The Space of Closed Abelian Subgroups of $\mathrm{PSL}_2(\mathbb{C})$

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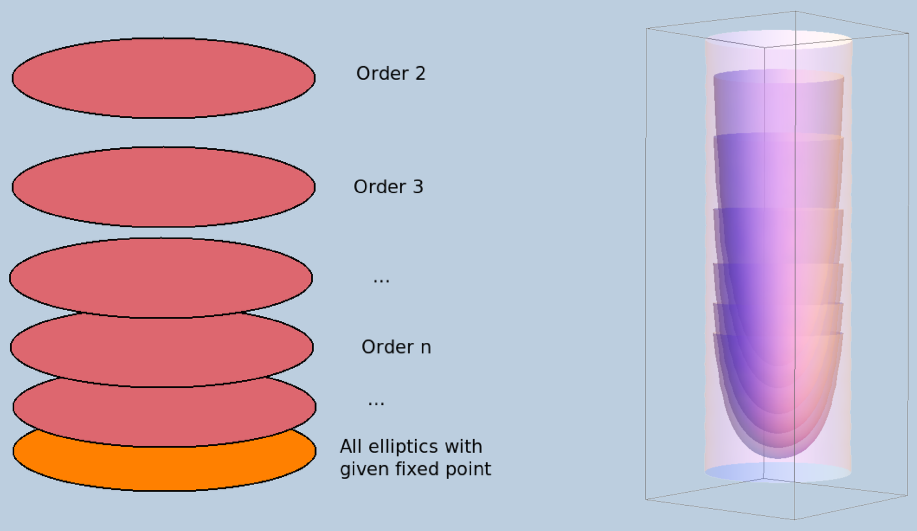
An instructive subcase: the boundary of the space of closed cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$

You can find a detailed treatment of this in [1]. An elliptic automorphism of $\mathbb{H}^2 = \mathbb{D}$ is determined by a fixed point in $\mathbb{D}$ and its order. Thus the space of closed cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$ generated by one elliptic element is a countable disjoint union of disks. To get a correct picture of its closure $\mathbf{E}$, we bend these disks by maps:

$$\mathbb{D}_n \to \mathbb{R}^3$$

$$p = re^{i\theta} \mapsto (r\cos\theta, r\sin\theta, \frac{|1-e^{2\pi i/n}|}{1-r^2})$$

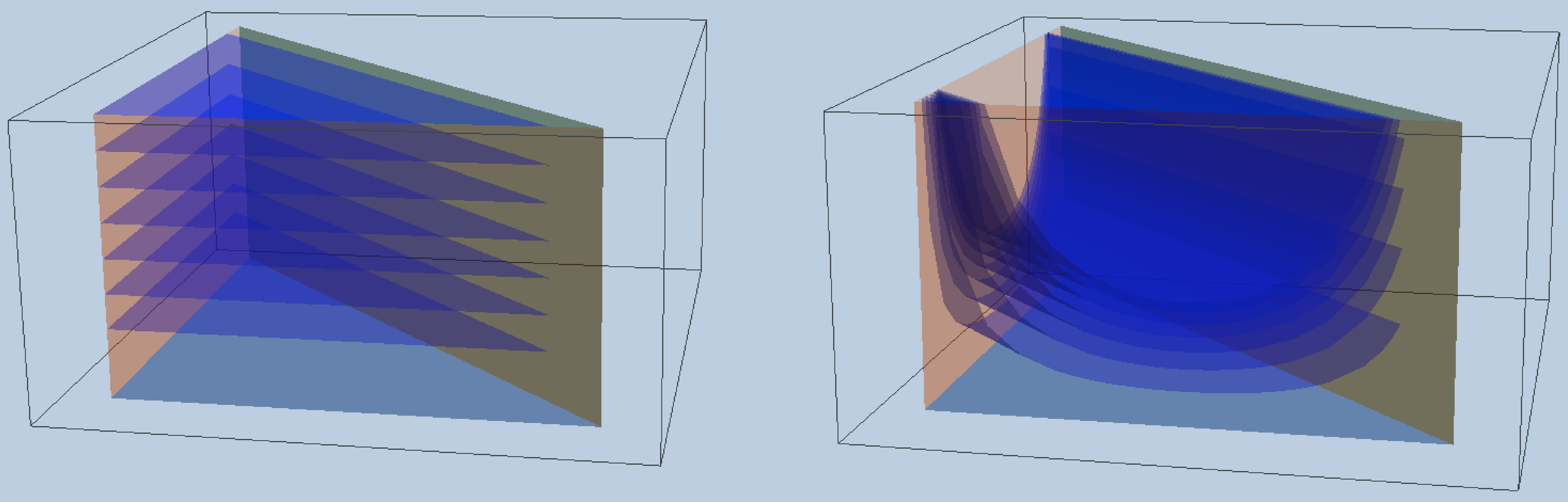


See Section 7 in [1].

