PREPOISSON ALGEBRAS

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Abstract. A definition of prepoisson algebras is proposed, combining structures of prelie and zinbiel algebra on the same vector space. It is shown that a prepoisson algebra gives rise to a Poisson algebra by passing to the corresponding Lie and commutative products. Analogs of basic constructions of Poisson algebras (through deformations of commutative algebras, or from filtered algebras whose associated graded algebra is commutative) are shown to hold for prepoisson algebras. The Koszul dual of prepoisson algebras is described. It is explained how one may associate a prepoisson algebra to any Poisson algebra equipped with a *Baxter operator*, and a dual prepoisson algebra to any Poisson algebra equipped with an *averaging operator*. Examples of this construction are given. It is shown that the free zinbiel algebra (the shuffle algebra) on a prelie algebra is a prepoisson algebra. A connection between the graded version of this result and the classical Yang-Baxter equation is discussed.

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1. Prelie and zinbiel algebras

Throughout the paper, we deal with the left version of each type of algebras. All definitions and results admit a right version.

A left prelie algebra is a vector space A together with a bilinear map $\circ : A \times A \to A$ such that

$$x \circ (y \circ z) - (x \circ y) \circ z = y \circ (x \circ z) - (y \circ x) \circ z \ . \tag{1.1}$$

Defining $\{\ ,\ \} : A \times A \to A$ by $\{x, y\} = x \circ y - y \circ x$ one obtains a Lie algebra structure on A. Prelie algebras were introduced by Gerstenhaber [Ger]. See [C-L] for more references and examples, including an explicit description of the free prelie algebra on a vector space.

A left zinbiel algebra is a vector space A together with a bilinear map $* : A \times A \to A$ such that

$$x * (y * z) = (y * x) * z + (x * y) * z \ . \tag{1.2}$$

Defining $\cdot : A \times A \to A$ by $x \cdot y = x * y + y * x$ one obtains a commutative algebra structure on A. Zinbiel algebras were introduced by Loday [L1]. In that work, they were called *dual leibniz* algebras. See [L2, chapter 7] and [Liv] for more on zinbiel algebras.

A dendriform algebra [L2, chapter 5] is a vector space A together with bilinear maps $\succ : A \times A \to A$ and $\prec : A \times A \to A$ such that

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (y \succ z) \tag{1.3}$$

$$x \succ (y \prec z) = (x \succ y) \prec z \tag{1.4}$$

$$x \succ (y \succ z) = (x \prec y) \succ z + (x \succ y) \succ z \ . \tag{1.5}$$

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Defining $\cdot : A \times A \to A$ by $x \cdot y = x \succ y + x \prec y$ one obtains an associative algebra structure on A. In addition, defining $x \circ y = x \succ y - y \prec x$ one obtains a left prelie algebra structure on A.

Just as a commutative algebra is an associative algebra for which $x \cdot y = y \cdot x$, a left zinbiel algebra may be equivalently defined as a dendriform algebra for which $x \succ y = y \prec x$ (the zinbiel product being $x * y = x \succ y = y \prec x$). If one thinks of dendriform algebras as an analog of associative algebras, then one may view zinbiel and prelie algebras as the "dendriform analogs" of commutative and Lie algebras respectively. The situation is summarized by means of the following diagram, where one may think of the rows as "exact sequences":

$$\begin{tikzcd} Zinbiel \arrow[r] \arrow[d] & Dendriform \arrow[r] \arrow[d] & Prelie \arrow[d] \\ Commutative \arrow[r] & Associative \arrow[r] & Lie \end{tikzcd}$$

The main goal of this note is to define the "dendriform analog" of Poisson algebras, by suitably combining the notions of zinbiel and prelie algebras.

2. Definition

Recall that a Poisson algebra is a triple $(A, \cdot, \{\ ,\ \})$ where $(A, \cdot)$ is a commutative algebra, $(A, \{\ ,\ \})$ is a Lie algebra and the following condition holds:

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\} \ . \tag{2.1}$$

Combining zinbiel and prelie algebras into an appropriate notion of "prepoisson" algebras appears to be not completely straightforward. We propose the following.

Definition 2.1. A left prepoisson algebra is a triple $(A, *, \circ)$ where $(A, *)$ is a left zinbiel algebra, $(A, \circ)$ is a left prelie algebra and the following conditions hold:

$$(x \circ y - y \circ x) * z = x \circ (y * z) - y * (x \circ z) \tag{2.2}$$

$$(x * y + y * x) \circ z = x * (y \circ z) + y * (x \circ z) \ . \tag{2.3}$$

Observe that the associated Lie and commutative products $\{x, y\} = x \circ y - y \circ x$ and $x \cdot y = x * y + y * x$ intervene in the axioms, but these cannot be expressed solely in terms of them.

