Chapter 5, problem 10:

(a) \( \sqrt{2\pi} F(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt \)

\[ = \int_{0}^{1} (1-t) e^{-i\xi t} dt + \int_{0}^{1} (1+t) e^{-i\xi t} dt \]

\[ = \int_{0}^{1} e^{-i\xi t} dt - \int_{0}^{1} e^{-i\xi t} dt + \int_{0}^{1} e^{-i\xi t} dt + \int_{-1}^{0} e^{-i\xi t} dt \]

\[ = -\frac{1}{i\xi} e^{-i\xi t} \bigg|_{-1}^{1} - \left[ -\frac{1}{i\xi} e^{-i\xi t} \bigg|_{0}^{1} + \int_{0}^{1} \frac{1}{i\xi} e^{-i\xi t} dt \right] + \left[ -\frac{1}{i\xi} e^{-i\xi t} \bigg|_{-1}^{0} + \int_{-1}^{0} \frac{1}{i\xi} e^{-i\xi t} dt \right] \]

\[ = -\frac{1}{i\xi} e^{-i\xi} + \frac{1}{i\xi} e^{i\xi} + \frac{1}{i\xi} e^{-i\xi} - \frac{1}{i\xi} e^{i\xi} \bigg|_{0}^{1} - \frac{1}{i\xi} e^{-i\xi} \bigg|_{-1}^{0} + \frac{1}{i\xi} e^{-i\xi} \bigg|_{-1}^{1} \]

\[ = -\frac{1}{i\xi} e^{-i\xi} + \frac{1}{i\xi} + \frac{1}{i\xi} e^{-i\xi} - \frac{1}{i\xi} e^{i\xi} = \frac{1}{i\xi} \left[ e^{-i\xi} - e^{i\xi} \right] + \frac{2}{i\xi} \]

\[ = -\frac{1}{\xi^2} \left[ \cos \xi - i\sin \xi + \cos \xi + i\sin \xi - 2 \right] = -\frac{2\cos \xi + 2}{\xi^2} \quad \text{(*)} \]

\( \frac{1 - \cos x}{2} = \sin^2 \left( \frac{x}{2} \right) \) can be used to get

\[ \text{(*)} = \frac{2 \cdot \sin^2 \left( \frac{x}{2} \right)}{\xi^2} \]

\( \Rightarrow \ G(\xi) = 2 \cdot \frac{2}{\pi} \cdot \frac{\sin^2 \left( \frac{x}{2} \right)}{\xi^2} \) \hspace{1cm} \square
(b) (5.28) reads $\csc^2(\frac{\pi}{2}) = \sum_{k \in \mathbb{Z}} \frac{4}{(\xi + 2\pi k)^2}$

Differentiating twice gives

$$\frac{d}{d\xi} \left( -\cot \left( \frac{\pi}{2} \right) \csc^2 \left( \frac{\pi}{2} \right) \right) = \frac{d}{d\xi} \sum_{k \in \mathbb{Z}} \frac{4 \cdot (-2)}{(\xi + 2\pi k)^3}$$

$$\cot \left( \frac{\pi}{2} \right) \csc^2 \left( \frac{\pi}{2} \right) + \frac{1}{2} \csc^4 \left( \frac{\pi}{2} \right) = \sum_{k \in \mathbb{Z}} \frac{4 \cdot 6}{(\xi + 2\pi k)^4} = A$$

$$
\csc^2 \left( \frac{\pi}{2} \right) \left( \cot \left( \frac{\pi}{2} \right) + \frac{1}{2} \csc^2 \left( \frac{\pi}{2} \right) \right) = A
$$

$$
\frac{1}{\sin^2 \left( \frac{\pi}{2} \right)} \left( \frac{\cos^2 \left( \frac{\pi}{2} \right)}{\sin^2 \left( \frac{\pi}{2} \right)} + \frac{1}{2} \right) = A
$$

$$
\frac{1}{\sin^4 \left( \frac{\pi}{2} \right)} \left( 1 - \sin^2 \left( \frac{\pi}{2} \right) + \frac{1}{2} \right) = A
$$

$$
\frac{3 - 2 \sin^2 \left( \frac{\pi}{2} \right)}{4 \pi \sin^4 \left( \frac{\pi}{2} \right)} = \sum_{k \in \mathbb{Z}} \frac{1}{(\xi + 2\pi k)^4}
$$

