GALOIS CONNECTIONS FOR INCIDENCE HOPF ALGEBRAS OF PARTIALLY ORDERED SETS.

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ABSTRACT. An important well-known result of Rota describes the relationship between the Möbius functions of two posets related by a Galois connection. We present an analogous result relating the antipodes of the corresponding incidence Hopf algebras, from which the classical formula can be deduced. To motivate the derivation of this more general result, we first observe that a simple conceptual proof of Rota's classical formula can be obtained by interpreting it in terms of bimodules over the incidence algebras. Bimodules correct the apparent lack of functoriality of incidence algebras with respect to monotone maps. The theory of incidence Hopf algebras is reviewed from scratch, and centered around the notion of cartesian posets. Also, the universal multiplicative function on a poset is constructed and an analog for antipodes of the classical Möbius inversion formula is presented.

1. Introduction, notation and preliminaries

All posets to be considered are assumed to be locally finite. k is a fixed commutative ring, often omitted from the notation. I_P is the incidence algebra of the poset P over k:

$$I_P = \{\varphi : P \times P \to k \ / \ \varphi(x,y) = 0 \text{ if } x \not\leq y\} \ ;$$

with multiplication

$$(\varphi * \psi)(x,y) = \sum_{z \in P} \varphi(x,z)\psi(z,y) = \sum_{x \leq z \leq y} \varphi(x,z)\psi(z,y)$$

and unit element $\delta_P \in I_P$, $\delta_P(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if not} \end{cases}$.

The *zeta function* of P is the element $\zeta_P \in I_P$ defined by $\zeta_P(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if not} \end{cases}$. Its inverse $\mu_P \in I_P$ always exists and is called the *Möbius function* of P.

Let P and Q be two posets. Letters x, y and z will usually denote elements of the poset P, while for Q we will use u, v and w. A pair of maps $f : P \to Q$ and $g : Q \to P$ is called *adjoint* if both f and g are monotone and for any $x \in P$ and $w \in Q$,

$$f(x) \leq w \iff x \leq g(w) \ .$$

We will also refer to such a pair as a *Galois connection* between P and Q (even though this term is often used in the literature for an adjoint pair between P and Q^{op}, the opposite poset of Q). In this

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situation, the Möbius functions of P and Q are related by the following formula due originally to Rota ([Rot, theorem 1], or, for the formulation below [Gre, 5.4]):

$$\forall\ x \in P \text{ and } \ w \in Q, \quad \sum_{\substack{y \in P \\ f(y)=w}} \mu_P(x,y) = \sum_{\substack{v \in Q \\ g(v)=x}} \mu_Q(v,w)\ . \tag{1}$$

The main purpose of this note is to present a generalization of this formula to the context of Hopf algebras. It will be convenient to first reinterpret Rota's formula, as well as a simple extension of it involving more general incidence functions, in terms of bimodules over the incidence algebras of P and Q. This will be done in section 3.

Recall that if A and B are k-algebras and M an A-B-bimodule, then

$$a \cdot (m \cdot b) = (a \cdot m) \cdot b \ \ \forall\ a \in A,\ b \in B \text{ and } m \in M\ .$$

Rota's formula can be seen as the following equality in a certain I_P-I_Q-bimodule $M_{f,g}$ associated to the Galois connection:

$$\mu_P \cdot (\zeta \cdot \mu_Q) = (\mu_P \cdot \zeta) \cdot \mu_Q\ ,$$

where $\zeta \in M_{f,g}$ is the *zeta function of the connection* (section 3).

The incidence coalgebra of P will be denoted by I^P, and the (standard) reduced incidence coalgebra by R^P. These definitions will be reviewed in section 4, where the basics of the theory of incidence Hopf algebras will be discussed from scratch. This theory was initiated by Joni and Rota [J-R] and later received further development and clarification in works of Schmitt [Sch1,2]. The main point is that when a poset carries some additional multiplicative structure, its incidence coalgebra becomes a Hopf algebra. Moreover, the Möbius function can be obtained from the antipode as the composite

$$\mu_P = \zeta_P \circ S_P\ .$$

Schmitt also finds an analog of Hall's formula for Möbius functions [Sch2, theorem 4.1] that reads

$$S_P[x,y] = \sum_{n \geq 0} (-1)^n \sum_{x=x_0<x_1<\ldots<x_n=y} [x_0,x_1] \cdot \ldots \cdot [x_{n-1},x_n]\ ; \tag{2}$$

from where the classical formula, giving μ_P as an alternating sum of numbers of chains, is recovered simply by applying ζ_P.

It was suggested by Rota and Schmitt that there should be antipode analogues of other classic formulas for the Möbius function (for a summary of these properties, see [Gre]). In this note we present a very natural analog of Rota's formula (1) for a Galois connection (theorem 5.4). It reads

$$\sum_{\substack{y \in P \\ f(y)=w}} S_P[x,y] \otimes 1 = \sum_{\substack{v \in Q \\ g(v)=x}} 1 \otimes S_Q[v,w]\ .$$

The above equality is between elements of the tensor product of the incidence Hopf algebras of P and Q, over a certain algebra. This algebra is non-other than the dual version of the bimodule $M_{f,g}$ mentioned above. The classical formula is obtained simply by applying $\zeta_P \otimes \zeta_Q$ to both sides of this equality.

In section 2 we construct a bimodule over the incidence algebras associated to any monotone map between the posets. This bimodule replaces in some sense the morphism of algebras that the monotone map in general fails to induce. In section 3 we use this construction to present the proof of Rota's formula (1) announced above. This approach to the classical case suggests the constructions and results for the Hopf algebra case, that are carried out in section 5. The necessary background on incidence Hopf algebras is discussed in section 4. Our presentation emphasizes the relevance of the notion of cartesian posets to the construction of incidence Hopf algebras. We also discuss multiplicative functions on arbitrary posets and construct the universal multiplicative function of a poset. In section

6 we illustrate how one can use the notion of multiplicative functions as a bridge between classical formulas for Möbius functions and their antipode analogues, through two examples: the formula for Galois connections and the classical Möbius inversion formula. The analog of the latter reads

$$g(x) = \sum_{\substack{y\in P\\ x\leq y}} [x,y]\cdot f(y)\ \forall\ x\in P \iff f(x) = \sum_{\substack{y\in P\\ x\leq y}} S_P[x,y]\cdot g(y)\ \forall\ x\in P\ ,$$

where f and g are functions defined on P with values on a module over the incidence Hopf algebra of P (corollary 6.4).

2. The bimodule associated to a monotone map

Let $f : P \to Q$ be a monotone map between posets. In general, f does not induce any morphism between the incidence algebras of P and Q. This lack of functoriality can be corrected by considering bimodules over the incidence algebras, as we now describe.

Recall that if A and B are k-algebras, then an A-B-bimodule is a k-space M together with k-linear actions

$$A\times M \to M,\ (a,m)\mapsto a\cdot m \ \text{ and } \ M\times B\to M,\ (m,b)\mapsto m\cdot b\ ,$$

that make M into a left A-module and a right B-module in such a way that

$$a\cdot(m\cdot b) = (a\cdot m)\cdot b\ \ \forall\ a\in A,\ b\in B \text{ and } m\in M\ .$$

Assume that f satisfies the following condition (to be implicitly assumed from now on): for any $x\in P$ and $w\in Q$, the set $\{y\in P\ /\ x\leq y \text{ and } f(y)\leq w\}$ is finite.

The space

$$M_f = \{\alpha : P\times Q\to k\ /\ \alpha(x,w) = 0 \text{ if } f(x)\not\leq w\}$$

carries a natural I_P-I_Q-bimodule structure as follows. Take $\alpha\in M_f$, $\varphi\in I_P$, $\psi\in I_Q$ and $(x,w)\in P\times Q$. We define

$$(\varphi\cdot\alpha)(x,w) = \sum_{y\in P}\varphi(x,y)\alpha(y,w) = \sum_{\substack{x\leq y\\ f(y)\leq w}}\varphi(x,y)\alpha(y,w) \tag{3}$$

$$(\alpha\cdot\psi)(x,w) = \sum_{v\in Q}\alpha(x,v)\psi(v,w) = \sum_{f(x)\leq v\leq w}\alpha(x,v)\psi(v,w)\ . \tag{4}$$

Notice that if $f(x)\not\leq w$ then both sums above are zero, so these assignments define elements $\varphi\cdot\alpha$ and $\alpha\cdot\psi\in M_f$. The bimodule axioms are now straightforward. For instance, for the associativity axiom for the left action of I_P on M_f, we have

$$\begin{aligned}(\varphi_1 * \varphi_2)\cdot\alpha(x,w) &= \sum_{y\in P}(\varphi_1*\varphi_2)(x,y)\alpha(y,w)\\ &= \sum_{y,z\in P}\varphi_1(x,z)\varphi_2(z,y)\alpha(y,w)\\ &= \sum_{z\in P}\varphi_1(x,z)(\varphi_2\cdot\alpha)(z,w) = \varphi_1\cdot(\varphi_2\cdot\alpha)(x,w)\ .\end{aligned}$$

Similarly, the crucial axiom $(\varphi\cdot\alpha)\cdot\psi = \varphi\cdot(\alpha\cdot\psi)$ holds because both sides evaluate on (x,w) to

$$\sum_{\substack{y\in P\\ v\in Q}}\varphi(x,y)\alpha(y,v)\psi(v,w)\ .$$

Two extreme cases of this construction are as follows. First, for the identity map $\mathsf{id}_P : P \to P$, $M_{\mathsf{id}_P} = I_P$ viewed as I_P-bimodule via left and right multiplication. At the other end, if $f : P \to \{\bullet\}$ is the constant map, then

$$M_f = \{\alpha : P \to k \ / \ \alpha \text{ is any function } \}$$

with its usual left I_P-module structure

$$(\varphi \cdot \alpha)(x) = \sum_{x \leq y} \varphi(x,y)\alpha(y) \ .$$

Composition of maps corresponds to tensor product of bimodules. In this sense, the assignments

$$P \mapsto I_P \ \text{ and } \ f \mapsto M_f$$

define a functor from the category of posets and monotone maps to the category of algebras and bimodules (where composition is tensor product). Explicitly, if $P \xrightarrow{f} Q \xrightarrow{g} R$ are monotone maps, then

$$M_{gf} \cong M_f {\otimes_{I_Q}} M_g \text{ as } I_P\text{-}I_R\text{-bimodules .}$$

Moreover, if $f_i : P \to Q$ are monotone maps for $i = 1, 2$ such that $f_1(x) \leq f_2(x) \ \forall \ x \in P$, then $M_{f_2} \subseteq M_{f_1}$ as I_P-I_Q-bimodules. This means that this assignment is really a 2-functor between 2-categories, but this terminology will not be needed here.

