Of course, I should add a disclaimer that I don’t actually know what I’m talking about. [Looking back, I really didn’t know what I was talking about!]

1 Poles

We’re going to study poles. First of all is the following:

**Theorem 1.1.** Let \( f \neq 0 \) be a holomorphic function so that \( f(z_0) = 0 \). Then in a neighborhood \( U \) containing \( z_0 \), there exists a holomorphic \( g \) with \( g(z_0) \neq 0 \) so that

\[
 f(z) = (z - z_0)^n g(z)
\]

for some integer \( n \).

I guess the fact that \( n \) is an integer was kind of surprising but

**Proof.** Recall \( f(z) = \sum a_i (z - z_0)^i \). Let \( n \) be the smallest integer so that \( a_n \neq 0 \). Then

\[
 f(z) = (z - z_0)^n (a_n + a_{n+1}(z - z_0) + \ldots) = (z - z_0)^n g(z)
\]

as desired. \( \square \)

We say that \( f \) has a pole at \( z_0 \) if \( 1/f \), defined to be zero at \( z_0 \), is holomorphic in a neighborhood of \( z_0 \).

**Theorem 1.2.** If \( f \) has a pole at \( z_0 \), then in a neighborhood \( U \) containing \( z_0 \) we have

\[
 f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + G(z)
\]

for some holomorphic function \( G \).

**Proof.** Because \( 1/f \) satisfies the condition of the previous theorem, we have \( f(z) = (z - z_0)^{-n} h(z) \) for some holomorphic \( h \). Then \( h = \sum a_i (z - z_0)^i \).

I guess this is called a Laurent series. If \( f \) has a pole at \( z_0 \), then

\[
 f(z) = \underbrace{\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)}}_{\text{"principal part"}} + G(z)
\]
2 Residue Formula

The following is a celebrated result, according to Stein/Shakarchi.

**Theorem 2.1.** Suppose that $f$ is holomorphic in an open set containing a circle $C$ and its interior, except for a pole at $z_0$ inside $C$. Then

$$\int_C f(z) \, dz = 2\pi i \text{res}_{z_0} f$$

**Proof.** Secretly, take the Laurent expansion of $f$, say

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)} + G(z).$$

We use our favorite keyhole contour, and get

$$\frac{1}{2\pi i} \int_{C_\epsilon} \frac{a_{-1}}{z - z_0} \, dz = a_{-1};$$

the book cites Cauchy integral formula but I think if you’re willing to believe that $\int_C 1/z \, dz = 2\pi i$ then you get this result too. Similarly,

$$\frac{1}{2\pi i} \int_{C_\epsilon} \frac{a_{-k}}{(z - z_0)^k} \, dz = 0$$

so by additivity of integrals we have $\int_{C_\epsilon} f \, dz = a_{-1}$. □

Using multiple keyholes you can get additivity, i.e. with poles at $z_1, \ldots, z_N$ we have $\int_C f \, dz = 2\pi i \sum \text{res}_{z_k} f$.

Here’s a computation that will apparently be important for Chapter 6, which we will maybe come close to later on. We will show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{\pi}{\sin(\pi a)}$$

for $0 < a < 1$.

To do this, we consider $\gamma_R$ the rectangle on

$$\Im(z) = 0 \to \Re(z) = R \to \Im(z) = 2\pi \to \Re(z) = -R \to \Im(z) = 0.$$

I’m too lazy to tikz this.

Let $f(z) = \frac{e^{az}}{1 + e^z}$; notice that $f$ has a pole at $z = \pi i$, so we should probably calculate that residue:

$$\lim_{z \to \pi i} (z - \pi i) f(z) = e^{a\pi i} \lim_{z \to \pi i} \frac{z - \pi i}{e^z - e^{\pi i}} = -e^{a\pi i} \frac{1}{(\text{deriv of } e^z)}$$

so the residue formula says $\int_{\gamma_R} f = -2\pi i e^{a\pi i}$.

Observe that $f$ over the vertical sides of the rectangle vanishes as $R \to \infty$. If $I_R$ denotes the integral over the bottom side of the rectangle, then the integral over the top of the rectangle is $-e^{2\pi i} I_R$ (the minus sign is due to opposite orientation). So in the limit

$$I - e^{2\pi i} I = -2\pi i e^{a\pi i}$$

and solving gives $I = \frac{\pi}{\sin(\pi a)}$. 

2
3 Argument Principle

We’ll skip over meromorphic functions, because we can. I will just define it in the way that I want to, and we’ll discuss it more later, because that’s what I actually want to talk about.

**Definition 3.1.** A meromorphic function is a quotient of two holomorphic functions.

This is different from the definition in the book, but you’ll see that it has the two properties that the book uses to define them, namely: at most countably many poles and holomorphic elsewhere.