Axioms (2.2) and (2.3) do in fact guarantee that the associated products satisfy the Poisson axiom (2.1):

Proposition 2.2. *Let $(A, *, \circ)$ be a left prepoisson algebra. Define*

$$x \cdot y = x * y + y * x \ \text{ and } \{x, y\} = x \circ y - y \circ x \ .$$

Then $(A, \cdot, \{\ ,\ \})$ is a Poisson algebra.

3. Deformations

Let A be a commutative algebra. A *deformation* of A is an associative multiplication on the space $A[[\hbar]]$ of formal power series, that is $\hbar$-linear and coincides with the original commutative multiplication modulo $\hbar$ [C-P, section 1.6.A]. Denote the original multiplication of two elements a and b of A by the symbol $a \cdot_0 b$ and the multiplication of two formal series f and g by the symbol $f \cdot_\hbar g$. The latter is determined by its effect on elements of A, for which it takes the form

$$a \cdot_\hbar b = a \cdot_0 b + (a \cdot_1 b)\hbar + (a \cdot_2 b)\hbar^2 + \dots \ .$$

Since $a \cdot_0 b = b \cdot_0 a$, one may define an element of A by the formula

$$\{a, b\} = \frac{a \cdot_\hbar b - b \cdot_\hbar a}{\hbar}\Big|_{\hbar=0} \ ,$$

or equivalently by

$$\{a,b\} = a \cdot_1 b - b \cdot_1 a \ .$$

It is well known that then $(A, \cdot_0, \{\ ,\ \})$ is a Poisson algebra (which is referred to as the *classical limit* of the deformation) [Dri, section 2].

Similarly, one may look for deformations of a zinbiel algebra into dendriform algebras, and wonder what additional structure this imposes on A. In support of definition 2.1, we find that this is the structure of a prepoisson algebra.

Proposition 3.1. *Let $(A, *)$ be a left zinbiel algebra, that is, a dendriform algebra $(A, \succ_0, \prec_0)$ for which $a \succ_0 b = a * b = b \prec_0 a$. Suppose $(A[[\hbar]], \succ, \prec)$ is a deformation into dendriform algebras, that is, a dendriform algebra structure on the space of formal power series, with the properties that both operations $\succ$ and $\prec$ are $\hbar$-linear and coincide with $\succ_0$ and $\prec_0$ modulo $\hbar$. Define a new operation on A by*

$$a \circ b = \frac{a \succ b - b \prec a}{\hbar}|_{\hbar=0} \ .$$

Then $(A, *, \circ)$ is a left prepoisson algebra.

4. Prepoisson algebras from filtered dendriform algebras

One of the basic constructions of Poisson algebras goes as follows. Starting from an associative algebra equipped with a filtration, if the corresponding graded algebra happens to be commutative, then it carries a canonical Lie bracket that turns it into a Poisson algebra.

There is a dendriform analog of this construction. A *filtration* of a dendriform algebra $(A, \succ, \prec)$ is an increasing sequence of subspaces $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ such that

$$A = \bigcup_{n=0}^{\infty} A_n \quad \text{and} \quad A_n \succ A_m + A_n \prec A_m \subseteq A_{n+m} \ .$$

In this situation, there is an associated graded dendriform algebra

$$\mathsf{Gr}(A) = \bigoplus_{n=0}^{\infty} A_{n+1}/A_n$$

with operations $(a+A_n) \succ (b+A_m) = a \succ b + A_{n+m+1} \in A_{n+m+2}/A_{n+m+1}$ and $(a+A_n) \prec (b+A_m) = a \prec b + A_{n+m+1} \in A_{n+m+2}/A_{n+m+1}$, for elements $a \in A_{n+1}$ and $b \in A_{m+1}$.

Suppose that $\mathsf{Gr}(A)$ happens to be zinbiel, i.e. $(a + A_n) \succ (b + A_m) = (b + A_m) \prec (a + A_n)$, for a and b as above. Then $a \succ b - b \prec a \in A_{n+m+1}$ and one may well define a new operation on $\mathsf{Gr}(A)$ by

$$(a + A_n) \circ (b + A_m) = a \succ b - b \prec a + A_{n+m} \in A_{n+m+1}/A_{n+m} \ .$$

This operation on $\mathsf{Gr}(A)$ satisfies the left prelie axiom (1.1), because so does the product $a \succ b - b \prec a$ on the dendriform algebra A. Moreover:

Proposition 4.1. *In this situation, $\mathsf{Gr}(A)$ is a left prepoisson algebra.*

The prepoisson axioms for $\mathsf{Gr}(A)$ follow from the following identities, that hold in the dendriform algebra A:

$$(a \circ b - b \circ a) \succ c = a \circ (b \succ c) - b \succ (a \circ c) \qquad \text{(this implies (2.2))}$$

$$(a \succ b + a \prec b) \circ c = a \succ (b \circ c) + (a \circ c) \prec b \qquad \text{(this implies (2.3))}$$

The fact that these identities hold in any dendriform algebra is analogous to the fact that in any associative algebra the commutator bracket $[x, y] = xy - yx$ satisfies $[x, yz] = [x, y]z + y[x, z]$.

