A NOTE ON POINCARÉ, SOBOLEV, AND HARNACK INEQUALITIES

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1. Introduction. Let $M$ be a $C^\infty$-connected manifold. Let $L$ be a second-order differential operator with real $C^\infty$ coefficients on $M$ and such that $L1 = 0$ (i.e., $L$ has no zero-order term). Assume that there exists a positive $C^\infty$ measure $\mu$ on $M$ such that

$$\langle L\varphi, \psi \rangle = \langle \varphi, L\psi \rangle, \quad \langle L\psi, \psi \rangle \geq 0$$

for all $\varphi, \psi \in C^\infty_0(M)$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $L^2(M, d\mu)$. We make the technical hypothesis that $L$ is locally subelliptic. Denote also by $L$ the Friedrichs extension of $L$ in $L^2(M, d\mu)$ and consider the symmetric submarkovian semigroup $H_t = e^{-tL}$ acting on the spaces $L^2(M, d\mu)$. The $C^\infty$ kernel $h_t(x, y)$ of $H_t$ is defined by

$$H_tf(x) = \int_M h_t(x, y)f(y)\,d\mu(y).$$

Since we assume that $L$ is locally subelliptic, there exists a genuine distance function $\rho$ canonically associated with $L$; see [6, 9]. This distance is continuous and defines the topology of $M$. We assume that $(M, \rho)$ is a complete metric space. Set $B(x, r) = \{ y \in M, \rho(x, y) < r \}$ and $V(x, r) = \mu(B(x, r))$. There is also a notion of gradient associated with $L$. At any rate, we can set

$$\Gamma(\varphi, \psi) = \frac{1}{2}(-L(\varphi\psi) + \varphi L\psi + \psi L\varphi)$$

and define the "length of the gradient" to be $|\nabla f| = \Gamma(f, f)^{1/2}$. ($\Gamma(f, f)$ is the "carré du champ" of Bakry-Emery [1].) See also [26] for an equivalent definition of $|\nabla f|$. It can be shown (see [3, 9] for instance) that, under our hypotheses,

$$\rho(x, y) = \sup \{ |f(x) - f(y)|, f \in C^\infty(M), |\nabla f| \leq 1 \}.$$  

What is really important for us is that, although $\rho$ is not smooth, we can formally apply the inequality $|\nabla \rho(x_0, x)| \leq 1$.

Given $0 < r_0 \leq +\infty$, consider the two properties

$$V(x, 2r) \leq C_1 V(x, r), \quad 0 < r < r_0, \quad x \in M \quad (1)$$

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and

\[ \int_{B(x, r)} |f - f_{x,r}|^2 \, d\mu \leq C_2 r^2 \int_{B(x,2r)} |\nabla f|^2 \, d\mu, \quad 0 < r < r_0, \quad x \in M, \quad f \in C_0^\infty(M) \] (2)

where \( f_{x,r} = \int_{B(x,r)} f \, d\mu \). Property (1) is the usual doubling property. The inequality appearing in property (2) is a (weak) form of Poincaré inequality. It follows from the work of D. Jerison \cite{Jer} that (1) and (2) imply the (strong) Poincaré inequality where the integral on the right-hand side of (2) is taken over the ball \( B(x, r) \) instead of \( B(x, 2r) \).

In this paper we show that a parabolic Harnack inequality is equivalent to the above two properties (see Section 3). In \([15,16]\), J. Moser proved a Harnack inequality for parabolic equations associated with second-order uniformly elliptic divergence form operators in Euclidean space. His approach has been used in many other situations because it rests only on two functional inequalities usually referred to as Sobolev and Poincaré inequalities. Here, we show that the doubling property (1) and the family of Poincaré inequalities (2) imply a family of Sobolev inequalities which is good enough to run Moser's iteration. It is well known that Harnack inequality is a powerful tool. Selected applications are presented which illustrate this fact.

