && 13 && 4 && 4 && 5 && 5 && 6 && 6) \\
\sigma_3 &= (& 6 && 6 && 7 && 7 && 8 && 8 && 9 && 9 && 10 && 10) && = \sigma_0
\end{aligned}$$
- Period 3, activity $1/3$.

An Example on K_{10}

- $$\begin{aligned}
\sigma_0 &= (6 \quad 6 \quad 7 \quad 7 \quad 8 \quad 8 \quad 9 \quad 9 \quad 10 \quad 10) \\
\sigma_1 &= (8 \quad 8 \quad 9 \quad 9 \quad 10 \quad 10 \quad 11 \quad 11 \quad 2 \quad 2) \\
\sigma_2 &= (12 \quad 12 \quad 13 \quad 13 \quad 4 \quad 4 \quad 5 \quad 5 \quad 6 \quad 6) \\
\sigma_3 &= (6 \quad 6 \quad 7 \quad 7 \quad 8 \quad 8 \quad 9 \quad 9 \quad 10 \quad 10) \quad = \sigma_0
\end{aligned}$$
- Period 3, activity $1/3$.
- $$\sigma_0 = (7 \quad 7 \quad 8 \quad 8 \quad 9 \quad 9 \quad 10 \quad 10 \quad 11 \quad 11)$$

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- $$\begin{aligned}
\sigma_0 &= (\ 6\ \ 6\ \ 7\ \ 7\ \ 8\ \ 8\ \ 9\ \ 9\ \ 10\ \ 10) \\
\sigma_1 &= (\ 8\ \ 8\ \ 9\ \ 9\ \ 10\ \ 10\ \ 11\ \ 11\ \ 2\ \ 2) \\
\sigma_2 &= (\ 12\ \ 12\ \ 13\ \ 13\ \ 4\ \ 4\ \ 5\ \ 5\ \ 6\ \ 6) \\
\sigma_3 &= (\ 6\ \ 6\ \ 7\ \ 7\ \ 8\ \ 8\ \ 9\ \ 9\ \ 10\ \ 10) \quad = \sigma_0
\end{aligned}$$
- Period 3, activity $1/3$.
- $$\begin{aligned}
\sigma_0 &= (\ 7\ \ 7\ \ 8\ \ 8\ \ 9\ \ 9\ \ 10\ \ 10\ \ 11\ \ 11) \\
\sigma_1 &= (\ 11\ \ 11\ \ 12\ \ 12\ \ 13\ \ 13\ \ 4\ \ 4\ \ 5\ \ 5)
\end{aligned}$$

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\sigma_2 &= (& 7 & & 7 & & 8 & & 8 & & 9 & & 9 & & 10 & & 10 & & 11 & & 11) & = \sigma_0
\end{aligned}$$
- Period 2, activity $1/2$.

How Does Adding More Chips Affect the Activity?

3	3	4	4	5	5	6	6	7	7	activity 0

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4	4	5	5	6	6	7	7	8	8	activity 0

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5	5	6	6	7	7	8	8	9	9	activity 0

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6	6	7	7	8	8	9	9	10	10	activity $1/3$

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7	7	8	8	9	9	10	10	11	11	activity $1/2$
8	8	9	9	10	10	11	11	12	12	activity $1/2$
9	9	10	10	11	11	12	12	13	13	activity $2/3$

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4	4	5	5	6	6	7	7	8	8	activity 0
5	5	6	6	7	7	8	8	9	9	activity 0
6	6	7	7	8	8	9	9	10	10	activity $1/3$
7	7	8	8	9	9	10	10	11	11	activity $1/2$
8	8	9	9	10	10	11	11	12	12	activity $1/2$
9	9	10	10	11	11	12	12	13	13	activity $2/3$
10	10	11	11	12	12	13	13	14	14	activity 1
11	11	12	12	13	13	14	14	15	15	activity 1
12	12	13	13	14	14	15	15	16	16	activity 1

An Example on K_{100}

- Let $\sigma = (25\ 25\ 26\ 26\ \ldots\ 74\ 74)$ on K_{100}.
- $(a(\sigma + k))_{k=0}^{100}$ =(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/6, 1/5, 1/5, 1/4, 1/4, 1/4, 2/7, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 2/5, 2/5, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 3/5, 3/5, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 5/7, 3/4, 3/4, 3/4, 4/5, 4/5, 5/6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).