5. Baxter operators

Gian-Carlo Rota advocated in many occasions for the study of some special types of operators defined on an associative algebra A, beyond the usual automorphisms and derivations [R1, R2]. Among these there are the *Baxter* operators, defined by the condition

$$\beta(x)\cdot\beta(y)=\beta\Big(\beta(x)\cdot y+x\cdot\beta(y)\Big)\ . \tag{5.1}$$

These operators are frequent in algebra. For instance, on the polynomial algebra, the indefinite integral

$$\beta(f)(\mathbf{x})=\int_0^{\mathbf{x}} f(t)\,dt$$

is a Baxter operator. The inverse of a bijective derivation is a Baxter operator.

Baxter operators are named after Glen Baxter, who introduced them in [Bax]. Surprisingly, there is a connection with the associative analog of the *classical Yang-Baxter equation* (named after C.N.Yang and R.J.Baxter), which was introduced in [A1] and further studied in [A2]. If A is an associative algebra, an element $r=\sum u_i{\otimes}v_i\in A{\otimes}A$ is a solution of the associative Yang-Baxter equation if

$$\mathbf{A}(r)=0 \tag{5.2}$$

where

$$\mathbf{A}(r)=r_{13}r_{12}-r_{12}r_{23}+r_{23}r_{13}=\sum u_iu_j{\otimes}v_j{\otimes}v_i-\sum u_i{\otimes}v_iu_j{\otimes}v_j+\sum u_j{\otimes}u_i{\otimes}v_iv_j\ .$$

The connection is as follows: if $r=\sum u_i{\otimes}v_i$ is a solution of the associative Yang-Baxter equation, then the map $\beta:A\to A$ defined by

$$\beta(x)=\sum u_ixv_i$$

is a Baxter operator. This follows from (5.2) by simply replacing the tensor symbols above by x and y.

The notion of a Baxter operator can be defined for algebras over any binary operad, in the obvious manner. For instance for the Associative and Commutative operad, Baxter operators are defined by condition (5.1), while for the Lie operad they are defined by

$$\{\beta(x),\beta(y)\}=\beta\Big(\{\beta(x),y\}+\{x,\beta(y)\}\Big)\ . \tag{5.3}$$

Starting from a Baxter operator on an associative algebra A, one may construct a dendriform algebra structure on A. This is an instance of a more general fact, of which we state the cases of present interest below.

Proposition 5.1. 1. *Let $(A,\cdot)$ be an associative algebra and $\beta:A\to A$ a Baxter operator. Define new operations on A by*

$$x\succ y=\beta(x)\cdot y\quad\textit{and}\quad x\prec y=x\cdot\beta(y)\ .$$

Then $(A,\succ,\prec)$ is a dendriform algebra.

2. *Let $(A,\{\ ,\ \})$ be a Lie algebra and $\beta:A\to A$ a Baxter operator. Define a new operation on A by*

$$x\circ y=\{\beta(x),y\}\ .$$

Then $(A,\circ)$ is a left prelie algebra.

3. *Let $(A,\cdot)$ be a commutative algebra and $\beta:A\to A$ a Baxter operator. Define a new operation on A by*

$$x*y=\beta(x)\cdot y\ .$$

Then $(A,*)$ is a left zinbiel algebra.

In view of the above results, one expects that a Baxter operator on a Poisson algebra will allow us to construct a prepoisson algebra structure on it. This is indeed the case, and this provides another reason in support of definition 2.1.

Proposition 5.2. *Let $(A, \cdot, \{\ ,\ \})$ be a Poisson algebra and $\beta : A \to A$ a Baxter operator, i.e. a map satisfying both* (5.1) *and* (5.3)*. Define new operations on A by*

$$x * y = \beta(x) \cdot y \quad \textit{and} \quad x \circ y = \{\beta(x), y\} \ .$$

Then $(A, *, \circ)$ is a left prepoisson algebra.