Proposition 2.1. *Let $f : P \to Q$ be a monotone map. The map $f^\sharp : I_Q \to M_f$,*

$$f^\sharp(\psi)(x,w) = \psi(f(x),w) \ ,$$

is a morphism of right I_Q-modules.

Proof. Take ψ_1, $\psi_2 \in I_Q$, then

$$\begin{aligned} f^\sharp(\psi_1 * \psi_2)(x,w) = \\ (\psi_1 * \psi_2)(f(x),w) = \sum_{v \in Q} \psi_1(f(x),v)\psi_2(v,w) = \sum_{v \in Q} f^\sharp(\psi_1)(x,v)\psi_2(v,w) \\ = (f^\sharp(\psi_1) \cdot \psi_2)(x,w) \ . \end{aligned}$$

□

Remarks 2.1.

(1) Consider the element $\delta_f = f^\sharp(\delta_Q) \in M_f$. It follows from proposition 2.1 that for any $\psi \in I_Q$,

$$f^\sharp(\psi) = \delta_f \cdot \psi \ .$$

Notice that $\delta_f(x,w) = \begin{cases} 1 & \text{if } f(x) = w \\ 0 & \text{otherwise} \end{cases}$. The analogous map $I_P \to M_f$, $\varphi \mapsto \tilde{\varphi} := \varphi \cdot \delta_f$ (a morphism of left I_P-modules) is explicitly given by $\tilde{\varphi}(x,w) = \sum_{\substack{x \leq y \\ f(y)=w}} \varphi(x,y)$. We will not make use of this map.

(2) One may consider linear maps $f^\flat : M_f \to I_P$ and $f^* : I_Q \to I_P$ given by $f^\flat(\alpha)(x,y) = \alpha(x, f(y))$ and $f^*(\psi)(x,y) = \psi(f(x), f(y))$ for $x \leq y$ in P. There is a commutative diagram $\begin{array}{ccc} & M_f & \\ {}^{f^\sharp}\nearrow & & \searrow{}^{f^\flat} \\ I_Q & \xrightarrow[f^*]{} & I_P \end{array}$. In general, $f^\flat$ fails to be a morphism of left I_P-modules and f^* a morphism of algebras.

(3) Graves has considered other type of bimodules over incidence algebras [Gra]. He also associates a bimodule to a monotone map $P \to Q$, but a I_P-I_P-bimodule instead of a I_P-I_Q-one. There is no relation between his construction and the one presented here.

3. Rota's formula for a Galois connection

Let $f : P \to Q$, $g : Q \to P$ be a Galois connection, as defined in section 1. Thus both f and g are monotone and for any $x \in P$ and $w \in Q$,

$$f(x) \leq w \iff x \leq g(w) \ . \tag{5}$$

According to the construction of section 2, there are two bimodules associated to this situation

$$M_f = \{\alpha : P \times Q \to k \ / \ \alpha(x,w) = 0 \text{ if } f(x) \not\leq w\}, \text{ an } I_P\text{-}I_Q\text{-bimodule,}$$

and

$$M_g = \{\alpha : Q \times P \to k \ / \ \alpha(w,x) = 0 \text{ if } g(w) \not\leq x\}, \text{ an } I_Q\text{-}I_P\text{-bimodule.}$$

We may also regard g as a monotone map between the opposite posets $g^{op} : Q^{op} \to P^{op}$ and therefore consider

$$M_{g^{op}} = \{\alpha : Q^{op} \times P^{op} \to k \ / \ \alpha(w,x) = 0 \text{ if } g^{op}(w) \not\leq^{op} x\}, \text{ an } I_{Q^{op}}\text{-}I_{P^{op}}\text{-bimodule.}$$

But this is not a new bimodule: under the obvious identification between B^{op}-A^{op}-bimodules and A-B-bimodules, $M_{g^{op}} = M_f$. In fact,

$$f(x) \leq w \iff x \leq g(w) \iff g^{op}(w) \leq^{op} x \ ,$$

hence the map

$$M_f \to M_{g^{op}}, \ \alpha \mapsto \tilde{\alpha}, \ \tilde{\alpha}(w,x) = \alpha(x,w) \ ,$$

is an isomorphism of I_P-I_Q-bimodules. For this reason, we use the notation $M_{f,g}$ for the bimodule $M_f = M_{g^{op}}$ in this context.

Therefore, according to proposition 2.1, there are maps

$$f^\sharp : I_Q \to M_{f,g}, \ f^\sharp(\psi)(x,w) = \psi(f(x),w), \text{ a map of right } I_Q\text{-modules} \tag{6}$$

and

$$g^\sharp : I_P \to M_{f,g}, \ g^\sharp(\varphi)(x,w) = \varphi(x,g(w)), \text{ a map of left } I_P\text{-modules.} \tag{7}$$

Consider the element $\zeta \in M_{f,g}$ defined by

$$\zeta(x,w) = \begin{cases} 1 & \text{if } f(x) \leq w, \text{ or equivalently, if } x \leq g(w), \\ 0 & \text{otherwise.} \end{cases}$$

We call it the *zeta function of the Galois connection.* Notice that

$$f^\sharp(\zeta_Q) = \zeta = g^\sharp(\zeta_P) \ . \tag{8}$$

We claim that Rota's formula (1) is precisely the following equality in $M_{f,g}$:

$$\mu_P \cdot (\zeta \cdot \mu_Q) = (\mu_P \cdot \zeta) \cdot \mu_Q \ . \tag{9}$$

To see this, we need only compute each side of (9) separately. We have,

$$\mu_P \cdot (\zeta \cdot \mu_Q) \overset{(8)}{=} \mu_P \cdot (f^\sharp(\zeta_Q) \cdot \mu_Q) \overset{(6)}{=} \mu_P \cdot f^\sharp(\zeta_Q * \mu_Q) = \mu_P \cdot f^\sharp(\delta_Q) \ ,$$

while

$$(\mu_P \cdot \zeta) \cdot \mu_Q \overset{(8)}{=} (\mu_P \cdot g^\sharp(\zeta_P)) \cdot \mu_Q \overset{(7)}{=} g^\sharp(\mu_P * \zeta_P) \cdot \mu_Q = g^\sharp(\delta_P) \cdot \mu_Q \ .$$

Hence, for any $x \in P$ and $w \in W$,

$$\mu_P \cdot (\zeta \cdot \mu_Q)(x, w) = \mu_P \cdot f^\sharp(\delta_Q)(x, w) \overset{(3)}{=} \sum_{y \in P} \mu_P(x, y) f^\sharp(\delta_Q)(y, w) = \sum_{y \in P} \mu_P(x, y)\delta_Q(f(y), w)$$
$$= \sum_{\substack{y \in P \\ f(y)=w}} \mu_P(x, y) \ ,$$

and

$$(\mu_P \cdot \zeta) \cdot \mu_Q(x, w) = g^\sharp(\delta_P) \cdot \mu_Q(x, w) \overset{(4)}{=} \sum_{v \in Q} g^\sharp(\delta_P)(x, v)\mu_Q(v, w) = \sum_{v \in Q} \delta_P(x, g(v))\mu_Q(v, w)$$
$$= \sum_{\substack{v \in Q \\ g(v)=x}} \mu_Q(v, w) \ .$$

Thus, the equality in (9) gives precisely Rota's formula (1).

This approach to Rota's formula suggests the following simple extension:

Proposition 3.1. *Let $f : P \to Q$ and $g : Q \to P$ be a Galois connection as above. Let $\gamma_P \in I_P$ and $\gamma_Q \in I_Q$ be two invertible functions such that*

$$\forall\ x \in P \text{ and } w \in Q, \quad \gamma_P(x, g(w)) = \gamma_Q(f(x), w) \ . \tag{10}$$

Then,

$$\forall\ x \in P \text{ and } w \in Q, \quad \sum_{\substack{y \in P \\ f(y)=w}} \gamma_P^{-1}(x, y) = \sum_{\substack{v \in Q \\ g(v)=x}} \gamma_Q^{-1}(v, w) \ . \tag{11}$$

Proof. Hypothesis (10) means that $g^\sharp(\gamma_P) = f^\sharp(\gamma_Q)$. Let $\gamma \in M_{f,g}$ denote this common image. As before, the result follows by computing separately each side of the equality

$$\gamma_P^{-1} \cdot (\gamma \cdot \gamma_Q^{-1}) = (\gamma_P^{-1} \cdot \gamma) \cdot \gamma_Q^{-1} \ .$$

Using (6) and (7) we see that each of the sides above is respectively equal to $\gamma_P^{-1} \cdot f^\sharp(\delta_Q)$ and $g^\sharp(\delta_P) \cdot \gamma_Q^{-1}$. Now computing each of these on $(x, w) \in P \times Q$ gives precisely (11). □

4. Cartesian posets, multiplicative functions and incidence Hopf algebras

In this section we recall the definitions of incidence coalgebras [J-R] and Hopf algebras [Sch1,2] and associated concepts. Only the most basic notions from Hopf algebra theory are needed, as found for instance in [Mon]. We also discuss the notion of multiplicative functions on a poset, from a slightly different point of view to the one in the literature. The *universal* multiplicative function on a poset is constructed. The central notion will be that of a cartesian poset.