One aspect of this work is that it unifies important results which were obtained in different settings by different means. For instance, consider the question whether or not harmonic positive functions are constant. S.-T. Yau proved that the answer is yes on manifolds with nonnegative Ricci curvature (here, \( L \) is the Laplace operator); see \([27]\). Y. Guivarc'h in \([7]\), T. Lyons and D. Sullivan in \([13]\) also gave a positive answer for manifold which are normal covering of a compact manifold with nilpotent deck transformation group. In \([24]\), N. Varopoulos obtained a similar result in the setting of Lie groups having polynomial volume growth. As explained in the last section, all these results can be seen as corollaries of Theorem 4.3 below.

2. Sobolev inequality. In this section we show that (1) and (2) imply a family of Sobolev inequalities on balls.

**Theorem 2.1.** Assume that \( M, L \) are as above and that (1), (2), hold for some fixed \( r_0 > 0 \). Then there exist \( \nu > 2 \) and \( C_3 > 0 \) depending only on \( C_1, C_2 \) such that

\[
\left( \int |f|^{2\nu/(\nu - 2)} \, d\mu \right)^{(\nu - 2)/\nu} \leq C_3 V(x, r)^{-2/\nu} r^2 \left( \int (|\nabla f|^2 + r^{-2} |f|^2) \, d\mu \right),
\]

\[ f \in C_0^\infty(B(x, r)) \]

for all \( x \in M \) and all \( 0 < r < r_0 \).
Note that for $0 < s \leq r$ we have

$$V(x, r) \leq 2V(x, s)\left(\frac{r}{s}\right)^{\nu_0}$$

for some $\nu_0 > 0$ depending only on $C_1, C_2$. Indeed, consider the integer $n$ such that $2^{n-1} < r/s \leq 2^n$. From the doubling property it follows that

$$V(x, r) \leq V(x, 2^n s) \leq C_1^n V(x, s) \leq 2^n V(x, s)\left(\frac{r}{s}\right)^{\nu_0}$$

where $\nu_0 = \log(C_1)/\log(2)$. The real $\nu$ appearing in Theorem 2.1 can be taken to be any number greater or equal to $\nu_0$ and strictly greater than 2. The first ingredient of the proof of Theorem 2.1 is an abstract result.

**Theorem 2.2.** Let $e^{-tA}$ be a symmetric submarkovian semigroup acting on the spaces $L^p(M, d\mu)$. Given $\nu > 2$, the following properties are equivalent.

1. $\|e^{-tA}f\|_{\infty} \leq C_t^{-\nu/2}\|f\|_1$ for $0 < t < t_0$.
2. $\|f\|_2^{2\nu/(\nu-2)} \leq C_5(\|A^{1/2}f\|_\infty^2 + t_0^{-1}\|f\|_\infty^2)$.
3. $\|f\|_2^{2+4\nu} \leq C_6(\|A^{1/2}f\|_\infty^2 + t_0^{-1}\|f\|_\infty^2)\|f\|_1^{4\nu}$.

Moreover, 3. implies 1. with $C_4 = (vCC_6)^{\nu/2}$ and 1. implies 2. with $C_5 = CC_2^{2\nu}$, where $C$ is some numerical constant.

The proof of 1. implies 2. follows easily from [22, Theorem 1. The equivalence with 3. follows from [3]. The other ingredients in the proof of Theorem 2.1 are the two following lemmas. Denote by $f_s(x)$ the mean of $f$ over the ball $B(x, s)$. Set $\chi_s(x, z) = V(x, s)^{-1}1_{B(x, s)}(z)$ so that

$$f_s(x) = \int \chi_s(x, z)f(z)\,d\mu(z).$$

**Lemma 2.3.** There exists a constant $C_7$ depending only of $C_1$ such that for all $y \in M$ and all $0 < s \leq r < r_0$ we have

$$\|f_s\|_2 \leq C_7 V^{-1/2}(r/s)^{\nu_0/2}\|f\|_1, \quad \text{for all } f \in C_0^\infty(B)$$

where $B = B(y, r)$, $V = V(y, r)$.