As noted before, a solution of the associative Yang-Baxter equation gives rise to a Baxter operator on an associative algebra. Surprisingly, if the algebra is Poisson, the operator automatically satisfies the Baxter identity with respect to the Lie bracket:

Proposition 5.3. *Let $(A, \cdot, \{\ ,\ \})$ be a Poisson algebra and $r = \sum u_i \otimes v_i$ a solution of the associative Yang-Baxter equation* (5.2)*. Then the map $\beta : A \to A$ given by $\beta(x) = \sum u_i x v_i$ is a Baxter operator on the Poisson algebra A.*

The proof is based on repeated applications of the Poisson axiom (2.1), together with the facts that

$$(a) \sum u_i u_j v_i v_j = 0 \ , \quad (b) \sum \{u_i, u_j\} v_i v_j = 0 \quad \text{and} \quad (c) \sum u_i u_j \{v_i, v_j\} = 0 \ .$$

(a) holds because it follows from (5.2) that $r^2 = 0$, while (b) and (c) hold simply by the skew symmetry of the Lie bracket and the commutativity of the product.

Example 5.4. An interesting family of examples can be obtained as follows. Let A be a finite dimensional Poisson algebra and $f : A \to k$ a linear functional for which the form

$$\omega_f : A \otimes A \to k \ , \quad \omega_f(a, b) = f(\{a, b\})$$

is non degenerate. The form ω_f is skew symmetric and, moreover, it satisfies

$$\omega_f(ab \otimes c) - \omega_f(a \otimes bc) + \omega_f(ca \otimes b) = 0 \ .$$

(This is an immediate consequence of the Poisson axiom (2.1).) In other words, ω_f is a cyclic cocycle in the sense of Connes. Now, since by assumption ω_f is non degenerate, it corresponds to some tensor $r_f \in A \otimes A$ (identifying $A \cong A^*$ through ω_f). According to [A2, proposition 2.1], r_f is a solution of the associative Yang-Baxter equation. Therefore, by means of propositions 5.3 and 5.2, f gives rise to a Baxter operator and a prepoisson structure on A.

6. Dual prepoisson algebras

It is well known that the Koszul dual of the Zinbiel operad is the Leibniz operad [L2, chapter 4] and the Koszul dual of the Prelie operad is the Permutative operad of [Cha]. The Koszul dual of the operad defining left prepoisson algebras is as follows.

Proposition 6.1. *An algebra over the Koszul dual of the operad defining left prepoisson algebras is a triple $(A, \bullet, \{\ ,\ \})$ where $(A, \bullet)$ is a left permutative algebra:*

$$a \bullet (b \bullet c) = (a \bullet b) \bullet c = (b \bullet a) \bullet c \ , \tag{6.1}$$

$(A, \{\ ,\ \})$ is a left Leibniz algebra:

$$\{\{a, b\}, c\} = \{a, \{b, c\}\} - \{b, \{a, c\}\} \tag{6.2}$$

and the following three conditions hold:

$$\{a, b \bullet c\} = \{a, b\} \bullet c + b \bullet \{a, c\} \tag{6.3}$$

$$\{a \bullet b, c\} = a \bullet \{b, c\} + b \bullet \{a, c\} \tag{6.4}$$

$$\{a, b\} \bullet c + \{b, a\} \bullet c = 0 \ . \tag{6.5}$$

In [L2, section 4.15], Loday introduced a notion of *Poisson dialgebras*, which combines a structure of a *diassociative* algebra and a (right) Leibniz algebra. The algebras of proposition 6.1 are a particular case of (the left version of) these Poisson dialgebras.

The construction of a prepoisson algebra from a Poisson algebra equipped with a Baxter operator (section 5) admits a dual counterpart. The role of Baxter operators is now played by another class of operators, which surprisingly are also singled out in Rota's papers [R1, R2]. They are called *averaging operators* in those works. As before, this notion makes sense for algebras over an arbitrary binary operad. For instance, an averaging operator on a commutative algebra $(A, \cdot)$ is a map $\alpha : A \to A$ such that

$$\alpha(x) \cdot \alpha(y) = \alpha\Big(\alpha(x) \cdot y\Big) \ . \tag{6.6}$$

Averaging operators over Lie and associative algebras are respectively defined by the conditions

$$[\alpha(x), \alpha(y)] = \alpha\Big([\alpha(x), y]\Big) \ , \tag{6.7}$$

$$\alpha\Big(x \cdot \alpha(y)\Big) = \alpha(x) \cdot \alpha(y) = \alpha\Big(\alpha(x) \cdot y\Big) \ . \tag{6.8}$$

Proposition 6.2. 1. *Let $(A, \cdot)$ be an associative algebra and $\alpha : A \to A$ an averaging operator. Define new operations on A by*

$$x \vdash y = \alpha(x) \cdot y \quad \textit{and} \quad x \dashv y = x \cdot \alpha(y) \ .$$

Then $(A, \vdash, \dashv)$ is a diassociative algebra.