The incidence coalgebra of a (locally finite) poset P is the k-space I^P with basis

$$\{(x, y) \in P \times P \ / \ x \leq y\} \ ,$$

comultiplication Δ_P and counit δ_P defined by

$$\Delta_P(x,y) \sum_{\substack{z\in P\\ x\leq z\leq y}} (x,z)\otimes(z,y) \quad \text{and} \quad \delta_P(x,y) = \begin{cases} 1 & \text{if } x=y, \\ 0 & \text{if not} \end{cases} .$$

Notice that $(I^P)^* = I_P$ as k-algebras; in particular, the counit of I^P is the unit δ_P of I_P—this is the reason for departing from the more usual ε-notation for counits.

An interval of P is a subset of the form

$$[x,y]_P = \{z \in P \;/\; x \leq z \leq y\} ,$$

often viewed as poset in its own right with the order induced by that of P.

The reduced incidence coalgebra R^P is the quotient of I^P by the subspace

$$\mathsf{Span}\{(x,y) - (x',y') \in R^P \;/\; [x,y]_P \cong [x',y']_P\} ,$$

where $\cong$ denotes isomorphism of posets (bijections ϕ such that $\phi(a) \leq \phi(b) \iff a \leq b$).

The subspace in question is a coideal of I^P and thus R^P is indeed a coalgebra. The image of $(x,y) \in I^P$ on R^P is denoted by $[x,y]$. It follows that

$$[x,y] = [x',y'] \iff [x,y]_P \cong [x',y']_P .$$

Definition 4.1. Let P be a poset and A a commutative k-algebra. A *multiplicative* function on P with values on A is a k-linear map $\alpha : R^P \to A$ with the following properties

$$\alpha[x,x] = 1 \text{ for any } x \in P , \tag{12}$$

$$\alpha[x,y]\alpha[u,v] = \alpha[a,b] \text{ whenever } [x,y]_P \times [u,v]_P \cong [a,b]_P \text{ as posets.} \tag{13}$$

Notice that, in general, the cartesian product of two intervals of P will not be isomorphic to another interval of P. As first noted by Schmitt [Sch1], when the poset P does satisfy this property, its reduced incidence coalgebra R^P carries a natural structure of Hopf algebra, in such a way that multiplicative functions are precisely morphisms of algebras $R^P \to A$. We turn our attention to this class of posets now.

Definition 4.2. A poset P is called *cartesian* if given x, y, u and $v \in P$ such that $x \leq y$ and $u \leq v$, there exist a and $b \in P$ with $a \leq b$ and an isomorphism

$$[x,y]_P \times [u,v]_P \cong [a,b]_P .$$

cartesian posets are called *hereditary* in [S-O] (definition 3.3.3). At first, the notion of cartesian posets may seem too restrictive. For instance, it follows from the definition that a non-discrete cartesian poset must be infinite. However, quite the opposite is true. For, first, "infinite versions" of several familiar posets are indeed cartesian (cf. examples below), and second, and perhaps more importantly, every poset embeds canonically into a universal cartesian poset (proposition 4.3 below). It is this fact that allows us to derive a given classical formula for Möbius functions of *arbitrary* posets from its counterpart for antipodes of Hopf algebras of cartesian posets. We will elaborate on this after discussing the Hopf algebra structure on the reduced incidence coalgebra of a cartesian poset. First, the announced examples.

Examples 4.1.

The following posets are cartesian:

1. The poset of positive integers ordered by divisibility;
2. The poset of finite subsets of a countable set;
3. The poset of partitions of a countable set into a finite number of blocks;
4. The cartesian product of two cartesian posets.

A chain is not cartesian, unless it is trivial.

The following important result is due to Schmitt.

Proposition 4.1. *Let P be a cartesian poset. Define a multiplication, unit element and antipode in R^P as follows. Given two basis elements $[x,y]$ and $[u,v]$, we choose a and b as in definition 4.2 and set*

$$[x,y]\cdot[u,v]=[a,b]\in R^P \ . \tag{14}$$

Also, for any $x_0\in P$ let

$$1=[x_0,x_0]\in R^P \ . \tag{15}$$

Finally, the antipode $S_P : R^P \to R^P$ is defined by induction on the length of $[x,y]_P$ by

$$S_P[x,x]=1\ \forall\ x\in P \tag{16}$$

and

$$S_P[x,y]=-\sum_{\substack{z\in P\\ x\leq z<y}} S_P[x,z]\cdot[z,y] \text{ for } x<y\in P \ . \tag{17}$$

With this structure, the reduced incidence coalgebra R^P becomes a commutative Hopf algebra.

Proof. The multiplication, unit element and antipode are clearly well-defined. The Hopf algebra axioms are checked without difficulty. In the process one finds that S_P also satisfies the recursion

$$S_P[x,y]=-\sum_{\substack{z\in P\\ x<z\leq y}}[x,z]\cdot S_P[z,y] \text{ for } x<y\in P \ .$$

For more details see [Sch2, theorem 4.1]. □

When P is a cartesian poset, we refer to R^P as the incidence Hopf algebra of P. Notice that, as announced, a multiplicative function α on a cartesian poset P with values on A is precisely a morphism of algebras $\alpha : R^P \to A$. In order to obtain an adequate algebraic understanding of multiplicative functions on arbitrary posets we are led to consider the following construction.

Definition 4.3. Let P be an arbitrary poset. The *cartesian envelope* of P is the poset:

$$P^{(\infty)}=\coprod_{n\geq 1}P^n=\{\mathbf{x}=(x_1,x_2,\ldots,x_n)\in P^n\ /\ n\geq 1\}$$

with the order

$$\mathbf{x}\leq\mathbf{y}\iff \mathbf{x},\ \mathbf{y}\in P^n \text{ for some } n \text{ and } x_i\leq y_i\ \forall\ i=1,\ldots,n\ .$$

Remark 4.1. The cartesian envelope $P^{(\infty)}$ is closely related to the poset denoted by X^w in [S-O, page 148]: the families of isomorphism classes of intervals of both posets coincide, so both posets serve the same purpose from the point of view of reduced incidence coalgebras. Definition 4.3 has the advantage of eliminating the choice of a base point in P, which is needed for X^w. However, X^w may be more natural in some situations. For instance, if X is the chain of natural numbers and the base point is 0, then X^w is identified with the poset of positive integers under divisibility.

For any poset P, its envelope $P^{(\infty)}$ is clearly a cartesian poset (in particular, $P^{(\infty)}$ is locally finite if so is P, as we always implicitly assume). Therefore, $R^{P^{(\infty)}}$ is a Hopf algebra. Moreover, the map

$$P\hookrightarrow P^{(\infty)},\ x\mapsto\mathbf{x}=(x)\in P^1\subseteq P^{(\infty)}\ ,$$

embeds P as a *convex* subposet of $P^{(\infty)}$, and hence there is a corresponding morphism of coalgebras

$$\theta_P : R^P \hookrightarrow R^{P^{(\infty)}}$$

that simply views an interval of P as an interval of $P^{(\infty)}$. Clearly, θ_P is a multiplicative function on P, with values on the commutative algebra $R^{P^{(\infty)}}$ (definition 4.1). Proposition 4.3 below shows that θ_P is actually the *universal multiplicative function* on P. This result shows the relevance of the envelope construction. The proof is fairly simple, but some care is needed when dealing with isomorphic factorizations of posets. To properly address these details we first state a lemma.

Recall that a poset P is called *indecomposable* if it is connected, $P \neq \{\bullet\}$ and $P \cong P_1 \times P_2$ implies $P_1 = \{\bullet\}$ or $P_2 = \{\bullet\}$.

Lemma 4.2. (1) *Let $[a,b]_P$ be an interval of a poset P such that*

$$[a,b]_P \cong P_1 \times \ldots \times P_n \text{ for some posets } P_i \ .$$

Then each P_i is isomorphic to an interval of P.

(2) *Let α be a multiplicative function on a poset P and x_i, y_i, a and $b \in P$ be such that*

$$[x_1, y_1]_P \times \ldots \times [x_n, y_n]_P \cong [a,b]_P \ .$$

Then

$$\alpha[x_1, y_1] \cdot \ldots \cdot \alpha[x_n, y_n] = \alpha[a,b] \ .$$

(3) *Each finite non-trivial connected poset admits a decomposition into a product of indecomposable posets, which is unique up to order (and isomorphism).*

Proof. (1) If $\phi : [a,b]_P \to P_1 \times \ldots \times P_n$ is an isomorphism, $\phi(a) = (x_1, \ldots, x_n)$ and $\phi(b) = (y_1, \ldots, y_n)$, then

$$P_i \cong [(x_1, \ldots, x_i, \ldots, x_n), (x_1, \ldots, y_i, \ldots, x_n)] \cong [\phi^{-1}(x_1, \ldots, x_i, \ldots, x_n), \phi^{-1}(x_1, \ldots, y_i, \ldots, x_n)]_P \ .$$

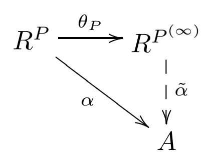
(2) Apply part 1 to $P_1 = [x_1, y_1]_P$ and $P_2 = [x_2, y_2]_P \times \ldots \times [x_n, y_n]_P$.Then use equation (13) and induction.

