Category theory in action

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Category theory has many widely recognized uses: it is a way to abstract various mathematical disciplines, to show the connections and interplay between them, and to prove or display existing theorems in a more elegant and succinct manner. Unfortunately, this can lead us to think that category theory is nothing but a formal language, an abstract way to rewrite what we already know.

The goal of this talk will be to show a concrete problem where category theory truly brings something new to the table; an example where abstracting the concepts helps clear a path towards a solution that was not in sight before, and where later, tools from category theory are used to find that solution.

The problem

Any time we work with an algebraic structure (groups, vector spaces, rings, etc), there is a notion of what the "right functions" between them are: we are interested in the functions that preserve the algebraic structure on hand. For example, when working with groups, we care about functions

$$f : G \to H$$

that preserve the product; that is,

$$f(xy) = f(x)f(y)$$

The set of all such functions is denoted by $\mathrm{Hom}(G, H)$, short for *homomorphism*. It is then natural to wonder: when is the set $\mathrm{Hom}(G, H)$ of all group homomorphisms between G and H a group itself?

We can try the naive guess and see if it works: define a pointwise product by $(fg)(x) = f(x)g(x)$. This will be associative, have a unit given by the constant morphism $f(x) = 1_G$, and every f will have an inverse, given by $f^{-1}(x) = (f(x))^{-1}$ since inverses exist in G. But of course, for this to work out, (fg) needs to be an element of $\mathrm{Hom}(G, H)$ too, and we can see that

$$(fg)(xy) = f(xy)g(xy) = f(x)f(y)g(x)g(y) \neq f(x)g(x)f(y)g(y) = (fg)(x)(fg)(y)$$

since our groups are not presumed to be abelian.[1]

So, the obvious guess doesn't work, but how can we prove that there is no other approach that will work? And even if we're crafty enough to show that, what happens if we want to change from groups to rings? Do we need to come up with a new clever, ad hoc proof?

We will see that we can define what an *algebraic theory* is abstractly, and use category theory to give a systematic way to deal with this question.

A 5 minute intro to category theory

We start by introducing some of the very basics of category theory, naming only what we will need for the purposes of this talk.

Date: March 24th, 2018

[1]One could check that the function f^{-1} won't be an element of $\mathrm{Hom}(G, H)$ either.

Definition 1. A category consists of

- a collection of *objects*
- a collection of *morphisms*

such that

- each morphism has a *source* and a *target* given by objects of the category
- given two morphisms $f : X \to Y$ and $g : Y \to Z$ there exists a *composite morphism* $gf : X \to Z$
- this composition is *associative*
- each object has a special *identity morphism* that serves as a unit for the composition

Most of the mathematical structures that we work with every day assemble into categories.

Example 2.

- **Set** has sets as objects and functions as morphisms.
- **Grp** has groups as objects and group homomorphisms as morphisms.
- **Ab** has abelian groups as objects and group homomorphisms as morphisms.
- $\mathbf{Vect}_{\Bbbk}$ has $\Bbbk$-vector spaces as objects and $\Bbbk$-linear maps as morphisms.

Just like when working with groups, it is evident that the "correct" maps to consider are group homomorphisms, there is a right notion of maps between categories which preserves all the relevant structure.

Definition 3. A covariant (contravariant) *functor* $F : \mathcal{C} \to \mathcal{D}$ between two categories consists of the following data:

- for each object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$
- for each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ $(F(f) : F(Y) \to F(X))$ in $\mathcal{D}$

satisfying the following conditions:

- for each object $X \in \mathcal{C}$, $F(id_X) = id_{F(X)}$
- for any pair of composable maps f and g in $\mathcal{C}$, $F(gf) = F(g)F(f)$ $(F(gf) = F(f)F(g))$

Example 4. Given a category $\mathcal{C}$ and an object Y in $\mathcal{C}$, the contravariant Hom functor $\mathrm{Hom}_{\mathcal{C}}(-, Y) : \mathcal{C} \to \mathbf{Set}$ is defined as follows:

- for any object X in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the set of all morphisms in $\mathcal{C}$ with source X and target Y
- given a morphism $f : X \to X'$ in $\mathcal{C}$, $\mathrm{Hom}(f, Y) : \mathrm{Hom}_{\mathcal{C}}(X', Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ precomposes by f; that is, takes a morphism $\varphi : X' \to Y$ to the morphism $\varphi f : X \to Y$.