(c) Thm 5.18 will give the result if we can show that

$$\sum_{k \in \mathbb{Z}} \left| \hat{\phi}(\xi + 2\pi k) \right|^2 = \frac{1}{2\pi}$$


\[
\sum_{k \in \mathbb{Z}} \left| 2 \frac{\sin^2 \left( \frac{\xi + 2\pi k}{2} \right)}{\pi (\xi + 2\pi k)^2 - 1} \right|^2
\]
\[= \frac{4}{\pi} \sum_{k \in \mathbb{Z}} \frac{\sin^4 \left(\frac{3 + 2\pi k}{2}\right)}{(3 + 2\pi k)^4 (1 - \frac{3}{2} \sin^2 \left(\frac{3 + 2\pi k}{2}\right))}\]

\[= 3 \cdot \frac{3}{\pi} \sum_{k \in \mathbb{Z}} \frac{\sin^4 \left(\frac{3}{2} + \pi k\right)}{(3 + 2\pi k)^4 (3 - 2\sin^2 \left(\frac{3}{2} + \pi k\right) = \sin^2 \left(\frac{3}{2}\right) \right) \text{just changes sign by periodicity}\]

\[= \frac{24}{\pi} \left(\sum_{k \in \mathbb{Z}} \frac{1}{(3 + 2\pi k)^4}\right) \frac{\sin^4 \left(\frac{3}{2}\right)}{3 - 2\sin^2 \left(\frac{3}{2}\right)}\]

\[= \frac{24}{\pi} \cdot \frac{1}{48} = \frac{1}{2\pi}\]

as required to apply 5.18.

□
Chapter 5, problem 11:

We have to verify $P(1) = 1$, $|P(z)|^2 + |P(-z)|^2 = 1$ for $|z| = 1$

$|P(e^{it})| > 0$ for $|t| \leq \frac{\pi}{2}$

$P(1) = \frac{1}{2} \sum_{k=0}^{3} p_k = \frac{1}{2} \left[ \frac{1+13}{4} + \frac{3+13}{4} + \frac{3-13}{4} + \frac{1-13}{4} \right] = \frac{4}{8} \cdot [1+3+3+1] = 1$

For the other two parts we can use a CAS; see relevant printout on next page (Mathematica)
The next command loads a package to symbolically declare the type of variables (e.g. real or complex). We declare \( t \) to be real-valued and put in \( z = \text{Exp}[i \cdot t] \) to represent a complex number with unit modulus.

\[
\text{In}[42] := \text{<< Algebra`ReIm`}
\]

\[
\text{In}[75] := t /; \text{Im}[t] = 0;
\]

\[
\begin{align*}
A &= \text{Re}[\text{Exp}[i \cdot t]]^2 + \text{Im}[\text{Exp}[i \cdot t]]^2 \\
B &= \text{Re}[\text{Exp}[i \cdot t]]^2 + \text{Im}[\text{Exp}[i \cdot t]]^2
\end{align*}
\]

\[
\text{Out}[76] = \left(\frac{1}{8} + \frac{\sqrt{3}}{8} - \frac{3 \cos[t]}{8} + \frac{1}{8} \sqrt{3} \cos[t] +
\right.
\]

\[
\left.\frac{3}{8} \cos[2t] - \frac{1}{8} \sqrt{3} \cos[2t] + \frac{1}{6} \cos[3t] - \frac{1}{8} \sqrt{3} \cos[3t]\right)^2 +
\]

\[
\left(\frac{3 \sin[t]}{8} - \frac{1}{8} \sqrt{3} \sin[t] + \frac{3}{8} \sin[2t] - \frac{1}{8} \sqrt{3} \sin[2t] + \frac{1}{8} \sin[3t] - \frac{1}{8} \sqrt{3} \sin[3t]\right)^2
\]

\[
\text{Out}[77] = \left(\frac{1}{8} + \frac{\sqrt{3}}{8} - \frac{3 \cos[t]}{8} - \frac{1}{8} \sqrt{3} \cos[t] +
\right.
\]

\[
\left.\frac{3}{8} \cos[2t] - \frac{1}{8} \sqrt{3} \cos[2t] - \frac{1}{6} \cos[3t] + \frac{1}{8} \sqrt{3} \cos[3t]\right)^2 +
\]

\[
\left(-\frac{3 \sin[t]}{8} - \frac{1}{8} \sqrt{3} \sin[t] + \frac{3}{8} \sin[2t] - \frac{1}{8} \sqrt{3} \sin[2t] - \frac{1}{8} \sin[3t] + \frac{1}{8} \sqrt{3} \sin[3t]\right)^2
\]

\[
\text{In}[70] := \text{Simplify}[A + B]
\]

\[
\text{Out}[70] = 1
\]