A hyperbolic automorphism of $\mathbb{H}^2 = \mathbb{D}$ is determined by a pair of fixed points in $\partial\mathbb{D}$ and a translation quantity. Thus the space of closed cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$ generated by one hyperbolic element is a product of $\mathbb{R}_{>0}$ and the space of pairs of elements of S^1 (a Möbius band). To get a correct picture of its closure $\mathbf{H}$, we bend the leaves of constant translation quantity a by:

$$M \to \mathbb{R}^3$$

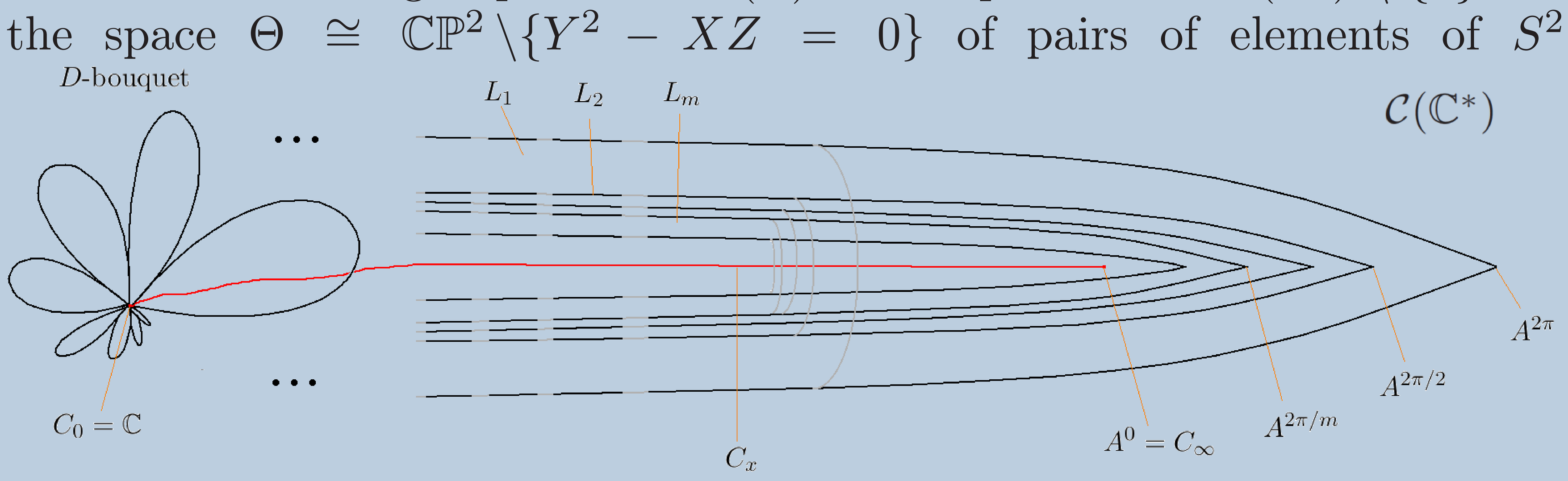
$$(\theta_1, \theta_2) \mapsto (\theta_1, \theta_2, \frac{a-1}{|1-e^{i(\theta_2-\theta_1)}|})$$



See Section 8 in [1].

Bending in the case of $\mathrm{PSL}_2(\mathbb{C})$

A non-trivial non-parabolic automorphism of $\mathrm{PSL}_2(\mathbb{C})$ is determined by two fixed points in $S^2 = \partial\mathbb{H}^3$, and a rotation/complex translation quantity $a \in \mathbb{C}^* \setminus 1$. Thus the space of non-trivial non-parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is the product of $\mathcal{C}(\mathbb{C}^*) \setminus \{1\}$ and the space $\Theta \cong \mathbb{CP}^2 \setminus \{Y^2 - XZ = 0\}$ of pairs of elements of S^2.