(3) This is a simple well-known result, which can be proved along the same lines as part 1. A more general version is given in [Sch2, lemma 6.1].

□

Proposition 4.3. *Let P be a poset and $P^{(\infty)}$ its cartesian envelope.*

(1) *Let $\alpha : R^P \to A$ be a multiplicative function with values on a commutative algebra A. Then there exists a unique morphism of algebras $\tilde{\alpha} : R^{P^{(\infty)}} \to A$ such that*



commutes.

(2) *If A is a bialgebra and α a morphism of coalgebras, then $\tilde{\alpha}$ is a morphism of bialgebras.*

Proof. (1) Given $\mathbf{x} \leq \mathbf{y} \in P^{(\infty)}$ we define

$$\tilde{\alpha}[\mathbf{x}, \mathbf{y}] = \prod_{n \geq 1} \alpha[x_n, y_n] \ .$$

We must verify that $\tilde{\alpha}$ is well-defined. Suppose $[\mathbf{x},\mathbf{y}]_{P^{(\infty)}} \cong [\mathbf{x}',\mathbf{y}']_{P^{(\infty)}}$, i.e. that

(a) $$[x_1,y_1]_P \times \ldots \times [x_n,y_n]_P \cong [x_1',y_1']_P \times \ldots \times [x_m',y_m']_P \ .$$

We need to check that

(b) $$\prod_{i=1}^{n} \alpha[x_i,y_i] = \prod_{j=1}^{m} \alpha[x_j',y_j'] \ .$$

Since α satisfies (12), we may assume that all factors above are non-trivial. Decompose each factor $[x_i,y_i]_P$ and $[x_j',y_j']_P$ into a product of indecomposables (lemma 4.2, part 3). By part 1 of the lemma, each of the new factors is again an interval of P, and hypothesis (a) holds for the new factors. Also, by part 2 of the lemma, equality (b) is equivalent to the corresponding equality for the new factors. In other words, we may assume that all intervals $[x_i,y_i]_P$ and $[x_j',y_j']_P$ are indecomposable. Then the uniqueness in part 3 of the lemma says that the same factors appear on both sides of each (a) and (b), possibly in different orders. Since A is commutative, (b) holds.

We prove now that $\tilde{\alpha}$ is a morphism of algebras. First,

$$\tilde{\alpha}(1) \overset{(15)}{=} \tilde{\alpha}[\mathbf{x},\mathbf{x}] = \prod_{n\geq 1} \alpha[x_n,x_n] \overset{(12)}{=} 1 \ .$$

Second, take elements $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{v} \in P^{(\infty)}$ as in definition (14), so that

$$[\mathbf{x},\mathbf{y}]\cdot[\mathbf{u},\mathbf{v}] = [\mathbf{a},\mathbf{b}] \in R^{P^{(\infty)}} \ .$$

From (14) we have an isomorphism

$$[x_1,y_1]_P \times \ldots \times [x_n,y_n]_P \times [u_1,v_1]_P \times \ldots \times [u_m,v_m]_P \cong [a_1,b_1]_P \times \ldots \times [a_l,b_l]_P \ .$$

This means that the following two intervals of $P^{(\infty)}$ are isomorphic:

$$[\mathbf{a},\mathbf{b}]_{P^{(\infty)}} \cong [(x_1,\ldots,x_n,u_1,\ldots,u_m),(y_1,\ldots,y_n,v_1,\ldots,v_m)]_{P^{(\infty)}} \ .$$

Hence, since $\tilde{\alpha}$ is well-defined,

$$\tilde{\alpha}\Big([\mathbf{x},\mathbf{y}]\cdot[\mathbf{u},\mathbf{v}]\Big) = \tilde{\alpha}[\mathbf{a},\mathbf{b}] = \prod_{h\geq 1}\alpha[a_h,b_h] = \prod_{i\geq 1}\alpha[x_i,y_i]\prod_{j\geq 1}\alpha[u_j,v_j] = \tilde{\alpha}[\mathbf{x},\mathbf{y}]\cdot\tilde{\alpha}[\mathbf{u},\mathbf{v}] \ ,$$

proving that $\tilde{\alpha}$ preserves multiplications.

By construction, $\tilde{\alpha}\circ\theta_P = \alpha$. Since $\tilde{\alpha}$ must be multiplicative, this commutativity uniquely determines $\tilde{\alpha}$. This completes the proof.

(2) Let Δ_A and δ_A be the comultiplication and counit of A. Take $\mathbf{x} \leq \mathbf{y} \in P^{(\infty)}$. Then, since δ_A is a morphism of algebras and α a morphism of coalgebras,

$$\delta_A\tilde{\alpha}[\mathbf{x},\mathbf{y}] = \delta_A\prod_{n\geq 1}\alpha[x_n,y_n] = \prod_{n\geq 1}\delta_A\alpha[x_n,y_n] = \prod_{n\geq 1}\delta_P[x_n,y_n] = \delta_{P^{(\infty)}}[\mathbf{x},\mathbf{y}] \ .$$

Also, since Δ_A is a morphism of algebras and α a morphism of coalgebras,

$$\Delta_A\tilde{\alpha}[\mathbf{x},\mathbf{y}] = \Delta_A \prod_{n\geq 1}\alpha[x_n,y_n] = \prod_{n\geq 1}\Delta_A\alpha[x_n,y_n] = \prod_{n\geq 1}(\alpha\otimes\alpha)\Delta_P[x_n,y_n] =$$

$$\prod_{n\geq 1}(\alpha\otimes\alpha)\sum_{z_n\in[x_n,y_n]_P}[x_n,z_n]\otimes[z_n,y_n] = \prod_{n\geq 1}\sum_{z_n\in[x_n,y_n]_P}\alpha[x_n,z_n]\otimes\alpha[z_n,y_n] =$$

$$\sum_{z_1\in[x_1,y_1]_P}\sum_{z_2\in[x_2,y_2]_P}\cdots\left[\Big(\alpha[x_1,z_1]\otimes\alpha[z_1,y_1]\Big)\cdot\Big(\alpha[x_2,z_2]\otimes\alpha[z_2,y_2]\Big)\cdot\ldots\right] =$$

$$\sum_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]_{P^{(\infty)}}}\Big(\prod_{n\geq 1}\alpha[x_n,z_n]\Big)\otimes\Big(\prod_{n\geq 1}\alpha[z_n,y_n]\Big) = \sum_{\mathbf{z}\in[\mathbf{x},\mathbf{y}]_{P^{(\infty)}}}\tilde{\alpha}[\mathbf{x},\mathbf{z}]\otimes\tilde{\alpha}[\mathbf{z},\mathbf{y}] = (\tilde{\alpha}\otimes\tilde{\alpha})\Delta_{P^{(\infty)}}[\mathbf{x},\mathbf{y}]\ .$$

This proves that $\tilde{\alpha}$ is a morphism of coalgebras and completes the proof.

□

Remark 4.2. A similar result to proposition 4.3 is stated in [S-O, proposition 3.3.14]. We chose to present a complete proof because a few details appear to be missing in this reference (in particular the crucial assumption that α be multiplicative).

We close this section by reinterpreting the notion of multiplicative functions in algebraic terms. We need to recall some basic facts from Hopf algebra theory.

If C is a coalgebra and A an algebra, the k-space $\mathsf{Hom}_k(C,A)$ of all linear maps becomes an algebra under the convolution product

$$\alpha * \beta = m_A\circ(\alpha\otimes\beta)\Delta_C$$

with unit element $u_A\circ\delta_C$, where m_A, u_A, Δ_C and δ_C are the structure maps of A and C. For instance if $C = I^P$ and $A = k$ then the convolution product on $I_P = \mathsf{Hom}_k(I^P,k)$ is the multiplication of incidence functions (section 1).

If H is a Hopf algebra and A a commutative algebra then the set $\mathsf{Alg}_k(H,A)$ of all morphisms of algebras is closed under the convolution product of $\mathsf{Hom}_k(H,A)$ and furthermore, a group; the inverse of $\alpha\in\mathsf{Alg}_k(H,A)$ being $\alpha\circ S_H$.

Let $G(P,A)$ denote the set of multiplicative functions on an arbitrary poset P, with values on a commutative algebra A. It is easy to see directly from definition 4.1 that any multiplicative function is invertible in $\mathsf{Hom}_k(R^P,A)$, since we can view it as a triangular matrix with 1's on the diagonal, by condition (12). It is not so clear that its inverse should be multiplicative again. However, this is an easy consequence of proposition 4.3, as we now explain. This can be seen as a generalization of the *product formula* for Möbius functions [Gre, 3.1].