**Proof.** Note that $\chi_s(x, z) \leq C_1 \chi_s(z, x)$. This shows that $\|f_s\|_1 \leq C_1 \|f\|_1$. Moreover, if $B \cap B(x, s) \neq \emptyset$ with $0 < s \leq r$, (3) yields

$$V(x, s)^{-1} \leq 2V(x, 2r + s)^{-1}(2r/s + 1)^{\nu_0} \leq V^{-1}(4r/s)^{\nu_0}.$$

Hence, $\|f\|_\infty \leq V^{-1}(4r/s)^{\nu_0}\|f\|_1$ for all $f \in C_0^\infty(B)$. The lemma follows by interpolation.
Lemma 2.4. There exists $C_8$ depending only on $C_1, C_2$, such that

$$\|f - f_s\|_2 \leq C_8 s \|\nabla f\|_2, \quad f \in \mathscr{C}_0^\infty(M)$$

for all $0 < s < r_0/4$.

Proof. Fix $0 < s < r_0/4$. Let $\{B_j, j \in J\}$ be a collection of balls of radius $s/2$ such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $M = \bigcup_{i \in J} 2B_i$, where $tB = B(x, tr)$ if $B = B(x, r)$. Such a collection always exists. Moreover, the doubling property implies that the overlapping number $N(z) = \# \{i \in J, z \in 8B_i\}$ is bounded by a number $N_0$ depending only on $C_1$. Now, write

$$\|f - f_s\|_2^2 \leq \sum_{i \in J} \int_{2B_i} |f(x) - f_s(x)|^2 \leq \sum_{i \in J} \left( \int_{2B_i} |f(x) - f_{4B_i}|^2 + |f_{4B_i} - f_s(x)|^2 \right)$$

where all the integrations are taken with respect to $\mu$ and where $f_B$ is the mean of $f$ over the ball $B$. Poincaré inequality (2) implies

$$\int_{2B_i} |f(x) - f_{4B_i}|^2 \leq \int_{4B_i} |f(x) - f_{4B_i}|^2 \leq C_2 s^2 \int_{8B_i} |\nabla f|^2.$$

Using (1) and (2), we also have

$$\int_{2B_i} |f_{4B_i} - f_s(x)|^2 \leq \int_{2B_i} \int_{4B_i} \chi_{B_i}(x, z) |f_{4B_i} - f(z)|^2 \, d\mu(z) \, d\mu(x) \leq C_9 V_i^{-1} \int_{2B_i} \int_{4B_i} |f_{4B_i} - f(z)|^2 \, d\mu(z) \, d\mu(x) \leq C_{10} s^2 \int_{8B_i} |\nabla f|^2.$$

Hence, we obtain

$$\|f - f_s\|_2^2 \leq C_{11} s^2 \sum_{i \in J} \int_{8B_i} |\nabla f|^2 \leq C_{11} N_0 s \|\nabla f\|_2^2.$$

This ends the proof of Lemma 2.4.

Proof of Theorem 2.1. Fix $x \in M, 0 < r < r_0$, and set $v = \max \{3, v_0\}$. Assume that $0 < s \leq r/4$ and $f \in \mathscr{C}_0^\infty(B(x, r))$. Following an idea of Robinson [17], write

$$\|f\|_2 \leq \|f - f_s\|_2 + \|f_s\|_2.$$
Using the above two lemmas, we obtain

\[ \|f\|_2^2 \leq C_8 s \|\nabla f\|_2 + C_7 V^{-1/2}(r/s)^{\nu/2} \|f\|_1 \]

where \( V = V(x, r) \). Hence, for all \( s > 0 \) and \( f \in \mathcal{C}^\infty_0(B(x, r)) \), we have

\[ \|f\|_2^2 \leq 4C_8 s (\|\nabla f\|_2 + r^{-1} \|f\|_2) + C_7 V^{-1/2}(r/s)^{\nu/2} \|f\|_1. \]