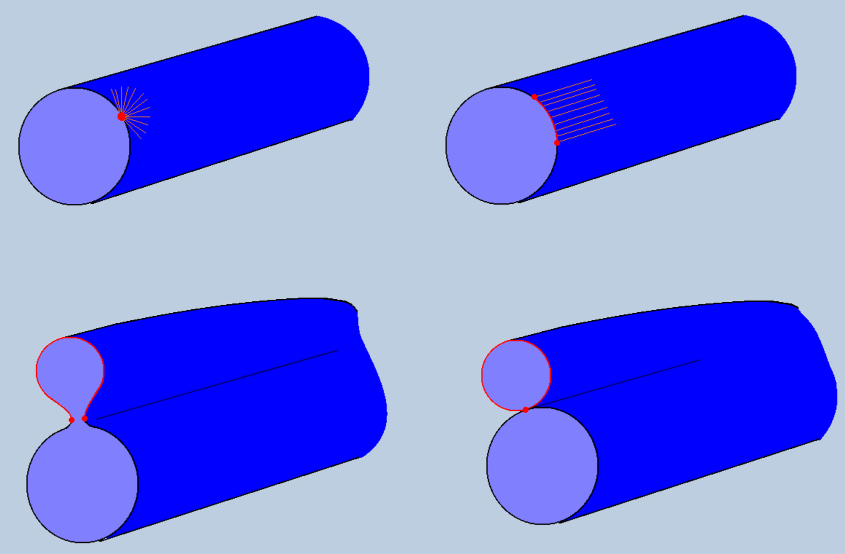


As above, a correct picture consists of bending this direct product, using the map

$$\mathcal{C}(\mathbb{C}^*) \times \Theta \to \mathcal{C}(\mathbb{C})$$

$$\Xi, (z_1, z_2) \mapsto Re^{i\omega}\,\mathrm{Log}(\Xi)$$

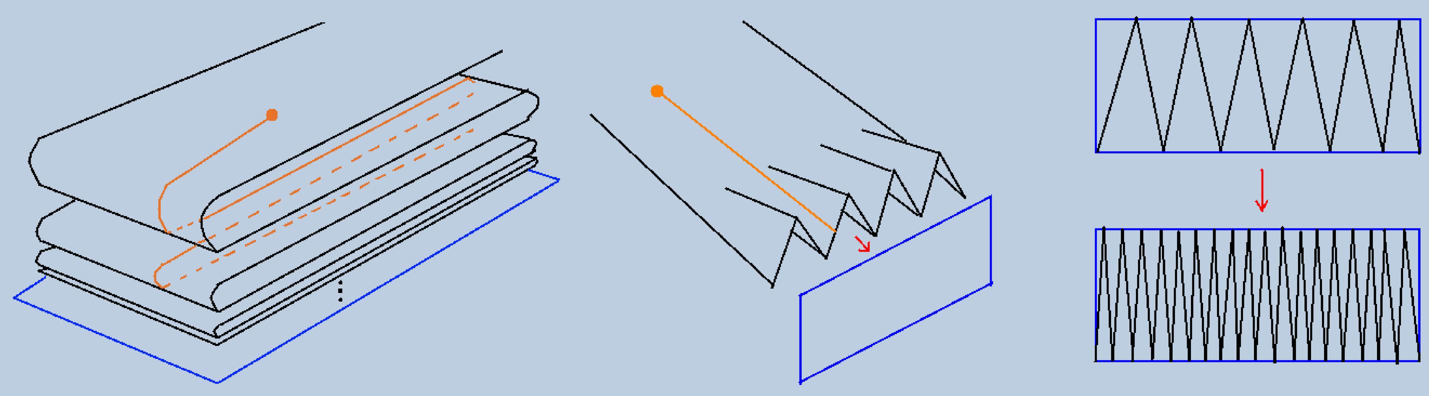
where $1/R$ is the spherical distance and $-\omega$ is the relative angle between z_1 and z_2. More precise statements can be found in Section 3 in [2].