2. *Let $(A, [\ ,\ ])$ be a Lie algebra and $\alpha : A \to A$ an averaging operator. Define a new operation on A by*

$$\{x, y\} = [\alpha(x), y] \ .$$

Then $(A, \{\ ,\ \})$ is a left Leibniz algebra.

3. *Let $(A, \cdot)$ be a commutative algebra and $\alpha : A \to A$ an averaging operator. Define a new operation on A by*

$$x \bullet y = \alpha(x) \cdot y \ .$$

Then $(A, \bullet)$ is a left permutative algebra.

4. *Let $(A, \cdot, [\ ,\ ])$ be a Poisson algebra and $\alpha : A \to A$ an averaging operator, i.e. a map satisfying both* (6.6) *and* (6.7). *Define new operations on A by*

$$x \bullet y = \alpha(x) \cdot y \quad \textit{and} \quad \{x, y\} = [\alpha(x), y] \ .$$

Then $(A, \bullet, \{\ ,\ \})$ is a dual left prepoisson algebra (as described in proposition 6.1).

Notice that a differential $d : A \to A$ (i.e. a derivation d such that $d^2 = 0$) is a special case of an averaging operator. According to Y.Kosmann-Schwarzbach, the observation that one may construct a Leibniz algebra from such a map d essentially goes back to Koszul [Kos]. The construction of a diassociative algebra from a differential is already mentioned in [L2, section 2.2].

7. The shuffle algebra of a prelie algebra

Let S_n denote the symmetric group and

$$Sh_{n,m} = \{\sigma \in S_{n+m} \ / \ \sigma(1) < \sigma(2) < \ldots < \sigma(n),\ \sigma(n+1) < \sigma(n+2) < \ldots < \sigma(n+m)\}$$

denote the set of (n,m)-shuffles. Recall that the shuffle algebra on a vector space V is the space

$$Sh(V) := \bigoplus_{n \geq 1} V^{\otimes n}$$

equipped with the *shuffle product*:

$$(x_1 \otimes \ldots \otimes x_n) \pitchfork (x_{n+1} \otimes \ldots x_{n+m}) = \sum_{\sigma \in Sh_{n,m}} x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n+m)} \ . \tag{7.1}$$

$(Sh(V), \pitchfork)$ is an associative, commutative, nonunital algebra.

The free left zinbiel algebra on a vector space V is the space $Sh(V)$ equipped with the product

$$(x_1 \otimes \ldots \otimes x_n) * (x_{n+1} \otimes \ldots x_{n+m}) = \sum_{\sigma \in Sh_{n,m-1}} x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n+m-1)} \otimes x_{n+m} \ ; \tag{7.2}$$

see [L2, section 7.1]. The corresponding commutative product (section 1) is precisely the shuffle product.

One of the most basic examples of a Poisson algebra is provided by the symmetric algebra $S(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In view of the above, the "dendriform analog" of this fact should state that the shuffle algebra of a prelie algebra is a prepoisson algebra. This is our main result. In order to describe the prelie product, we introduce some notation. For $\sigma \in Sh_{n,m-1}$, let

$$I(\sigma) = \{i \ / \ 1 \leq i \leq n+m-2,\ \sigma^{-1}(i) \leq n \text{ and } \sigma^{-1}(i+1) \geq n+1\} \ .$$

Also, let

$$Sh^2_{n,m-1} = \{\sigma \in Sh_{n,m-1} \ / \ \sigma(n) = n+m-1\} \ .$$

Theorem 7.1. *Let $(P, \circ)$ be a left prelie algebra. Then $Sh(P)$ is a prepoisson algebra, with the zinbiel product* (7.2) *and the prelie product given by*

$$\begin{aligned} &(x_1 \otimes \ldots \otimes x_n) \circ (x_{n+1} \otimes \ldots \otimes x_{n+m}) \\ &= \sum_{\sigma \in Sh_{n,m-1}} \sum_{i \in I(\sigma)} x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(i-1)} \otimes \{x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)}\} \otimes x_{\sigma^{-1}(i+2)} \otimes \ldots \otimes x_{\sigma^{-1}(n+m-1)} \otimes x_{n+m} \\ &\qquad + \sum_{\sigma \in Sh^2_{n,m-1}} x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n+m-2)} \otimes (x_n \circ x_{n+m}) \end{aligned} \tag{7.3}$$

for any $n, m \geq 1$, where $\{x, y\} = x \circ y - y \circ x$ for $x, y \in P$.