Corollary 4.4. *Let P be a poset, A a commutative k-algebra and $G(P,A)$ the set of multiplicative functions.*

1. *If P is cartesian, $G(P,A) = \mathsf{Alg}_k(R^P,A)$, a group under convolution.*
2. *For arbitrary P, $G(P,A)$ is still a group under convolution*

$$(\alpha*\beta)[x,y] = \sum_{\substack{z\in P\\ x\leq z\leq y}}\alpha[x,z]\otimes\beta[z,y]\ .$$

Moreover, $G(P,A)$ is canonically isomorphic to $G(P^{(\infty)},A)$ and the inverse of $\alpha\in G(P,A)$ is

$$\alpha^{-1}[x,y] = \sum_{n\geq 0}(-1)^n\sum_{x=x_0<x_1<\ldots<x_n=y}\alpha[x_0,x_1]\cdot\ldots\cdot\alpha[x_{n-1},x_n]\ .$$

Proof. We have already noted that if P is cartesian then $G(P,A) = \mathsf{Alg}_k(R^P, A)$. Since in this case R^P is a Hopf algebra, part 1 follows. For an arbitrary poset P, proposition 4.3 says that there is a bijection

$$G(P^{(\infty)}, A) \xrightarrow{\cong} G(P, A)$$

given by restriction along $\theta_P : P \hookrightarrow P^{(\infty)}$. Since θ_P is a morphism of coalgebras, this bijection preserves convolution products and units. In particular, any multiplicative function α on P is invertible and its inverse is multiplicative. This proves that $G(P,A)$ is a group, isomorphic to $G(P^{(\infty)}, A)$. The inverse of $\beta \in G(P^{(\infty)}, A)$ is $\beta^{-1} = \beta \circ S_{P^{(\infty)}}$. It follows that the inverse of $\alpha \in G(P,A)$ is given by $\tilde{\alpha} \circ S_{P^{(\infty)}} \circ \theta_P$. Using Schmitt's analog (2) of Hall's formula for the antipode of a cartesian poset, we deduce that $\alpha^{-1}[x,y]$ is as stated. □

Remarks 4.3.

Often in the literature, the case of a the Hopf algebra of a *family* of isomorphism classes of posets is considered [Sch1,2, Ehr]. The family should be closed under subintervals and cartesian products in order to obtain a Hopf algebra. This is really an equivalent point of view to the one adopted in this note in terms of cartesian posets. In fact, given such a family $\mathfrak{F}$, the poset

$$\mathfrak{P}_{\mathfrak{F}} = \coprod_{P \in \mathfrak{F}} P$$

obtained as the disjoint union of its members is cartesian and the corresponding Hopf algebra $R^{\mathfrak{P}_{\mathfrak{F}}}$ is the Hopf algebra associated to the family $\mathfrak{F}$ by those authors.

In this note we have chosen to consider the case of incidence coalgebras reduced by the isomorphism relation only. Other equivalence relations on the set of intervals of a poset can be considered, as studied in detail by Schmitt [Sch1,2, S-O]. With some technical modifications, the constructions of this note, including the statement and proof of the Galois connection formula (theorem 5.4) can be extended to this context. This situation is more general but more cumbersome and it does not seem to add many new examples of interest. For instance, the *Faa di Bruno Hopf algebra* is already obtained as the Hopf algebra corresponding to the poset of partitions of example 4.1.

5. The formula for the antipodes

The derivation of Rota's formula presented in section 3 should serve as a guide for the derivation of the formula for the antipodes.

We start by describing the dual constructions and results to those of section 2.

Proposition 5.1. *Let $f : P \to Q$ be a monotone map, satisfying the same finiteness assumptions as in section 2.*

(1) *The k-space M^f with basis*

$$\{(x,w) \in P \times Q \;/\; f(x) \leq w\}$$

is an I^P-I^Q-bicomodule with structure maps

$$c_P : M^f \to I^P \otimes M^f, \quad (x,w) \mapsto \sum_{\substack{y \in P \\ x \leq y,\ f(y) \leq w}} (x,y) \otimes (y,w)$$

and

$$c_Q : M^f \to M^f \otimes I^Q, \quad (x,w) \mapsto \sum_{\substack{v \in Q \\ f(x) \leq v \leq w}} (x,v) \otimes (v,w) \ .$$

(2) *The map $f_\sharp : M^f \to I^Q$, $f_\sharp(x,w) = (f(x), w)$ is a morphism of right I^Q-comodules.*

Proof. The proofs are dual to those in section 2 and offer no difficulty. □

Remark 5.1. The map $f_\sharp$ may be seen as the composite $M^f \xrightarrow{c_Q} M^f \otimes I^Q \xrightarrow{\delta_f \otimes \mathsf{id}} I^Q$, where $\delta_f \in (M^f)^* = M_f$ is as in remark 2.1. Maps of these type (for arbitrary functionals in place of δ_f) are sometimes called *coaction* maps and are always morphisms of comodules.

Let $f : P \to Q$ and $g : Q \to P$ be a Galois connection between P and Q. As in section 3, the map $[x, w] \mapsto [w, x]$ identifies M^f and $M^{g^{op}}$ as I^P-I^Q-bicomodules. We use $M^{f,g}$ to denote this bicomodule. Therefore, according to proposition 5.1, there are maps

$$f_\sharp : M^{f,g} \to I^Q,\ \ f_\sharp(x, w) = (f(x), w), \text{ a morphism of right } I^Q\text{-comodules,}$$

and

$$g_\sharp : M^{f,g} \to I^P,\ \ g_\sharp(x, w) = (x, g(w)), \text{ a morphism of left } I^P\text{-comodules.}$$

We turn our attention now to reduced incidence coalgebras, since this is the framework where Hopf algebras arise. First we check that the objects constructed above admit natural "reduced" versions.

Consider the following relation on the set $B = \{(x, w) \in P \times Q \ / \ f(x) \le w\}$ (the k-basis of $M^{f,g}$):

$$(x, w) \sim (x', w') \text{ if there exist order-preserving bijections}$$
$$\phi_Q : [f(x), w]_Q \to [f(x'), w']_Q \ \text{ and } \ \phi_P : [x, g(w)]_P \to [x', g(w')]_P \ \text{ such that}$$

(18)
$$\begin{array}{ccc} [x,g(w)]_P & \xrightarrow{\phi_P} & [x',g(w')]_P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ [f(x),w]_Q & \xrightarrow[\phi_Q]{} & [f(x'),w']_Q \end{array} \quad \text{and} \quad \begin{array}{ccc} [x,g(w)]_P & \xrightarrow{\phi_P} & [x',g(w')]_P \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ [f(x),w]_Q & \xrightarrow[\phi_Q]{} & [f(x'),w']_Q \end{array} \quad \text{commute.}$$

Clearly, $\sim$ is an equivalence relation on B. The equivalence class represented by (x, w) will be denoted by $[x, w]$, in agreement with our previous conventions. Let $R^{f,g}$ be the k-space with basis consisting of these equivalence classes. In other words, $R^{f,g}$ is the quotient of $M^{f,g}$ by the subspace

$$K^{f,g} = \mathsf{Span}\{(x, w) - (x', w') \ / \ (x, w) \sim (x', w'), \text{ for } (x, w),\ (x', w') \in B\}\ .$$

Let $\pi_P : I^P \to R^P$ and $\pi_Q : I^Q \to R^Q$ denote the canonical projections. They allow us to view $M^{f,g}$ as a R^P-R^Q-bicomodule.

Proposition 5.2. *In the above situation:*

(1) $K^{f,g}$ *is a* R^P-R^Q*-subbicomodule of* $M^{f,g}$*. Therefore,* $R^{f,g}$ *is a* R^P-R^Q*-bicomodule with structure maps*

(19)
$$c_P : R^{f,g} \to R^P \otimes R^{f,g}, \quad [x, w] \mapsto \sum_{\substack{y \in P \\ x \le y \le g(w)}} [x, y] \otimes [y, w]$$

and

(20)
$$c_Q : R^{f,g} \to R^{f,g} \otimes R^Q, \quad [x, w] \mapsto \sum_{\substack{v \in Q \\ f(x) \le v \le w}} [x, v] \otimes [v, w]\ .$$

(2) *The maps* $f_\sharp$ *and* $g_\sharp$ *factor through* $K^{f,g}$ *yielding morphisms*

(21)
$$f_\sharp : R^{f,g} \to R^Q,\ \ f_\sharp[x, w] = [f(x), w]$$

and

$$g_\sharp : R^{f,g} \to R^P, \;\; g_\sharp[x,w] = [x, g(w)] \tag{22}$$

of right R^Q-comodules and left R^P-comodules respectively.

Proof. (1) We must show that $(\pi_P \otimes \mathsf{id})c_P(K^{f,g}) \subseteq R^P \otimes K^{f,g}$ (and similarly for R^Q). Take two equivalent elements (x,w) and (x',w') in B. Choose isomorphisms $\phi_P : [x,g(w)]_P \to [x',g(w')]_P$ and $\phi_Q : [f(x),w]_Q \to [f(x'),w']_Q$ as in 18. For each $y \in [x,g(w)]_P$, ϕ_P induces isomorphisms

$$(a)\; [x,y]_P \cong [x',\phi_P(y)]_P \text{ and } (b)\; [y,g(w)]_P \cong [\phi_P(y), g(w')]_P \; .$$

Also, ϕ_Q induces an isomorphism

$$(c)\; [f(y),w]_Q \cong [\phi_Q f(y), w']_Q = [f\phi_P(y), w']_Q \; ,$$

the last equality in virtue of the commutativity of the first diagram in (18).

From (a), $[x,y] = [x',\phi_P(y)] \in R^P$, and from (b) and (c), $(y,w) \sim (\phi_P(y),w')$, since these isomorphisms commute with f and g by construction.