An Example on K_{1000}

- Let $\sigma = (250\ 250\ 251\ 251\ \ldots 749\ 749)$ on K_{1000}.
- $(a(\sigma + k))_{k=0}^{1000} =$

(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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1/6, 1/6, 2/11, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 1/5, 2/9, 1/4, 1/4, 1/4,
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2/3, 7/10, 5/7, 5/7, 5/7, 5/7, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4, 3/4,
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4/5, 4/5, 4/5, 4/5, 4/5, 4/5, 4/5, 4/5, 4/5, 4/5, 9/11, 5/6, 5/6, 5/6, 5/6, 5/6, 5/6, 5/6, 5/6, 6/7, 6/7, 6/7, 6/7,
7/8, 7/8, 8/9, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
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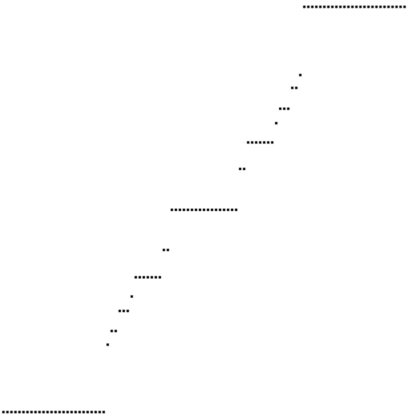
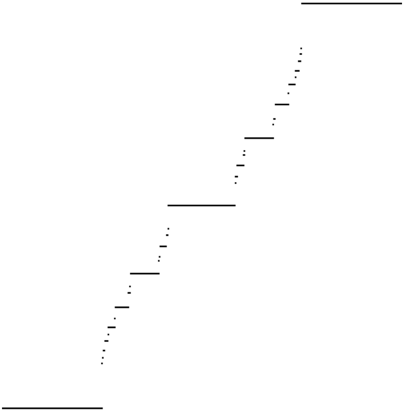
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)

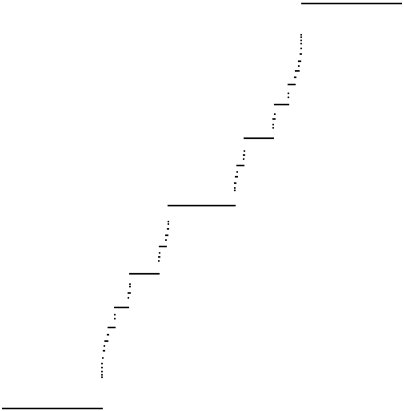
K_{10}



K_{100}



K_{1000}



K_{10000}

Questions

- Why such small denominators?
- Is there a limiting behavior as $n \longrightarrow \infty$?

The Large n Limit

- Sequence of stable chip configurations $(\sigma_n)_{n \geq 2}$ with σ_n defined on K_n.
- **Activity phase diagram** $s_n : [0,1] \to [0,1]$

$$s_n(y) = a(\sigma_n + \lfloor ny \rfloor).$$

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- **Activity phase diagram** $s_n : [0,1] \to [0,1]$

$$s_n(y) = a(\sigma_n + \lfloor ny \rfloor).$$

- Suppose there is a continuous function $F : [0,1] \to [0,1]$, such that for all $0 \leq x \leq 1$

$$\frac{1}{n}\#\{v \in [n] \,|\, \sigma_n(v) < nx\} \to F(x)$$

as $n \to \infty$.

Main Result: The Devil's Staircase

- **Theorem** (L., 2008): There is a continuous, nondecreasing function $s : [0,1] \to [0,1]$, depending on F, such that for each $y \in [0,1]$

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  Moreover
  - If $y \in [0,1]$ is irrational, then $s^{-1}(y)$ is a point.
  - For "most" choices of F, the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0,1]$.