Optimizing over \( s > 0 \) yields

\[ \|f\|_2^{2+4/v} \leq C_{12} V^{-2/v} r^{2} (\|\nabla f\|_2^2 + r^{-2} \|f\|_2^2) \|f\|_1^{4/v}. \]

Theorem 2.1 follows from the above and Theorem 2.2. In [5], Th. Coulhon and the author use variations of the above arguments to study isoperimetric questions on Riemannian manifolds. In the present setting, the method of [5] shows that (1) and the \( L^1 \) version of (2), namely

\[ \int_{B(x, r)} |f - f_r(x)| \leq C_2 r \int_{B(x, 2r)} |\nabla f|, \quad 0 < r < r_0, \quad x \in M, \quad f \in \mathcal{C}^\infty(M) \]

imply the \( L^1 \) version of Theorem 2.1 which reads

\[ \left( \int |f|^{v/(v-1)} \, d\mu \right)^{(v-1)/v} \leq C_4 V(x, r)^{-1} \left( \int (|\nabla f| + r^{-1} |f|) \, d\mu \right), \quad f \in \mathcal{C}^\infty_0(B(x, r)) \]

for all \( x \in M \) and all \( 0 < r < r_0 \).

3. Harnack inequality. The power of properties (1) and (2) is better understood through the result presented below. Indeed, we show in this section that the conjunction of (1) and (2) is equivalent to a parabolic Harnack inequality.

**Theorem 3.1.** Let \( M \) and \( L \) be as in Section 1. The following two properties are equivalent.

1. The properties (1) and (2) hold for \( M, L, \) and some \( r_0 > 0 \).
2. There exists \( r_1 > 0, \) and there exists a constant \( C \) depending only on the parameters \( 0 < \varepsilon < \eta < \delta < 1, \) such that, for any \( x \in M, \) any real \( s, \) and any \( 0 < r < r_1, \) any nonnegative solution \( u \) of \((\partial_t + L)u = 0 \) in \( Q = ]s - r^2, s[ \times B(x, r) \) satisfies

\[ \sup_{Q_-} \{ u \} \leq C \inf_{Q_+} \{ u \} \]

where \( Q_- = [s - \delta r^2, s - \eta r^2] \times B(x, \delta r) \) and \( Q_+ = [s - \varepsilon r^2, s[ \times B(x, \delta r). \)
Proof of 1. implies 2. This part of the theorem follows from Moser's iteration: assume that (1) and (2) hold, Theorem 2.1 yields the family of Sobolev inequalities

\[ \| f \|_{2/(v-2)}^2 \leq CV^{-2/v}r^2 (\| \nabla f \|_2^2 + r^{-2} \| f \|_2^2) \]

\[ f \in C^0_0(B(x, r)), \quad y \in M, \quad 0 < r < r_0. \]

As explained in [20] in a Riemannian setting, such a family of Sobolev inequalities is enough to run the first part of Moser's iteration. Hence, we have (see [15]) the following theorem.

**Theorem 3.2.** Assume that (1), (2), hold for some \( r_0 > 0 \). Given \( 0 < \delta < 1 \), there exists a constant \( C \) depending on \( C_1, C_2, \) and \( \delta \), such that, for any \( x \in M \), any real \( s \), and any \( 0 < r < r_0 \), any nonnegative solution of \( (\partial_t + L)u \leq 0 \) in \( Q = ]s - r^2, s[ \times B(x, r) \) satisfies

\[ \sup_{Q_s} \{ u^2 \} \leq C(r^2V)^{-1} \int_Q u^2 \]

where \( Q_s = ]s - \delta r^2, s[ \times B(x, \delta r) \).