Therefore

$$(\pi_P \otimes \mathsf{id})c_P\Big((x,w) - (x',w')\Big) =$$

$$\sum_{\substack{y\in P\\ x\le y\le g(w)}} [x,y]\otimes(y,w) \; - \sum_{\substack{y'\in P\\ x'\le y'\le g(w')}} [x',y']\otimes(y',w') = \sum_{\substack{y\in P\\ x\le y\le g(w)}} [x,y]\otimes(y,w) \; - \sum_{\substack{y\in P\\ x\le y\le g(w)}} [x',\phi_P(y)]\otimes(\phi_P(y),w')$$

$$= \sum_{\substack{y\in P\\ x\le y\le g(w)}} [x,y]\otimes\Big((y,w) - (\phi_P(y),w')\Big) \in R^P\otimes K^{f,g} \; ,$$

as needed.

(2) We need to check that $\pi_Q f_\sharp(K^{f,g}) = 0$ (and similarly for $g_\sharp$). But this is immediate from the definitions: if $(x,w) \sim (x',w')$ then in particular $[f(x),w]_Q \cong [f(x'),w']_Q$, hence

$$\pi_Q f_\sharp\Big((x,w) - (x',w')\Big) = [f(x),w] - [f(x'),w'] = 0 \in R^Q \; .$$

□

We consider now those Galois connections that are compatible with the multiplicative structure of cartesian posets. It is for this type of connections that a formula for the antipodes can be stated and proved. Examples of these connections will be discussed below.

Definition 5.1. A Galois connection $f : P \to Q$ and $g : Q \to P$ between cartesian posets is called *multiplicative* if given $x,\, y \in P$ and $v,\, w \in Q$ such that $f(x) \le w$ and $f(y) \le v$, there are $z \in P$ and $u \in Q$ with $f(z) \le u$ and isomorphisms

$$\lambda_Q : [f(x),w]_Q \times [f(y),v]_Q \to [f(z),u]_Q \text{ and } \lambda_P : [x,g(w)]_P \times [y,g(v)]_P \to [z,g(u)]_P$$

such that

$$\begin{array}{ccc} [x,g(w)]_P \times [y,g(v)]_P & \xrightarrow{\lambda_P} & [z,g(u)]_P \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ [f(x),w]_Q \times [f(y),v]_Q & \xrightarrow[\lambda_Q]{} & [f(z),u]_Q \end{array} \quad\text{and}\quad \begin{array}{ccc} [x,g(w)]_P \times [y,g(v)]_P & \xrightarrow{\lambda_P} & [z,g(u)]_P \\ \uparrow{\scriptstyle g\times g} & & \uparrow{\scriptstyle g} \\ [f(x),w]_Q \times [f(y),v]_Q & \xrightarrow[\lambda_Q]{} & [f(z),u]_Q \end{array} \tag{23}$$

commute.

Examples 5.1.

(1) Let $P = Q$ be the poset of positive integers under divisibility. Then

$$f : P \to P,\ p_1^{x_1} \cdot \ldots \cdot p_r^{x_r} \mapsto p_1^{y_1} \cdot \ldots \cdot p_r^{y_r} \text{ where } y_i = \begin{cases} x_i/2 & \text{if } x_i \text{ is even} \\ (x_i+1)/2 & \text{if } x_i \text{ is odd} \end{cases}$$

and

$$g : P \to P,\ n \mapsto n^2$$

define a multiplicative Galois connection between P and itself.

(2) Let Q be again the poset of positive integers and P the poset of finite subsets of the set of positive integers (example 4.1). Fix an enumeration $\{p_i\}_{i\geq 1}$ of the prime numbers. Then

$$f : P \to Q,\ I \mapsto \prod_{i\in I} p_i \ \text{ and } \ g : Q \to P,\ n \mapsto \{\ i\ /\ p_i \text{ divides } n\ \}$$

define a multiplicative Galois connection between P and Q.

Given a multiplicative Galois connection as in definition 5.1, we define a multiplication and a unit element in $R^{f,g}$ as follows. Given two basis elements $[x, w]$ and $[y, v]$, we choose z and u as in (23) and set

$$[x, w] \cdot [y, v] = [z, u] \in R^{f,g}\ . \tag{24}$$

Also, for any $x_0 \in g(Q) \subseteq P$ and $w_0 \in f(P) \subseteq Q$, let

$$1 = [x_0, f(x_0)] = [g(w_0), w_0] \in R^{f,g}\ . \tag{25}$$

Proposition 5.3. *In the above situation:*

(1) *The multiplication and unit element of $R^{f,g}$ are well-defined and turn it into an associative commutative k-algebra.*
(2) *$c_P : R^{f,g} \to R^P \otimes R^{f,g}$ and $c_Q : R^{f,g} \to R^{f,g} \otimes R^Q$ are morphisms of algebras*
(3) *$f_\sharp : R^{f,g} \to R^Q$ and $g_\sharp : R^{f,g} \to R^P$ are morphisms of algebras.*

Proof. (1) Suppose $(x, w) \sim (x', w')$, $(y, v) \sim (y', v')$ and z' and u' are such that the conditions of (23) hold for the triple (x', w'), (y', v') and (z', u'). Using (23) and (18) we build commutative diagrams

$$\begin{array}{ccccccc}
[z, g(u)]_P & \xleftarrow{\lambda_P} & [x, g(w)]_P \times [y, g(v)]_P & \xrightarrow{\phi_P \times \phi_P} & [x', g(w')]_P \times [y', g(v')]_P & \xrightarrow{\lambda'_P} & [z', g(u')]_P \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\
[f(z), u]_Q & \xleftarrow[\lambda_Q]{} & [f(x), w]_Q \times [f(y), v]_Q & \xrightarrow[\phi_Q \times \phi_Q]{} & [f(x'), w']_Q \times [f(y'), v']_Q & \xrightarrow[\lambda'_Q]{} & [f(z'), u']_Q
\end{array}$$

and similar ones with g instead of f. These imply that $(z, u) \sim (z', u')$ and thus the multiplication is well-defined.

Now take any $x_0 \in g(Q)$ and $w_0 \in f(P)$. It follows from the definition of Galois connection that then $gf(x_0) = x_0$ and $fg(w_0) = w_0$. Hence there are unique isomorphisms $\phi_P : [x_0, gf(x_0)]_P \cong [g(w_0), g(w_0)]_P$ and $\phi_Q : [f(x_0), f(x_0)]_Q \cong [fg(w_0), w_0]_Q$, and they commute with f and g trivially. Thus $(x_0, f(x_0)) \sim (g(w_0), w_0)$ and the element $1 \in R^{f,g}$ is well-defined.

Associativity for the multiplication is reduced to associativity for the cartesian product of posets along the same lines as above. Commutativity is obvious. As for unitality, consider a basis element $[x, w] \in R^{f,g}$, then

$$[f(x), w]_Q \times [f(x_0), f(x_0)]_Q \cong [f(x), w]_Q \text{ and } [x, g(w)]_P \times [x_0, gf(x_0)]_P \cong [x, g(w)]_P\ ,$$

and these bijections commute with f and g trivially, so $[x,w]\cdot 1=[x,w]$.

(2) Let us prove the assertion for c_P, the other follows by symmetry. Take basis elements $[x,w]$ and $[x',w']$ of $R^{f,g}$ and let $[x,w]\cdot[x',w']=[x'',w'']$. Choose isomorphisms $\lambda_P:[x,g(w)]_P\times[x',g(w')]_P\to[x'',g(w'')]_P$ and $\lambda_Q:[f(x),w]_Q\times[f(x'),w']_Q\to[f(x''),w'']_Q$ satisfying (23). Then, for $y\in[x,g(w)]$ and $y'\in[x',g(w')]$, there are induced isomorphisms

$$[x,y]_P\times[x',y']_P\cong[x'',y'']_P \tag{a}$$

and

$$[y,g(w)]_P\times[y',g(w')]_P\cong[y'',g(w'')]_P \text{ and } [f(y),w]_Q\times[f(y'),w']_Q\cong[f(y''),w'']_Q\ . \tag{b}$$

From (a), $[x,y]\cdot[x',y']=[x'',y'']\in R^P$, while from (b) (since these isomorphisms commute with f and g by construction), $[y,w]\cdot[y',w']=[y'',w'']\in R^{f,g}$. Therefore,

$$c_P[x,w]\cdot c_P[x',w']=$$

$$\Big(\sum_{x\le y\le g(w)}[x,y]\otimes[y,w]\Big)\cdot\Big(\sum_{x'\le y'\le g(w')}[x',y']\otimes[y',w']\Big)=\sum_{\substack{x\le y\le g(w)\\ x'\le y'\le g(w')}}[x,y]\cdot[x',y']\otimes[y,w]\cdot[y',w']$$

$$=\sum_{x''\le y''\le g(w'')}[x'',y'']\otimes[y'',w'']=c_P[x'',w'']=c_P\Big([x,w]\cdot[x',w']\Big)\ .$$

Finally,

$$c_P(1)=c_P[g(w_0),w_0]=\sum_{g(w_0)\le y\le g(w_0)}[g(w_0),y]\otimes[y,w_0]=[g(w_0),g(w_0)]\otimes[g(w_0),w_0]=1\otimes 1\ ,$$

which completes the proof.

(3) These assertions follow immediately from definitions (24) and (25). For instance, for x, w, y, v, z and u as in (24), we have $[f(x),w]_Q\times[f(y),v]_Q\cong[f(z),u]_Q$, and hence

$$f_\sharp[x,w]\cdot f_\sharp[y,v]=[f(x),w]\cdot[f(y),v]=[f(z),u]=f_\sharp[z,u]=f_\sharp\Big([x,w]\cdot[y,v]\Big)\ .$$

□

Remarks 5.2.

(1) Propositions 5.2 and 5.3 say that $R^{f,g}$ is a R^P-R^Q-bicomodule algebra, and also that $f_\sharp$ and $g_\sharp$ are morphisms of comodule algebras.