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  - For "most" choices of F, the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0,1]$.
- So for most F, the limiting function s is a *devil's staircase*: it is locally constant on an open dense subset of $[0,1]$.
- Stay tuned for:
  - The construction of s.
  - What "most" means.

Confined States

- Call σ confined if
  - $\sigma(v) \leq 2n-1$ for all vertices v of K_n;
  - $\max_v \sigma(v) - \min_v \sigma(v) \leq n-1$.
- **Lemma**: If $a(\sigma_0) < 1$, then there is a time T such that σ_t is confined for all $t \geq T$.

Which Vertices Are Unstable At Time t?

- Let

$$\alpha_t = u_0 + \ldots + u_{t-1}$$

be the total number of firings before time t.

- **Lemma**: If σ is confined, then v is unstable at time t if and only if

$$\sigma(v) \equiv -j \pmod{n} \qquad \text{for some } \alpha_{t-1} < j \leq \alpha_t.$$

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- **Lemma**: If σ is confined, then v is unstable at time t if and only if
$$\sigma(v) \equiv -j \pmod{n} \qquad \text{for some } \alpha_{t-1} < j \leq \alpha_t.$$
- Proof uses the fact that for any two vertices v, w, the difference
$$\sigma_t(v) - \sigma_t(w) \mod n$$
doesn't depend on t.

A Recurrence For The Total Activity

- Get a three-term recurrence

$$\alpha_{t+1} = \alpha_t + \sum_{j=\alpha_{t-1}+1}^{\alpha_t} \phi(j)$$

where

$$\phi(j) = \#\{v \mid \sigma(v) \equiv -j \pmod{n}\}.$$

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A Recurrence For The Total Activity

- Get a three-term recurrence

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- $\alpha_{t+1} = f(\alpha_t)$, where

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- So $f - Id$ is periodic.

The Poincaré Rotation Number

- Renormalizing and interpolating

$$g(x) = \frac{(1-\{nx\})f(\lfloor nx \rfloor) + \{nx\}f(\lceil nx \rceil)}{n}$$

yields a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

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- So g descends to a circle map $S^1 \to S^1$ of degree 1.
- The **rotation number** of g is defined as the limit

$$\rho(g) = \lim_{t\to\infty} \frac{g^t(0)}{t}.$$

- **Theorem**: $\boxed{a(\sigma) = \rho(g)}$.

Devil's Staircase Revisited

- Sequence of stable chip configurations $(\sigma_n)_{n\geq 2}$ with σ_n defined on K_n.
- Recall: we assume there is a continuous function $F : [0,1] \to [0,1]$, such that for all $0 \leq x \leq 1$

$$\frac{1}{n}\#\{v \in [n] \,|\, \sigma_n(v) < nx\} \to F(x)$$

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  as $n \to \infty$.
- Extend F to all of $\mathbb{R}$ by

$$F(x+m) = F(x) + m, \qquad m \in \mathbb{Z},\ x \in [0,1].$$

  (Since $F(0) = 0$ and $F(1) = 1$, this extension is continuous.)

Devil's Staircase Revisited

- **Theorem**: For each $y \in [0,1]$

$$s_n(y) \to s(y) := \rho(R_y \circ G) \qquad \text{as } n \to \infty,$$

where $G(x) = -F(-x)$, and $R_y(x) = x + y$.

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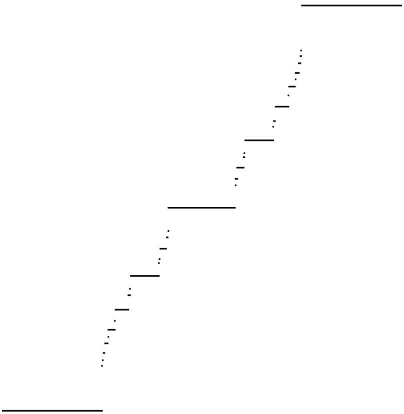
where $G(x) = -F(-x)$, and $R_y(x) = x + y$. Moreover,

  - s is continuous and nondecreasing.
  - If $y \in [0,1]$ is irrational, then $s^{-1}(y)$ is a point.
  - If

$$(\bar{R}_y \circ \bar{G})^q \neq Id : S^1 \to S^1$$

for all $y \in S^1$ and all $q \in \mathbb{N}$, then the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0,1]$.