In order to obtain the full Harnack inequality stated in Theorem 3.1, we first note that the technique presented in [21] applies here and allows us to deduce from (1) and (2) the following weighted form of Poincaré inequality. Set \( \Phi_{x,r}(z) = (1 - \rho(x, z)/r)^2 \) for \( z \in B(x, r) \) and \( \Phi_{x,r}(z) = 0 \) otherwise. Also, set \( \tilde{f}_r(x) = \int f \Phi_{x,r} \). We have

\[ \int |f - \tilde{f}_r(x)|^2 \Phi_{x,r} \leq C r^2 \int |\nabla f|^2 \Phi_{x,r} \]

for all \( x \in M \), \( 0 < r < r_0 \) and \( f \in C^\infty(M) \). Once we have such a weighted Poincaré inequality, we can prove statement 2. of Theorem 3.1 by using Moser's technique; see [15, 16, 20].

Proof of 2. implies 1. First, we show that 2. implies the doubling property of the volume. Recall that \( h_t \) is the kernel of \( H_t = e^{-tL} \). Applying 2. to \( h_t \), we obtain

\[ V(x, r)h_{2r}(x, x) \leq C \int_{B(x, r)} h_{2r}(x, y) d\mu(y) \leq C. \]

Consider now the function defined by \( u(s, z) = H_s 1_{B(x, r)}(z) \) when \( s > 0 \), and \( u(s, z) = 1 \) when \( s \leq 0 \). This function is a nonnegative solution of \( (\partial_t + L)u = 0 \) in \( ]-\infty, +\infty[ \times B(x, r) \). Hence, we have

\[ 1 = u(-r^2/4, x) \leq Cu(r^2/2, x) = \int_{B(x, r)} h_{2r}(x, y) d\mu(y) \leq C^2 V(x, r)h_{2r}(x, x). \]
The above yields

\[(C'V(x, r))^{-1} \leq h_{r^2}(x, x) \leq C'V(x, r)^{-1}.\]

Hence, 2. implies that \(V(x, 2r) \leq C''V(x, r)\) for all \(x \in M\) and all \(0 < r < r_1/2\).

The fact that 2. implies Poincaré inequality on balls follows from a remark of Kusuoka-Stroock [11] which we now explain. Denote by \(H_{B,t}\) the semigroup associated with the operator \(L\) and Neumann boundary condition on the ball \(B = B(x, r)\), where \(x \in M\) and \(0 < r < r_1\). Let \(h_{B,t}\) be the kernel of this semigroup. Applying Harnack inequality to \(h_{B,t}\) as above, we find that

\[h_{B,r^2}(z, y) \geq (CV)^{-1} \quad \text{for all } y, z \in B(x, r/2)\]

where \(V = V(x, r)\). Hence, for \(y \in B(x, r/2)\) we have

\[H_{B,r^2}(f - H_{B,r^2}f(y))^2(y) \geq (CV)^{-1} \int_{B(x, r/2)} |f(z) - H_{B,r^2}f(y)|^2 \, d\mu(z) \geq (CV)^{-1} \int_{B(x, r/2)} |f - f_{r/2}(x)|^2 \, d\mu.\]

Integrating over \(B(x, r/2)\), we obtain

\[
\int_B H_{B,r^2}(f - H_{B,r^2}f(y))^2(y) \, d\mu(y) \geq C'\int_{B(x, r/2)} |f - f_{r/2}(x)|^2 \, d\mu.
\]

But, we also have

\[
\int_B H_{B,r^2}(f - H_{B,r^2}f(y))^2(y) \, d\mu(y) = \|f\|^2_{2,B} - \|H_{B,r^2}f\|^2_{2,B} = -\int_0^{r^2} \partial_s\|H_{B,s}f\|^2_{2,B} \, ds \leq 2r^2 \int_B |\nabla f|^2 \, d\mu.
\]

This proves (2) with \(r_0 = r_1/2\) and also ends the proof of Theorem 3.1.