The first values of the prelie product are

$$a \circ b = a \circ b \tag{7.4}$$

$$a \circ (b_1 \otimes b_2) = \{a, b_1\} \otimes b_2 + b_1 \otimes (a \circ b_2) \tag{7.5}$$

$$(a_1 \otimes a_2) \circ b = a_1 \otimes (a_2 \circ b) \tag{7.6}$$

To compute the prelie product of two elements $a_1 \otimes \ldots \otimes a_n$ and $b_1 \otimes \ldots \otimes b_m$ one may proceed as follows. First, one performs the shuffle product of $a_1 \otimes \ldots \otimes a_n$ and $b_1 \otimes \ldots \otimes b_{m-1}$ and tensors the result with b_m. Now, each term in the shuffle product gives rise to several new ones, obtained by replacing a pair of consecutive factors of the form $a_i \otimes b_j$ for $\{a_i, b_j\}$, with the exception of the pair $a_n \otimes b_m$, which may occur only at the right end, and which should be replaced by $a_n \circ b_m$. The prelie product is the expression obtained as the sum of all these terms. For instance, the product $(a_1 \otimes a_2) \circ (b_1 \otimes b_2)$ is computed as follows:

1. Shuffle $a_1 \otimes a_2$ and b_1.	2. Tensor with b_2.	3. Pair up consecutive $a_i \otimes b_j$.
$a_1 \otimes a_2 \otimes b_1$	$a_1 \otimes a_2 \otimes b_1 \otimes b_2$	$a_1 \otimes \{a_2, b_1\} \otimes b_2$
$a_1 \otimes b_1 \otimes a_2$	$a_1 \otimes b_1 \otimes a_2 \otimes b_2$	$\{a_1, b_1\} \otimes a_2 \otimes b_2 + a_1 \otimes b_1 \otimes (a_2 \circ b_2)$
$b_1 \otimes a_1 \otimes a_2$	$b_1 \otimes a_1 \otimes a_2 \otimes b_2$	$b_1 \otimes a_1 \otimes (a_2 \circ b_2)$

Thus,

$$(a_1 \otimes a_2) \circ (b_1 \otimes b_2) = a_1 \otimes \{a_2, b_1\} \otimes b_2 + \{a_1, b_1\} \otimes a_2 \otimes b_2 + a_1 \otimes b_1 \otimes (a_2 \circ b_2) + b_1 \otimes a_1 \otimes (a_2 \circ b_2)\ .$$

Remark 7.2. While $S(\mathfrak{g})$ is the free Poisson algebra on a Lie algebra $\mathfrak{g}$, $Sh(P)$ is *not* the free prepoisson algebra on a prelie algebra P. Suppose this were the case. Then, for any prepoisson algebra A, there would be a morphism of prepoisson algebras $Sh(A) \to A$, extending the identity of A. Relation (7.6) would then imply that for any elements a_1, a_2, b of A one would have

$$(a_1 * a_2) \circ b = a_1 * (a_2 \circ b)$$

(since $a_1 \otimes a_2 = a_1 * a_2$ in $Sh(A)$). However, this relation does not hold in an arbitrary prepoisson algebra. In fact, it fails for $A = Sh(P)$, for elements of degree higher than one.

In this connection, it is worth mentioning a closely related distinction between prepoisson and Poisson algebras. The axiom defining Poisson algebras, equation (2.1), tells us how to distribute the Lie product over the commutative product. More precisely, the Poisson operad is obtained from the Lie and Commutative operad by means of a *distributive law* in the sense of Markl [Mar] (see also [F-M, section 9]). This implies that the free Poisson algebra on a Lie algebra $\mathfrak{g}$ is $S(\mathfrak{g})$.

Notice that, while axiom (2.2) in the definition of prepoisson algebras tells us how to distribute the prelie product over the zinbiel product (from the left), axiom (2.3) is not distributive in nature. For this reason, the Prepoisson operad cannot be described by means of a distributive law between the Prelie and the Zinbiel operads.

Remark 7.3. It is known that the shuffle algebra $Sh(\mathfrak{g})$ of a Lie algebra is a Poisson algebra. The commutative product is the shuffle product (7.1), while the Lie bracket is given by the formula

$$\begin{aligned}&\{x_1 \otimes \ldots \otimes x_n, x_{n+1} \otimes \ldots \otimes x_{n+m}\}\\ &= \sum_{\sigma \in Sh_{n,m}} \sum_{i \in I(\sigma)} x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(i-1)} \otimes \{x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)}\} \otimes x_{\sigma^{-1}(i+2)} \otimes \ldots \otimes x_{\sigma^{-1}(n+m-1)} \otimes x_{\sigma^{-1}(n+m)}\end{aligned}$$

where, as before, $I(\sigma) = \{i \ / \ 1 \leq i \leq n+m-1,\ \sigma^{-1}(i) \leq n \text{ and } \sigma^{-1}(i+1) \geq n+1\}$. The graded version of this result appears in work of Fresse [Fre, chapter 3], where the Lie bracket on $Sh(\mathfrak{g})$ is called the *shuffle Poisson bracket*. We will come back to this in section 8

In characteristic zero, $S(\mathfrak{g})$ embeds as a Poisson subalgebra of $Sh(\mathfrak{g})$, by viewing $S^n(\mathfrak{g})$ as the subspace of S_n-invariants of $\mathfrak{g}^{\otimes n}$.