And that is precisely what we had to prove. Please note that if you use a Computer Algebra System (CAS) you HAVE TO supply a printed version of the commands you used. Now for the plot of the second part we get:

\[
\text{In}[74] := \text{Plot}[\text{Abs}[\text{Exp}[i \cdot x]]], \{x, -\pi / 2, 4, \pi / 2 + 4\}
\]

\[
\text{Out}[74] = \text{Graphics}
\]

So we are clearly positive inside the interval \([-2,2]\) and since \(\pi/2 < 2\) we get the result.
Chapter 5, problem 13:

see p. 213, it should read

\[ P(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} p_k z^k \]

Now define \( Q(z) = -z \overline{P(-z)} \)

\[ |z| = 1 \text{ so set } z = e^{it} \quad t \in \mathbb{R} \quad \text{then we get} \quad -z \overline{P(-z)} = \]

\[ -\frac{e^{it}}{2} \sum_{k \in \mathbb{Z}} p_k (e^{it})^k = \frac{e^{it}}{2} \sum_{k \in \mathbb{Z}} \overline{p_k} e^{ikt} (-1)^{1-k} \]

\[ = \frac{1}{2} \sum_{k \in \mathbb{Z}} e^{(1-k)it} p_k = \frac{1}{2} \sum_{m \in \mathbb{Z}} e^{mit} \overline{p_{1-m}} (-1)^m \quad m = 1-k \]
Chapter 5, problem 14:

Thm. Suppose $\phi$ satisfies $\int \phi(x-k) \phi(x-l) \, dx = \delta_{k,l}$ and $\phi(x) = \sum_k p_k \phi(2x-k)$; set $\psi(x) = \sum_k q_k \phi(2x-k)$ and let $Q(z) = \sum_k q_k z^k$ then TFAE

1. $\int \psi(x-k) \phi(x-l) \, dx = 0 \quad \forall k, l \in \mathbb{Z}$
2. $P(z) Q(z) + P(-z) \overline{Q(-z)} = 0 \quad \forall z \text{ st. } |z| = 1$

Proof: First we use Thm. 5.18. which gives that (1) $\Rightarrow$ we have

$$\sum_{k \in \mathbb{Z}} \hat{\phi}(\delta + 2\pi k) \hat{\psi}(\delta + 2\pi k) = 0 \quad (3)$$

since if $\Psi(x)$ is orthogonal to $\phi(x-l) \quad \forall l \in \mathbb{Z}$ this means

$$\int \phi(x-l) \Psi(x) \, dx = 0$$

$x = z - m$ gives $\int \phi(z - m - l) \Psi(x - m) \, dx = 0$

so that indeed (1) iff (3).

So if we can construct a calculation which shows that (3) $\Rightarrow$ (4) we are done

$$\sum_{k \in \mathbb{Z}} \hat{\phi}(\delta + 2\pi k) \overline{\hat{\psi}(\delta + 2\pi k)} = 0$$
By theorem 5.18, we have
\[
\begin{align*}
\phi(\xi) &= \hat{\phi}(\frac{\xi}{2}) \mathcal{P}(e^{-i\xi/2}) \\
\psi(\xi) &= \hat{\psi}(\frac{\xi}{2}) Q(e^{-i\xi/2})
\end{align*}
\] (same argument as in proof to 5.18)