**Theorem 3.2.** Let $f = g/h$ be meromorphic in an open set containing a circle $C$ and its interior. If $g, h \neq 0$ on $C$, then
\[
\frac{1}{2\pi i} \int_C \frac{f'}{f} \, dz = \#\{z \in \text{int}(C) : g(z) = 0\} - \#\{z \in \text{int}(C) : h(z) = 0\},
\]
counted with multiplicity.

The result follows from argument principle (handwaving a little here; I should argue like in the book, but oh well). We have Rouché’s Theorem:

**Theorem 3.3.** Suppose that $f, g$ are holomorphic in an open set containing a circle $C$ and its interior. If
\[|f(z)| > |g(z)| \quad \text{for all } z \in C\]
then $f$ and $f + g$ have the same number of zeros inside the circle $C$.

**Proof.** For $t \in [0, 1]$, define $f_t(z) = f(z) + tg(z)$. Define $n_t$ to be the number of zeros inside $C$, that is,
\[n_t = \frac{1}{2\pi i} \int_C \frac{f_t'(z)}{f_t(z)} \, dz\]
since $f_t$ has no zeros on $C$ (this uses $|f| > |g|$). Observe that $n_t$ is a continuous function in $t$.

We also have the open mapping theorem:

**Theorem 3.4.** If $f$ is holomorphic and nonconstant, then $f$ maps open sets to open sets.

**Proof.** Let $w_0$ be in the image of $f$, so $w_0 = f(z_0)$. Find a circle $|z - z_0| = \delta$ so that $f(z) \neq w_0$ for every $z$ (one exists because $f - w_0$ only has countably many zeros xd); pick $\varepsilon$ so that $|f - w_0| \geq \varepsilon$. Then let $w$ be any point so that $|w - w_0| < \varepsilon$. Consider
\[g(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) \equiv F(z) + G(z)\]
so that $|F| > |G|$. Rouché says that $g$ has a zero inside the circle.

We also have the maximum modulus principle.

**Theorem 3.5.** If $f$ is a nonconstant holomorphic function on an open set then it does not attain a maximum.

**Proof.** At any point $z_0$, pick an open ball $D \ni z_0$ and observe that $f(D)$ is an open set containing $f(z_0)$. So $f(z_0)$ is not a maximum.

4 Meromorphic functions

Let me first observe that a holomorphic function $f$ contains at most countably many zeros.
Proof. Observe that

\[ \mathbb{C} = \bigcup_{k \in \mathbb{N}} B_k \]

so if \( f \) has uncountably many zeros on \( \mathbb{C} \) then it has in particular countably many zeros on some ball \( B_k \); this is compact so there is some accumulation point. We had a theorem that says holomorphic functions vanishing on a sequence of distinct points with a limit point on \( \Omega \) is identically 0 in \( \Omega \).

Also notice that holomorphic functions \( \mathcal{H} \) form a commutative ring under pointwise addition/multiplication; the additive identity is 0 everywhere so if \( fg = 0 \) then either \( f = 0 \) or \( g = 0 \). Such an algebraic structure is called an integral domain and whenever you see one you should think “oh, so it’s something like \( \mathbb{Z} \)”.

The set of meromorphic functions \( \mathcal{M} \), then, is the fraction field of the integral domain \( \mathcal{H} \). The relevant analogy is that \( \mathcal{M} \) is like \( \mathbb{Q} \).

Let me define meromorphic functions in their usual way:

A Riemann surface \( X \) is a 1-dimensional complex manifold, so it comes with an atlas. An example of a Riemann surface is \( \mathbb{C} \cup \{ \infty \} \). A map \( f: X \to X' \) is holomorphic if \( \varphi_{X'} \circ f \circ \varphi_X^{-1} \) is a holomorphic map \( \mathbb{C} \to \mathbb{C} \) for all \( \varphi \) in the atlases.

A holomorphic function on \( X \) is a holomorphic map \( X \to \mathbb{C} \). A meromorphic function on \( X \) is a holomorphic map \( X \to \mathbb{C} \cup \{ \infty \} \) that is not identically \( \infty \). The following dichotomy is nuts:

If \( X \) is a non-compact Riemann surface, then \( \mathcal{M}(X) \) is isomorphic to the fraction field of \( \mathcal{H}(X) \).

If \( X \) is a compact Riemann surface, then \( \mathcal{H}(X) \) consists only of constant functions; \( \mathcal{M}(X) \) is a finite dimensional vector space over \( \mathbb{C} \), and the precise dimension is given by some Riemann-Roch theorem. [Actually \( \mathcal{M}(X) \) is a transcendence degree 1 vector space over \( \mathbb{C} \), so it’s an algebraic extension of \( \mathbb{C}(x) \). Riemann-Roch tells you the dimension of these \( \mathcal{L}(D) \) thingies that are talked about in thingy #2. There are more details about this dichotomy here.]

Hopefully this motivates the study of meromorphic functions. In the first case, meromorphic functions seem to be “convenient”, at least structurally. Like, it would be nice if we were closed under division. In the second case, there is still a lot of structure to unpack, and it seems like there are phenomena to understand.