If P is a prelie algebra, then the two Poisson structures on $Sh(P)$ (one coming from the prepoisson structure of theorem 7.1, the other from the above result of Fresse applied to the Lie algebra associated to P), agree.

8. Pregerstenhaber algebras

Gerstenhaber algebras are a certain graded version of the notion of Poisson algebras. Specifically, a Gerstenhaber algebra is a graded vector space equipped with a commutative multiplication of degree 0:

$$x(yz) = (xy)z \text{ and } xy = (-1)^{|x||y|} yx$$

and with a Lie bracket of degree -1:

$$\{x,y\} = -(-1)^{(|x|-1)(|y|-1)}\{y,x\}\ ,$$
$$(-1)^{(|x|-1)(|z|-1)}\{\{x,y\},z\} + (-1)^{(|y|-1)(|x|-1)}\{\{y,z\},x\} + (-1)^{(|z|-1)(|y|-1)}\{\{z,x\},y\} = 0$$

which are compatible:

$$\{x,yz\} = \{x,y\}z + (-1)^{(|x|-1)|y|}y\{x,z\}\ .$$

Definition 8.1. A left pregerstenhaber algebra is a triple $(A,*,\circ)$ where A is a graded vector space, the operation $*$ preserves degrees and satisfies the graded zinbiel axiom:

$$x*(y*z) = (-1)^{|x||y|}(y*x)*z + (x*y)*z\ , \tag{8.1}$$

the operation $\circ$ lowers degrees by one and satisfies the graded prelie axiom:

$$x\circ(y\circ z) - (x\circ y)\circ z = (-1)^{(|x|-1)(|y|-1)}(y\circ(x\circ z) - (y\circ x)\circ z) \tag{8.2}$$

and the following conditions hold:

$$(x\circ y - (-1)^{(|x|-1)(|y|-1)}y\circ x)*z = x\circ(y*z) - (-1)^{(|x|-1)|y|}y*(x\circ z) \tag{8.3}$$

$$(x*y + (-1)^{|x||y|}y*x)\circ z = (-1)^{|x|(|z|-1)}x*(y\circ z) + (-1)^{(|x|+|z|-1)|y|}y*(x\circ z)\ . \tag{8.4}$$

The graded version of proposition 2.2 is:

Proposition 8.2. *Let $(A,*,\circ)$ be a left pregerstenhaber algebra. Define*

$$x\cdot y = x*y + (-1)^{|x||y|}y*x \ \text{and}\ \{x,y\} = x\circ y - (-1)^{(|x|-1)(|y|-1)}y\circ x\ .$$

Then $(A,\cdot,\{\ ,\ \})$ is a Gerstenhaber algebra.

The other results presented in this note also admit a graded version. Perhaps the most interesting is the one concerning the shuffle algebra. The graded shuffle algebra is the space

$$Sh_\lambda(V) := \bigoplus_{n\geq 1} V^{\otimes n}$$

equipped with the *graded shuffle product*:

$$(x_1\otimes\ldots\otimes x_n) \pitchfork (x_{n+1}\otimes\ldots\otimes x_{n+m}) = \sum_{\sigma\in Sh_{n,m}} \epsilon(\sigma)x_{\sigma^{-1}(1)}\otimes\ldots\otimes x_{\sigma^{-1}(n+m)}\ , \tag{8.5}$$

where ϵ is the sign of a permutation. $(Sh_\lambda(V),\pitchfork)$ is a graded commutative algebra. It is actually the graded commutative algebra associated to the following graded zinbiel structure on $Sh_\lambda(V)$:

$$(x_1\otimes\ldots\otimes x_n) * (x_{n+1}\otimes\ldots\otimes x_{n+m}) = \sum_{\sigma\in Sh_{n,m-1}} \epsilon(\sigma)x_{\sigma^{-1}(1)}\otimes\ldots\otimes x_{\sigma^{-1}(n+m-1)}\otimes x_{n+m}\ . \tag{8.6}$$