Remark. One can wonder whether the parabolic Harnack inequality 2. could be replaced by an elliptic Harnack inequality for \(L\)-harmonic functions on balls. I do not know the answer to this question.
4. Applications. The preceding section made it clear that (1) and (2) are enough to obtain powerful results concerning the operator $L$. In this section we present some further consequences of the hypothesis that $L$ satisfies (1) and (2). Since these results are obtained by arguments which have been explained elsewhere, I will be sketchy. A classical corollary of Harnack inequality is the Hölder continuity of the solutions of the given equation. Namely, we have the following theorem.

**Theorem 4.1.** Assume that (1), (2), hold for some $r_0 > 0$. Fix $0 < \delta < 1$. There exist $0 < a < 1$ and $C$ depending only on $C_1$, $C_2$, $\delta$, and such that, for any $x \in M$, $s \in ]-\infty, +\infty[$, and any $0 < r < r_0$, any solution $u$ of $(\partial_t + L)u = 0$ in $Q = ]s - r^2, s[ \times B(x, r)$ satisfies

$$|u(t', y') - u(t, y)| \leq C(\bar{\rho}/r)^a \|u\|_{\infty, Q}$$

where $\bar{\rho} = \max\{|t - t'|^{1/2}, \rho(y, y')\}$ and $(t, y), (t', y') \in Q_\delta$.

See Moser’s article [15] for a proof. Another important corollary of Theorems 3.1 and 3.2 is as follows.

**Theorem 4.2.** Assume that (1), (2), hold for some $r_0 > 0$. Then there exist constants $C_k$, $k = 0, 1, 2, \ldots$, such that

$$|\partial_t^k h_t(x, x')| \leq C_k V(x, t^{1/2} \wedge r_0)^{-1} t^{-k} (1 + \rho^2/t)^{\nu_0/2 + k} \exp(-\rho^2/4t)$$

for all $x, x' \in M$, all $t > 0$, and $\rho = \rho(x, x')$. Also, there exist $C, C'$, such that

$$h_t(x, x') \geq (C V(x, t^{1/2}))^{-1} \exp(-C'\rho^2/t)$$

for all $x, x' \in M$, $t > 0$, such that $\rho \leq r_0$ and $t \leq r_0^2$.

The proof can be adapted from the arguments in [20]. Note that, when $r_0 = +\infty$, we obtain a global two-sided Gaussian estimate for $h_t$. This implies that, under the hypothesis that (1), (2), hold with $r_0 = +\infty$, the Green function $G(x, y)$ of $L$ exists if and only if $\int_1^{+\infty} V(x, t^{1/2})^{-1} dt < +\infty$. Moreover, $G$ satisfies

$$C^{-1} \int_{\rho^2}^{+\infty} V(x, t^{1/2})^{-1} dt \leq G(x, y) \leq C \int_{\rho^2}^{+\infty} V(x, t^{1/2})^{-1} dt$$

where $\rho = \rho(x, y)$; see [12].

Consider the bottom of the spectrum of $L$ defined by $\lambda_0 = \inf\{\langle Lf, f \rangle/\|f\|_2^2, f \in C_0^\infty(M)\}$. In the case when $r_0 < +\infty$, it is possible that $\lambda_0 > 0$. See [20] for a Gaussian upper bound on $h_t$ which can be adapted to the present setting and takes $\lambda_0$ into account.

In [10], Koranyi and Taylor give elegant arguments which show that the uniqueness property for the positive Cauchy problem associated with $\partial_t + L$ follows from a local uniform Harnack inequality. Hence, their results apply to operators $L$ which
satisfy (1), (2) for some $r_0 > 0$. In the process they show that any minimal solution $u > 0$ of $(\partial_t^2 + L)u = 0$ on $]-\infty, \infty[ \times M$ is of the form $u(t, x) = e^{\gamma t}v(x)$, where $v$ is a minimal solution of $Lv = \gamma v$ on $M$. (Recall that a solution $u > 0$ is minimal if any solution $v$ such that $0 \leq v \leq u$ is a constant multiple of $u$.)