So
\[
\sum_{\ell \in \mathbb{Z}} \hat{\phi}(\frac{\xi}{2} + 2\ell \pi(2\ell)) \overline{\hat{\psi}(\frac{\xi}{2} + 2\ell \pi(2\ell))} + \\
\sum_{\ell \in \mathbb{Z}} \hat{\phi}(\frac{\xi}{2} + 2\ell \pi(2\ell+1)) \overline{\hat{\psi}(\frac{\xi}{2} + 2\ell \pi(2\ell+1))} = 0
\]

\[
\sum_{\ell \in \mathbb{Z}} \hat{\phi}(\frac{\xi}{2} + 2\ell \pi) \overline{\hat{\phi}(\frac{\xi}{2} + 2\ell \pi)} \mathcal{P}(e^{-i\xi/2}) Q(e^{-i\xi/2})
\]
\[
+ \sum_{\ell \in \mathbb{Z}} \hat{\phi}(\frac{\xi}{2} + \pi(2\ell+1)) \overline{\hat{\phi}(\frac{\xi}{2} + \pi(2\ell+1))} \mathcal{P}(e^{-i(3\pi/2)}) Q(-e^{-i\xi/2}) = 0
\]

\[
\frac{1}{2\pi} \mathcal{P}(e^{-i\xi/2}) Q(e^{-i\xi/2}) + \frac{1}{2\pi} \mathcal{P}(e^{-i3\pi/2}) Q(-e^{-i\xi/2}) = 0
\]

using Thm 5.18 with \( \frac{\xi}{2} + 2\pi \ell \) \& \( (\frac{\xi}{2} + \pi) + 2\pi \ell \)

Therefore the result follows if we multiply the last equation by \( 2\pi \) \& realize that \( \xi \) was arbitrary so that \( e^{i\xi/2} \) gives all elements in \( \mathbb{C} \) s.t. \( 1, 1 = 1 \).
Chapter 5, problem 17:

Let us explicitly calculate the support ...

Suppose \( p_k = 0 \) for \(-N \leq k \leq M\) \( N, M \in \mathbb{N}_0 \).

\[
\phi_n(x) = \sum_{k \in \mathbb{Z}} p_k \phi_{n-1}(2x-k) \quad \text{from p.217 (5.34)}
\]

\[
= \sum_{-N \leq k \leq M} p_k \phi_{n-1}(2x-k) = \sum_{-N \leq k \leq M} p_k \sum_{-N \leq m \leq M} p_m \phi_{n-2}(2(2x-k)-m)
\]

\[
= \sum_{-N \leq k \leq M} p_k p_m \phi_{n-2}(4x-2k-m)
\]

\[
= \sum_{-N \leq k, m \leq M} p_k p_m p_e \phi_{n-3} \left( \frac{2(4x-2k-m) - e}{e} \right) = 8x - 4k - 2m - e
\]

Using a more general indexing we can write this as

\[
\phi_n(x) = \sum_{-N \leq j_k \leq M} \prod_{k=1}^{n} p_k \phi_0 \left( \frac{2^n x - 2^{n-1} j_n - 2^{n-2} j_{n-1} - \ldots - j_1}{j_n} \right)
\]

\[
= 2^n x - \sum_{m=0}^{n-1} 2^m j_m + 1
\]

This implies that the function \( \phi_0 \) is first dilated by a factor \( 2^n \) which reduces the support from \([0,1]\) to \([0, \frac{1}{2^n}]\). The maximum translates of this dilated function to the left & right are determined then
by \(-N\) \& \(M\) respectively, i.e. the maximum translation of the support to the left is

\[ -\sum_{m=0}^{n-1} (-N) 2^m = N \sum_{m=0}^{n-1} 2^m = N \frac{2^n - 1}{2 - 1} = N(2^n - 1) \]

Similarly to the right we get \(-M(2^n - 1)\)

so that the support of \(\phi_n\) is given by

\[ \left[ 0 - \frac{N(2^n - 1)}{2^n}, \frac{1}{2^n} + \frac{M(2^n - 1)}{2^n} \right] \]

\[ = \left[ -N + \frac{N}{2^n}, \frac{1}{2^n} + M - \frac{M}{2^n} \right] \]

which upon taking limits as \(n \to \infty\) gives for \(\phi(x)\) (if it exists \& the sequence of \(\phi_n\)'s converges)

\[ \text{supp}(\phi) \in [-N, M] \] clearly this is compact.

\[ \square \]

Chapter 5, problem 18:

We read off from our previous work above

\[ \text{supp}(\phi) \in [0, N] \] (which is compact)

\[ \square \]