Theorem 8.3. *Let $(P,\circ)$ be a left prelie algebra. Then $Sh_\lambda(P)$ is a left pregerstenhaber algebra, with the zinbiel product* (8.6) *and the prelie product given by*

$$\begin{aligned}
&(x_1\otimes\ldots\otimes x_n)\circ(x_{n+1}\otimes\ldots\otimes x_{n+m}) \\
&= \sum_{\sigma\in Sh_{n,m-1}}\sum_{i\in I(\sigma)} \epsilon(\sigma)(-1)^i x_{\sigma^{-1}(1)}\otimes\ldots\otimes x_{\sigma^{-1}(i-1)}\otimes\{x_{\sigma^{-1}(i)},x_{\sigma^{-1}(i+1)}\}\otimes x_{\sigma^{-1}(i+2)}\otimes\ldots\otimes x_{\sigma^{-1}(n+m-1)}\otimes x_{n+m} \\
&\qquad + \sum_{\sigma\in Sh^2_{n,m-1}} \epsilon(\sigma)(-1)^{n+m-1}x_{\sigma^{-1}(1)}\otimes\ldots\otimes x_{\sigma^{-1}(n+m-2)}\otimes(x_n\circ x_{n+m})
\end{aligned} \tag{8.7}$$

for any $n,m\geq 1$, where $\{x,y\} = x\circ y - (-1)^{(|x|-1)(|y|-1)}y\circ x$ for $x,y\in P$.

The first values of the prelie product are

$$a \circ b = a \circ b$$
$$a \circ (b_1 \otimes b_2) = (a \circ b_1 - b_1 \circ a) \otimes b_2 + b_1 \otimes (a \circ b_2)$$
$$(a_1 \otimes a_2) \circ b = a_1 \otimes (a_2 \circ b)$$
$$(a_1 \otimes a_2) \circ (b_1 \otimes b_2) = a_1 \otimes (a_2 \circ b_1 - b_1 \circ a_2) \otimes b_2 + (a_1 \circ b_1 - b_1 \circ a_1) \otimes a_2 \otimes b_2 + a_1 \otimes b_1 \otimes (a_2 \circ b_2) - b_1 \otimes a_1 \otimes (a_2 \circ b_2)$$

Remark 8.4. It is also true that the graded shuffle algebra $Sh_\lambda(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Gerstenhaber algebra. The Lie bracket is given by the formula

(8.8) $\{x_1 \otimes \ldots \otimes x_n, x_{n+1} \otimes \ldots \otimes x_{n+m}\} =$

$$\sum_{\sigma \in Sh_{n,m}} \sum_{i \in I(\sigma)} \epsilon(\sigma)(-1)^i x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(i-1)} \otimes \{x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)}\} \otimes x_{\sigma^{-1}(i+2)} \otimes \ldots \otimes x_{\sigma^{-1}(n+m-1)} \otimes x_{\sigma^{-1}(n+m)}$$

where $I(\sigma)$ is as in remark 7.3. This is the result of Fresse mentioned before (although a different sign convention is used in that work).

If $\mathfrak{g}$ comes from a prelie algebra, then this Gerstenhaber structure on $Sh_\lambda(\mathfrak{g})$ agrees with the one corresponding to the pregerstenhaber structure of theorem 8.3.

There is a connection between these structures and the classical Yang-Baxter equation, that we explain next. For an element $r = \sum u_i \otimes v_i \in \mathfrak{g} \otimes \mathfrak{g}$, this equation is the equality $\mathbf{C}(r) = 0$, where

$$\mathbf{C}(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \sum [u_i, u_j] \otimes v_i \otimes v_j + \sum u_i \otimes [v_i, u_j] \otimes v_j + \sum u_i \otimes u_j \otimes [v_i, v_j] \ .$$

It follows readily from (8.8) that

$$\{r, r\} = 2\mathbf{C}(r) \ .$$

This is, in fact, an extension of a well known interpretation of the classical Yang-Baxter equation for a *skew symmetric* tensor $r \in \Lambda^2(\mathfrak{g})$: in characteristic zero, the exterior algebra $\Lambda(\mathfrak{g})$ embeds as a Gerstenhaber subalgebra of $Sh_\lambda(\mathfrak{g})$, by viewing $\Lambda^n(\mathfrak{g})$ as the subspace of skewed S_n-invariants of $\mathfrak{g}^{\otimes n}$, and it is well known that in this case $\{r, r\} = 2\mathbf{C}(r)$ [Dri, section 4]. Passing to the bigger algebra $Sh_\lambda(\mathfrak{g})$ allows us to get rid of the assumption that r be skew symmetric.

If P is a prelie algebra, then $\Lambda(P)$ is not a pregerstenhaber subalgebra of $Sh_\lambda(P)$, but one may still consider the prelie product $r \circ r$ in $Sh_\lambda(P)$, for $r \in P^{\otimes 2}$. One obtains, this time,

$$r \circ r = \mathbf{C}(r) \ .$$

We would like to know of a similar interpretation for the associative Yang-Baxter equation (5.2).

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