Concerning $L$-harmonic functions, Theorems 3.1 and 4.1 yield the following theorem.

**Theorem 4.3.** Assume that (1), (2), hold with $r_0 + \infty$. Then any solution of $Lv = 0$ on $M$ which is bounded below is constant. Moreover, there exists $0 < \alpha \leq 1$ depending only on $C_1, C_2$, such that any solution $v$ of $Lv = 0$ which satisfies

$$\lim_{r \to +\infty} \left( r^{-\alpha} \sup_{B(x_0, r)} \{ ||v|| \} \right) = 0$$

for some fixed $x_0 \in M$ is constant.

Finally, there is a further idea which, together with Theorem 3.2, yields interesting results. Namely, consider the wave equation $(\partial_t + L)u = 0$. When $L = \Delta$ is the Laplace operator of a complete Riemannian manifold, it is well known that, if $u(t, .)$ is supported in $B(x_0, r)$ and $s > t$, then $u(s, .)$ is supported in $B(x_0, r + s - t)$. In other words, waves have finite propagation speed. We claim that this is still true for the operator $L$ considered in this paper. Indeed, this can be seen by replacing $L$ by $L + \epsilon \Delta$, where $\Delta$ is the Laplace operator for some fixed Riemannian structure on $M$, and letting $\epsilon$ tend to zero; see [14]. The main point in this argument is to show that the distance associated with $L + \epsilon \Delta$ tends to the distance associated with $L$ when $\epsilon$ tends to zero; this follows from the qualitative hypothesis that $L$ is locally subelliptic. Once the above finite propagation speed property has been proved for $L$, we can follow Section 2 of [4] and obtain estimates on the kernels of operators $f(L^{1/2})$, where $f$ is a (nice) even function; see also [20], Section 8. Instead of writing a general theorem, we note the following application of this technique. (See [20] for more details in a Riemannian setting.)

**Theorem 4.4.** Assume that (1), (2), hold with $r_0 = +\infty$. Fix a positive integer $\sigma$. Then the kernel $h_{\epsilon, t}$ of the semigroup $e^{-tL^{1/2}}$ satisfies

$$|\partial_t^k h_{\epsilon, t}(x, x')| \leq C_k V(x, t^{1/2\sigma})^{-1} t^{-k} \exp\left( -\frac{\rho}{C_k t^{1/2\sigma}} \right)$$

for all $t > 0$, $x, x' \in M$, $\rho = \rho(x, x')$, and any fixed integer $k$. In particular, we have $\|\partial_t^k h_{\epsilon, t}(x, \cdot)\|_1 \leq C_k t^{-k}$ for all $t > 0$ and all $x \in M$, which shows that the semigroup $e^{-tL^{1/2}}$ is bounded analytic on $L^p$ for all $p \in [1, +\infty]$. Theorem 4.4.

Remarks. It is worth emphasizing the fact that the above results are very stable. For instance, if we assume that (1) and (2) hold for an operator $L$, then all that has been said about $L$ is also valid for any operator $L'$ symmetric with respect to a measure $\mu'$ and such that $C^{-1} \Gamma(f, f) \leq \Gamma'(f, f) \leq C \Gamma(f, f)$ for all $f \in C^0(M)$ and $C^{-1} V(x, r) \leq V'(x, r) \leq C V(x, r)$. Here, $L'$ does not even need to have smooth...
coefficients. For instance, Theorem 2.1 shows that Harnack inequality is stable under quasi-isometric changes of a metric on a Riemannian manifold.

Another instance of the stability of the above is as follows. Assume that \( M, L, \mu \) and \( M', L', \mu' \), are as in Section 1. Assume that \( L \) satisfies (1), (2), for some \( r_0 > 0 \). Assume also that \( \pi: M \to M' \) is a surjection such that \( L(u \circ \pi) = L' u \circ \pi \) for any smooth function \( u \) on \( M' \). Then (1), (2), also hold for \( L' \) and some \( r'_0 > 0 \). This is because Harnack inequality projects easily from \( M, L \) to \( M', L' \). Note that it does not seem easy to see more directly that the doubling property holds for \( M', L', \mu' \).