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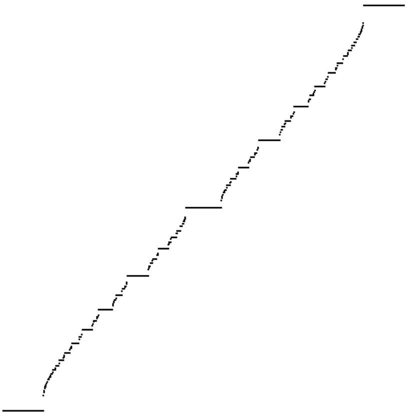

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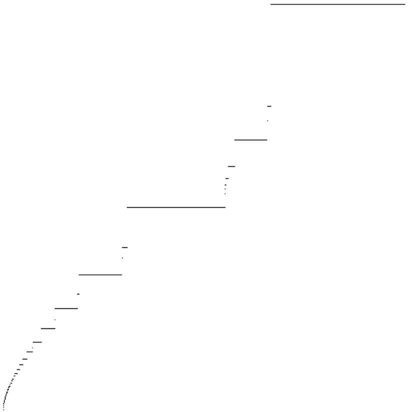

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Properties of the Rotation Number

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  $\Rightarrow$ $s_n \to s$, and s is continuous.

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  $\Rightarrow$ s is nondecreasing.
- **Instability of an irrational rotation number**. If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then $\rho(f_1) < \rho(f) < \rho(f_2)$.

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A Mystery: What About Other Graphs?

- Fixed-energy sandpile on the torus $\mathbb{Z}/n \times \mathbb{Z}/n$: **F. Bagnoli, F. Cecconi, A. Flammini, A. Vespignani** (*Europhys. Lett.* 2003).
  - Started with $m = \lambda n^2$ chips, each at a uniform random vertex.
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  - They found a devil's staircase!
- Is there a circle map hiding here somewhere??

Self-Organized Criticality

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- How to test out this idea in a concrete model?

Critical Density λ_c and Stationary Density λ_s

- Critical density for the fixed-energy sandpile on K_n:

$$\lambda_c(K_n) := \sup\{\lambda | \mathbb{P}(\sigma_\lambda \text{ is stabilizable}) \le \frac{1}{2}\}.$$

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- Stationary density for the dissipative sandpile on K_n:

$$\lambda_s(K_n) := \frac{1}{n^{n-2}} \sum_{\sigma \text{ recurrent on } K_n} \sigma(1).$$

Critical and stationary densities are far from equal!

- Self-organized criticality would predict $\lambda_c \approx \lambda_s$. But...
- **Theorem**. There are constants C_1, C_2 such that

$$\lambda_c(K_n) \geq n - C_1\sqrt{n}\log n.$$

$$\lambda_s(K_n) \leq \frac{n}{2} + C_2\sqrt{n}.$$

The Tutte Polynomial

- $G = (V, E)$ finite connected graph, $s \in V$ sink vertex.
- **Theorem** (C. Merino Lopez, 1997):

$$\sum_{\sigma \text{ recurrent on } G} y^{|\sigma|} = y^{\#E - d_s} T_G(1, y)$$

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where

$$T_G(x, y) = \sum_{\text{spanning trees } T} x^{\text{int}(T)} y^{\text{ext}(T)}$$

is the Tutte polynomial of G.

Unicycles

- A unicycle is a connected spanning subgraph of K_n with exactly n edges.
- Bijection

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- $\Rightarrow \lambda_s(K_n) = \frac{n}{2} + \left(\sqrt{\frac{\pi}{8}} + o(1) \right) \sqrt{n}.$

Thank You!