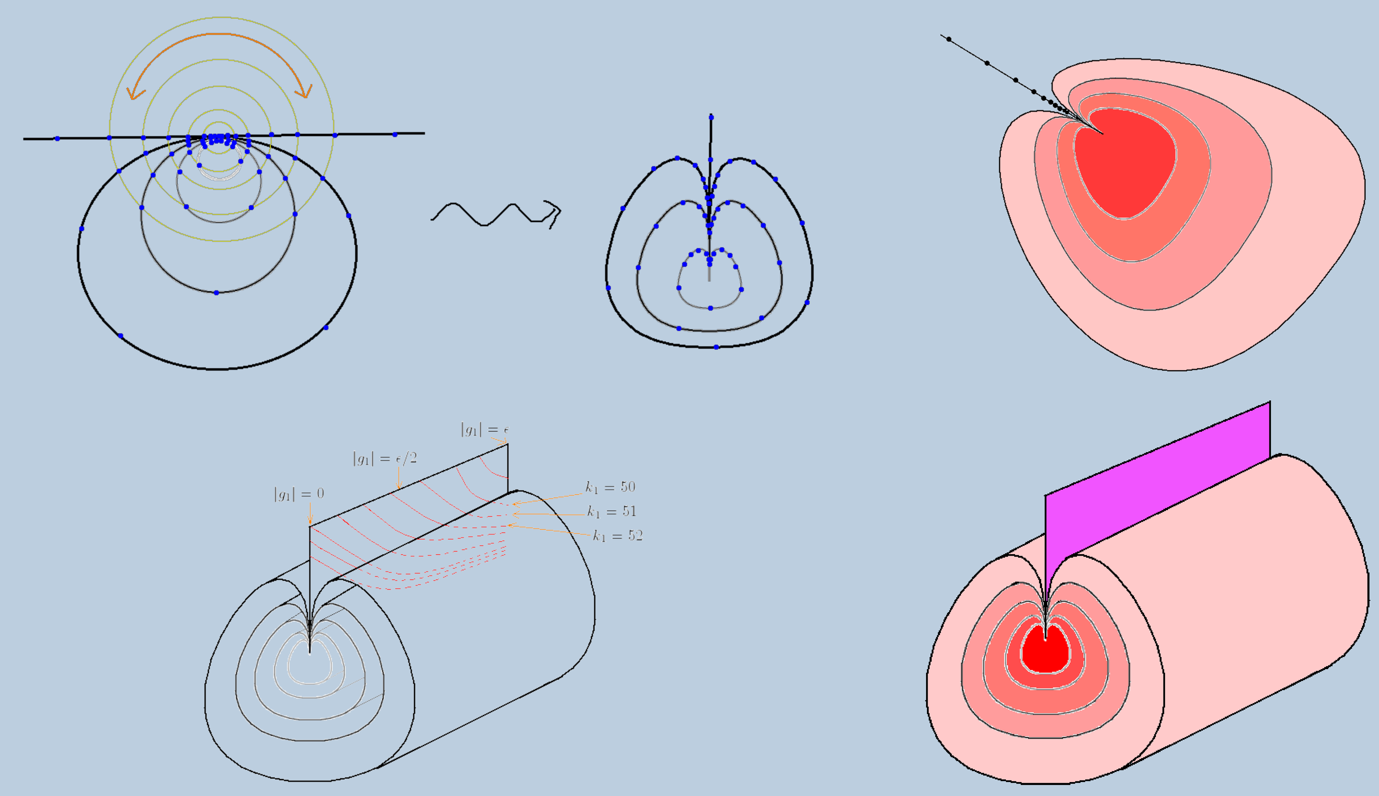
Pinching occurring around the 'bouquet' in the previous picture.

Local models around parabolic groups

When an n-dimensional space X accumulates on another n-dimensional space Y, one can consider a small neighborhood U of a chosen point y in Y. Then there may or may not be a finite path in U connecting some point in X to y. We call these behaviors non-spiraling and spiraling, respectively.



In our case, X is the space of non-parabolic groups and Y is the space of parabolic groups. If y is isomorphic to a lattice $\mathbb{Z}^2$, then there is a spiraling behavior around it. If y is a non-lattice subgroup, then different non-spiraling cases happen.



The picture shows the schematic drawings of the neighborhoods of y which is isomorphic to $\mathbb{Z}^2$, $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{R}$, $\mathbb{R}$, respectively. See Section 6 in [2] for details.

Along the way

Along the way, we investigated the following:

- how to change problems of Hausdorff convergence of (complicated) closed subsets of some space into simpler problems of Hausdorff convergence (Reduction Lemma, in Section 3 in [1]);
- the space of parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ has a twist (Subsections 2.4 and 2.5 in [2]);
- explicit geometric limits for sequences of non-parabolic closed abelian subgroups of $\mathrm{PSL}_2(\mathbb{C})$ converging to a parabolic subgroup; a nice relation with continued fractions came up (Subsection 5.2 in [2]);
- a geometrical interpretation of such converging sequences, using cylinders getting wider and wider (Subsection 3.3 in [2]).

References

[1] Hyungryul Baik, Lucien Clavier: *The Space of Geometric Limits of One-generator Closed Subgroups of PSL2(R)*, arXiv:1202.1365 (2012)
[2] Hyungryul Baik, Lucien Clavier: *The Space of Geometric Limits of Abelian Subgroups of PSL2C*, arXiv:1204.5269 (2012)