(2) Recall the functional $\delta_f:M^{f,g}\to k$ from remark 5.1. It is easy to see that it descends to the quotient $\delta_f:R^{f,g}\to k$ and that as such it is a morphism of algebras. Moreover, $f_\sharp$ is the composite $R^{f,g}\xrightarrow{c_Q}R^{f,g}\otimes R^Q\xrightarrow{\delta_f\otimes\mathrm{id}}R^Q$. $g_\sharp$ can be expressed similarly in terms of the functional $\delta_g:R^{f,g}\to k$, $\delta_g(x,w)=\begin{cases}1 & \text{if } x=g(w)\\ 0 & \text{otherwise}\end{cases}$.

We are now ready to state our main result. Viewing R^P and R^Q as $R^{f,g}$-algebras via $f_\sharp$ and $g_\sharp$ respectively, we form the tensor product of commutative algebras

$$R^P\otimes_{R^{f,g}}R^Q\ .$$

Theorem 5.4. *Let $f : P \to Q$ and $g : Q \to P$ be a multiplicative Galois connection between cartesian posets. Let S_P and S_Q be the antipodes of the reduced incidence Hopf algebras of P and Q. Then, for any $x \in P$ and $w \in Q$,*

$$\sum_{\substack{y\in P\\ f(y)=w}} S_P[x,y] \otimes 1 = \sum_{\substack{v\in Q\\ g(v)=x}} 1 \otimes S_Q[v,w] \in R^P \otimes_{R^{f,g}} R^Q \ . \tag{26}$$

Proof. Consider the composite

$$R^{f,g} \xrightarrow{c_P} R^P {\otimes} R^{f,g} \xrightarrow{S_P\otimes f_\sharp} R^P{\otimes}R^Q \xrightarrow{\mathsf{id}_P\otimes u_Q\delta_Q} R^P{\otimes}R^Q \twoheadrightarrow R^P \otimes_{R^{f,g}} R^Q \ ,$$

where $u_Q : k \to R^Q$ and $\delta_Q : R^Q \to k$ are the unit and counit of R^Q and the last map is the canonical projection. The image of $[x, w] \in R^{f,g}$ under this map is computed below:

$$[x,w] \mapsto \sum_{\substack{x\le y\\ f(y)\le w}} [x,y]{\otimes}[y,w] \mapsto \sum_{\substack{x\le y\\ f(y)\le w}} S_P[x,y]{\otimes}[f(y),w] \mapsto \sum_{\substack{x\le y\\ f(y)=w}} S_P[x,y]{\otimes}1 \mapsto \sum_{\substack{y\in P\\ f(y)=w}} S_P[x,y] \otimes 1 \ .$$

Similarly, under the composite

$$R^{f,g} \xrightarrow{c_Q} R^{f,g}{\otimes}R^Q \xrightarrow{g_\sharp\otimes S_Q} R^P{\otimes}R^Q \xrightarrow{u_P\delta_P\otimes\mathsf{id}_Q} R^P{\otimes}R^Q \twoheadrightarrow R^P \otimes_{R^{f,g}} R^Q \ ,$$

the element $[x, w] \in R^{f,g}$ maps to $\displaystyle\sum_{\substack{v\in Q\\ g(v)=x}} 1 \otimes S_Q[v,w] \in R^P \otimes_{R^{f,g}} R^Q$.

Therefore, equality (26) is equivalent to the commutativity of the boundary of the following diagram. Below we finish the proof by proving this commutativity.

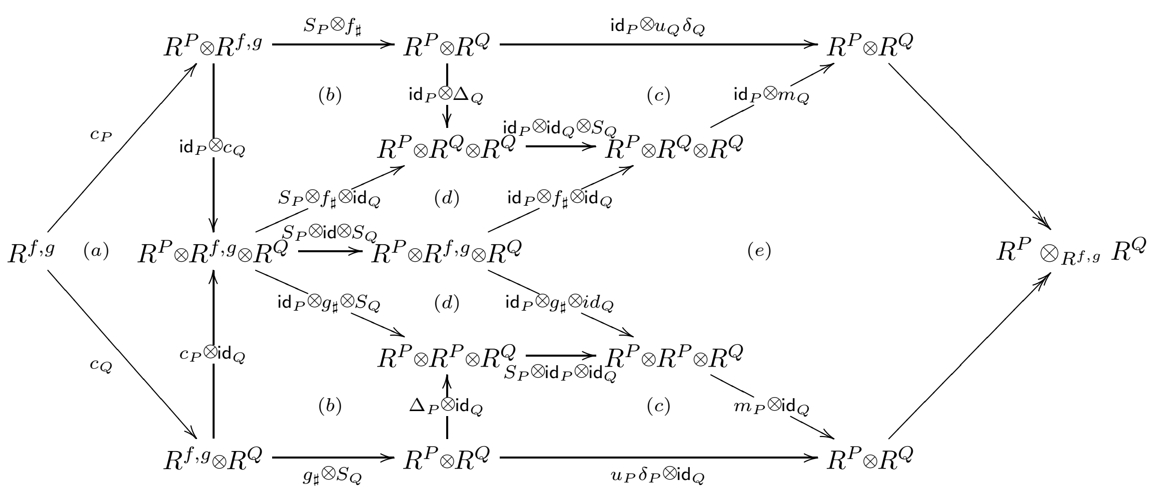


Diagram (a) commutes because $R^{f,g}$ is a R^P-R^Q-bicomodule (proposition 5.2). Diagrams (b) commute because $f_\sharp$ and $g_\sharp$ are morphisms of comodules (also by proposition 5.2). Diagrams (c) commute by definition of antipode for the Hopf algebras R^P and R^Q. Diagrams (d) commute trivially. Finally, diagram (e) commutes precisely by definition of tensor product of algebras. This completes the proof. □

Remark 5.3. Equality (26) can also be written as

$$\sum_{\substack{y\in P\\ f(y)=w}} S_P[x,y]\cdot[y,g(w)]\otimes 1 = \sum_{\substack{v\in Q\\ g(v)=x}} 1\otimes[f(x),v]\cdot S_Q[v,w] \in R^P\otimes_{R^{f,g}} R^Q \ .$$

This is simply because by definition of tensor product we have, for each y and w with $f(y)=w$,

$$1\otimes 1 = 1\otimes[f(y),w] = 1\otimes f_\sharp[y,w] = g_\sharp[y,w]\otimes 1 = [y,g(w)]\otimes 1 \ \in R^P\otimes_{R^{f,g}} R^Q \ ,$$

and similarly for the right hand side.

6. From antipodes to Möbius functions

In this section we illustrate, by means of two examples, the principle that a formula for antipodes is essentially the same as a formula for multiplicative functions, and hence in particular one for Möbius functions. The first example is for the case of Galois connections, the main topic of this paper. The second is about Möbius inversion formula.

For any poset P, the zeta function is clearly multiplicative: $\zeta_P\in G(P,k)$. It follows from the general considerations of section 4 that its inverse the Möbius function is then given by

$$\mu_P = \zeta_P\circ S_P \ . \tag{27}$$

Given a multiplicative Galois connection between cartesian posets, one recovers Rota's formula formula (1) by applying $\zeta_P\otimes\zeta_Q$ to (26) and using (27). Notice that $\zeta_P\otimes\zeta_Q$ is well-defined on the tensor product $R^P\otimes_{R^{f,g}} R^Q$, because $\zeta_P\circ g_\sharp = \zeta_Q\circ f_\sharp$. In fact, this holds precisely by definition of Galois connection.

We will explain now how the general case of Rota's formula (for arbitrary posets) can also be deduced from the formula for the antipodes.

Lemma 6.1. *Let $f: P\to Q$ and $g: Q\to P$ be a Galois connection between arbitrary posets P and Q. Then f and g extend to a multiplicative connection*

$$f^{(\infty)}: P^{(\infty)}\to Q^{(\infty)} \quad \textit{and} \quad g^{(\infty)}: Q^{(\infty)}\to P^{(\infty)}$$

between the cartesian envelopes.

Proof. For $\mathbf{x}\in P^{(\infty)}$ and $\mathbf{w}\in Q^{(\infty)}$ set

$$f^{(\infty)}(\mathbf{x}) = (f(x_1), f(x_2),\dots)\in Q^{(\infty)} \text{ and } g^{(\infty)}(\mathbf{w}) = (g(w_1), g(w_2),\dots)\in Q^{(\infty)} \ .$$

Then

$$f^{(\infty)}(\mathbf{x})\le\mathbf{w} \iff f(x_n)\le w_n \ \forall\ n \iff x_n\le g(w_n)\ \forall\ n \iff \mathbf{x}\le g^{(\infty)}(\mathbf{w}) \ ,$$

so $f^{(\infty)}$ and $g^{(\infty)}$ define a Galois connection.