In the same spirit as the above remarks, note that operators of the form \( L + \text{lower-order terms} \) can also be studied using Moser’s iteration; see [20] and the references given there. In fact, most of the results described in [20] could be adapted to the present setting.

5. Examples. In this section we describe different settings where the above results apply.

**Example 1.** Let \( M, \gamma, \) be a complete Riemannian manifold of dimension \( n \) and \( L = \Delta \) be the corresponding Laplace operator. Assume that there exists \( K > 0 \) such that the Ricci curvature satisfies \( \text{Ric} \geq -K \gamma \) on \( M \). Classical comparison theorems imply that \( \frac{V(x, r)}{V(x, s)} \leq \frac{(r/s)^n}{e^{r(n-1)K}} \) for \( r > s \); see [4], for instance. Moreover, P. Buser proved in [2] that

\[
\int_{B(x,r)} |f - f_r(x)|^2 \, dv \leq r^2 C r^{K/2} \int_{B(x,r)} |\nabla f|^2 \, dv
\]

for \( f \in C^\infty(M), x \in M, \) and \( r > 0 \). Hence, we can apply the above in this setting. Note that, when \( K = 0, (1) \) and (2) hold with \( r_0 = +\infty \). This gives an alternative approach to most of the results of Li-Yau [12]. (The above method cannot yield gradient estimates but only Hölder continuity estimates.) At the same time, we also recover the results of [19] concerning manifolds which are quasi-isometric to a manifold with nonnegative Ricci curvature.

**Example 2.** Let \( G \) be a Lie group having polynomial volume growth; see [23, 18], for instance. Let \( L = -\sum X_i^2 \), where \( \{X_1, \ldots, X_k\} \) is a family of left invariant vector fields having the Hörmander property (see [23]). Then (1) and (2) hold with \( r_0 = \infty \); see [23, 24]. Hence, we recover the results of [23, 24, 18, 21]. The conclusion of Theorem 4.4 is new in this setting. Note that, in this context, Poincaré inequality is very easy to obtain; see [24].

**Example 3.** Let \( N \) be a compact Riemannian manifold and let \( M \) be a normal covering of \( N \). Assume that the deck transformation group \( G \) of this covering has polynomial volume growth. It follows from the arguments in [25] that (1), (2), hold with \( r_0 = \infty \) for the Laplace operator on \( M \). Moreover, thanks to the second remark at the end of last section, if \( H \subset G \) is closed subgroup of \( G \) (not necessarily normal), (1), (2), also holds on \( M' = M/H \) with \( r_0 = +\infty \). Many of the results obtained above,
including the uniform Harnack inequality which follows from Theorem 3.1, are new in this setting.

**Example 4.** Consider again a normal covering $M$ of a compact Riemannian manifold $N$ with the deck transformation group $G$ having polynomial volume growth. Let $L_0$ be the Laplace operator on $M$, $\mu_0$ be the Riemannian volume, and $V_0$ be the Riemannian gradient. Now let $L$, $\mu$ be as in Section 1. Assume that $L$ is uniformly subelliptic with respect to the Laplace operator $L_0$ and that $d\mu = md\mu_0$ with $C^{-1} < m < C$. Also, assume that $|Vf| \leq C|V_0f|$. (In the case when $M$ is just the euclidean space, this last hypothesis means that $L$ has bounded coefficients.) It follows from the local results concerning subelliptic operators (see [9] for details and references) and the arguments in [25] that (1), (2), holds for $L$ with $r_0 = \infty$. Note that our hypotheses are satisfied whenever $L$ is the pullback of a subelliptic operator on the compact manifold $N$.

**References**