In order to check the conditions in definition 5.1, take $\mathbf{x}$, $\mathbf{y}\in P^{(\infty)}$ and $\mathbf{v},\mathbf{w}\in Q^{(\infty)}$ such that $f^{(\infty)}(\mathbf{x})\le\mathbf{w}$ and $f^{(\infty)}(\mathbf{y})\le\mathbf{v}$. It follows that $\mathbf{x}\in P^n$ and $\mathbf{w}\in Q^n$ for some n, $\mathbf{y}\in P^m$ and $\mathbf{v}\in Q^m$ for some m and

$$f(x_i)\le w_i \text{ for } i=1,\dots,n \text{ and } f(y_j)\le v_j \text{ for } j=1,\dots,m \ .$$

Let $\mathbf{z}\in P^{(\infty)}$ and $\mathbf{u}\in Q^{(\infty)}$ be

$$\mathbf{z} = (x_1,\dots,x_n,y_1,\dots,y_m) \text{ and } \mathbf{u} = (w_1,\dots,w_n,v_1,\dots,v_m) \ .$$

Then the identity maps define isomorphisms

$$\lambda_{Q^{(\infty)}}: [f^{(\infty)}(\mathbf{x}),\mathbf{w}]_{Q^{(\infty)}}\times[f^{(\infty)}(\mathbf{y}),\mathbf{v}]_{Q^{(\infty)}}\to[f^{(\infty)}(\mathbf{z}),\mathbf{u}]_{Q^{(\infty)}}$$

and

$$\lambda_{P^{(\infty)}}: [\mathbf{x},g^{(\infty)}(\mathbf{w})]_{P^{(\infty)}}\times[\mathbf{y},g^{(\infty)}(\mathbf{v})]_{P^{(\infty)}}\to[\mathbf{z},g^{(\infty)}(\mathbf{u})]_{P^{(\infty)}}$$

and diagrams (23) commute trivially. Thus the Galois connection is multiplicative. □

Corollary 6.2. *Let Let $f : P \to Q$ and $g : Q \to P$ be a Galois connection between arbitrary posets P and Q. Let $\gamma_P : R^P \to A$ and $\gamma_Q : R^Q \to A$ be multiplicative functions on P and Q with values on a commutative algebra A, such that*

$$\forall\ x \in P \text{ and } w \in Q, \quad \gamma_P[x, g(w)] = \gamma_Q[f(x), w]\ . \tag{28}$$

Then,

$$\forall\ x \in P \text{ and } w \in Q, \quad \sum_{\substack{y\in P\\ f(y)=w}} \gamma_P^{-1}[x, y] = \sum_{\substack{v\in Q\\ g(v)=x}} \gamma_Q^{-1}[v, w]\ . \tag{29}$$

Proof. Consider first the case when both P and Q are cartesian and the connection is multiplicative. In this situation, R^P and R^Q are algebras and γ_P and γ_Q are morphisms of algebras (corollary 4.4). Hypothesis (28) says precisely that $\gamma_Q \circ f_\sharp = \gamma_P \circ g_\sharp$. Hence $\gamma_P {\otimes} \gamma_Q$ descends to a morphism of algebras

$$\gamma_P \otimes \gamma_Q : R^P \otimes_{R^{f,g}} R^Q \to A\ .$$

The result follows by applying this morphism to equation (26) and recalling that in this case $\gamma_P^{-1} = \gamma_P \circ S_P$ and $\gamma_Q^{-1} = \gamma_Q \circ S_Q$.

If P and Q are arbitrary then we extend the Galois connection to a multiplicative Galois connection between $P^{(\infty)}$ and $Q^{(\infty)}$ as in lemma 6.1. According to corollary 4.4, γ_P and γ_Q extend to multiplicative functions $\gamma_{P^{(\infty)}}$ and $\gamma_{Q^{(\infty)}}$ on $P^{(\infty)}$ and $Q^{(\infty)}$. Moreover, $\forall\ \mathbf{x} \in P^{(\infty)}$ and $\mathbf{w} \in Q^{(\infty)}$,

$$\begin{aligned}\gamma_{P^{(\infty)}}[\mathbf{x}, g^{(\infty)}(\mathbf{w})] &= \\ \gamma_{P^{(\infty)}} \prod_{n\geq 1}[x_n, g(w_n)] = \prod_{n\geq 1} \gamma_P[x_n, g(w_n)] &= \prod_{n\geq 1} \gamma_Q[f(x_n), w_n] = \gamma_{Q^{(\infty)}} \prod_{n\geq 1}[f(x_n), w_n] \\ &= \gamma_{Q^{(\infty)}}[f^{(\infty)}(\mathbf{x}), \mathbf{w}]\ ;\end{aligned}$$

so $\gamma_{P^{(\infty)}}$ and $\gamma_{Q^{(\infty)}}$ satisfy hypothesis (28). Hence (29) holds for them, by the case just proved. Restricting to P and Q we obtain the result for γ_P and γ_Q as well. □

Remark 6.1. Taking $\gamma_P = \zeta_P$ and $\gamma_Q = \zeta_Q$ one recovers Rota's formula (1) for Möbius functions.

Corollary 6.2 is actually a particular case of proposition 3.1. It was included as an illustration of how one may rederive a classical formula for Möbius functions from a version for antipodes. In fact, it was the result of that proposition that suggested the consideration of the tensor product $R^P \otimes_{R^{f,g}} R^Q$. In general, a result for the antipode of an incidence Hopf algebra will specialize to a result for multiplicative functions (in particular, the Möbius function), but not for arbitrary incidence functions.

As a last application we present a Hopf algebraic version of the classical Möbius inversion formula. This finds its natural place in the framework of cartesian posets and multiplicative functions discussed in section 4—but is not related to the previous material on Galois connections.

We need to recall one more basic fact from Hopf algebra theory. Suppose C is a coalgebra, A an algebra, M a left C-comodule via $c_M : M \to C {\otimes} M$ and N a left A-module via $a_N : A {\otimes} N \to N$. Then the convolution monoid $\mathsf{Hom}_k(C, A)$ acts on the space $\mathsf{Hom}_k(M, N)$; the action of $\alpha \in \mathsf{Hom}_k(C, A)$ on $f \in \mathsf{Hom}_k(M, N)$ is $\alpha \cdot f \in \mathsf{Hom}_k(M, N)$ defined as the composite

$$M \xrightarrow{c_M} C{\otimes}M \xrightarrow{\alpha\otimes f} A{\otimes}N \xrightarrow{a_N} N\ .$$

Proposition 6.3. *Let P be a poset such that for any $x \in P$, the set $\{y \in P \;/\; x \leq y\}$ is finite. Let α be a multiplicative function on P with values on a commutative algebra A. Let $f, g : P \to N$ be arbitrary functions with values on a A-module N. Then*

$$g(x) = \sum_{\substack{y \in P \\ x \leq y}} \alpha[x,y] \cdot f(y) \;\forall\; x \in P \iff f(x) = \sum_{\substack{y \in P \\ x \leq y}} \alpha^{-1}[x,y] \cdot g(y) \;\forall\; x \in P\ .$$

Proof. Let M be the left I^P-comodule arising from the map $P \to \{\bullet\}$, as in proposition 5.1—our assumption on P guarantees that this is well-defined. Thus, M has a k-basis that can be identified with the set P and the structure map is

$$M \to I^P \otimes M, \quad x \mapsto \sum_{\substack{y \in P \\ x \leq y}} (x,y) \otimes y\ .$$

Composing with the projection $I^P \to R^P$, we may view M as a left R^P-comodule as well. Extend f and g to linear maps $M \to N$.

We know from corollary 4.4 that $G(P,A)$ is a subgroup of the monoid $\mathsf{Hom}_k(R^P, A)$. It follows from the previous remark that $G(P,A)$ acts on $\mathsf{Hom}_k(M,N)$ and that the action is as follows:

$$(\alpha \cdot f)(x) = \sum_{\substack{y \in P \\ x \leq y}} \alpha[x,y] \cdot f(y)\ .$$

Thus the result boils down to the trivial assertion

$$g = \alpha \cdot f \iff f = \alpha^{-1} \cdot g\ .$$

□

The classical inversion formula is obtained by specializing to $A = N = k$, $\alpha = \zeta_P$ and (hence) $\alpha^{-1} = \mu_P$. The analog for antipodes also follows immediately.

Corollary 6.4. *Let P be a cartesian poset satisfying the same finiteness assumption as in proposition 6.3. Let N be a left R^P-module and $f, g : P \to N$ arbitrary functions. Then*

$$g(x) = \sum_{\substack{y \in P \\ x \leq y}} [x,y] \cdot f(y) \;\forall\; x \in P \iff f(x) = \sum_{\substack{y \in P \\ x \leq y}} S_P[x,y] \cdot g(y) \;\forall\; x \in P\ .$$

Proof. Apply proposition 6.3 to $A = R^P$, $\alpha = \mathsf{id}$. □

Bibliography

[Ehr]: R.Ehrenborg, *On Posets and Hopf Algebras,* Advances in Mathematics 119, (1996), 1-25.

[Gra]: W.Graves, *A Sheaf of Bicomodules over the Incidence Coalgebra,* Journal of Combinatorial Theory, Series A 13, (1972) 28-39.

[Gre]: C.Greene, *The Möbius function of a partially ordered set,* Nato Advanced Study Institute, Series C (1981), 555-581.

[J-R]: S.A.Joni and G.C.Rota, *Coalgebras and Bialgebras in Combinatorics,* Studies in Applied Mathematics 61 (1979), 93-139.

[Mon]: S.Montgomery, *Hopf Algebras and Their Actions on Rings,* CBMS 82 (1993).

[Rot]: G.C.Rota, *On the foundations of Combinatorial Theory. I. Theory of Möbius functions,* Z. Wahrscheinlichkertstheorie und Verw. Gebiete 2, (1964) 340-368.

[Sch1]: W.Schmitt, *Antipodes and Incidence Coalgebras,* Journal of Combinatorial Theory, Series A 46, (1987) 264-290.

[Sch2]: W.Schmitt, *Incidence Hopf algebras,* Journal of Pure and Applied Algebra 96 (1994) 299-330.

[S-O]: E.Spiegel, C.J.O'Donnell, *Incidence Algebras,* Pure and Applied Mathematics 206, Marcel Dekker (1997).

References [J-R] and [Rot] are reprinted in *Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries* (Joseph P.S. Kung, Ed.), Birkhäuser, Boston,1995.

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