Of course, one would need to check that the two conditions in the definition of functor are satisfied.

Finally, there is also a notion of maps between functors.

Definition 5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two contravariant functors. A natural transformation $\tau : F \Rightarrow G$ consists of a morphism $\tau_X : F(X) \to D(X)$ in $\mathcal{D}$, for each object $X \in \mathcal{C}$, such that for each morphism $f : X \to X'$ in $\mathcal{C}$, the following commutes

$$\begin{tikzcd} F(X') \arrow[r, "\tau_{X'}"] \arrow[d, "F(f)"'] & G(X') \arrow[d, "G(f)"] \\ F(X) \arrow[r, "\tau_X"'] & G(X) \end{tikzcd}$$

Example 6. Given a category $\mathcal{C}$ and a morphism $f : Y \to Y'$ in $\mathcal{C}$, we can define a natural transformation $\tau : \mathrm{Hom}_{\mathcal{C}}(-, Y) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, Y')$ by

$$\tau_X : \mathrm{Hom}_{\mathcal{C}}(X, Y) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y')$$

$$\varphi : X \to Y \mapsto f\varphi : X \to Y'$$

Indeed, given a morphism $g : X \to X'$, the following commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X', Y) \arrow[r, "\tau_{X'} = f\circ -"] \arrow[d, "-\circ f"'] & \mathrm{Hom}_{\mathcal{C}}(X', Y) \arrow[d, "-\circ f"] \\ \mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\tau_X = f\circ -"'] & \mathrm{Hom}_{\mathcal{C}}(X, Y) \end{tikzcd}$$

In fact, one of the result in category theory shows that all natural transformations between Hom functors arise in this way.

Theorem 7 (Yoneda lemma)**.** Let $\mathcal{C}$ be a category and Y, Y' two objects in $\mathcal{C}$. Any natural transformation

$$\tau : \mathrm{Hom}_{\mathcal{C}}(-, Y) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, Y')$$

is given by postcomposition with some morphism $f : Y \to Y'$ in $\mathcal{C}$.

Algebraic theories and their algebras

We now introduce the language in which to better express our problem.

Definition 8. An *algebraic theory* $\mathbb{T}$ is a family of operator symbols $\{f_i\}$, together with non-negative integers $\{v_i\}$ and equations relating the f_i's, each of which looks in the equations as if it were a function on v_i arguments.

For example, the algebraic theory of groups will be given by operators m of arity 2, i or arity 1, and e of arity 0, together with equations

$$m(x, m(y, x)) = m(m(x, y), z) \qquad \text{(associativity)}$$

$$m(x, e) = x = m(e, x) \qquad \text{(unit)}$$

$$m(x, i(x)) = e = m(i(x), x) \qquad \text{(inverses)}$$

We say that $A \in \mathbf{Set}$ is a $\mathbb{T}$-*algebra* if each operator $f_i \in \mathbb{T}$ has an interpretation $\bar{f}_i : \prod_{v_i} A \to A$ such that all equations are true (if so, these interpretations form the $\mathbb{T}$-algebra structure of A). We will denote by $\mathbf{Set}^{\mathbb{T}}$ the category of $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms.

In the case when $\mathbb{T}$ is the theory of groups, a $\mathbb{T}$-algebra will be a set A together with functions

$$m : A \times A \to A, \quad e : \{*\} \to A, \quad i : A \to A$$

satisfying the previous equations; that is, a group.

Definition 9. A morphism of $\mathbb{T}$-algebras is a function $g : A \to B$ such that, for every operator $f_i \in \mathbb{T}$, the following diagram commutes

$$\begin{tikzcd} \prod A \arrow[r, "f_i"] \arrow[d, "\prod g"'] & A \arrow[d, "g"] \\ \prod B \arrow[r, "f_i"] & B \end{tikzcd}$$

Let's use all this new language to rewrite our main result. We want to study whether the category $\mathbf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras is such that $\mathrm{Hom}_{\mathbb{T}}(A, B)$ is a $\mathbb{T}$-algebra, for any $\mathbb{T}$-algebras A and B. This is equivalent to saying the functor

$$\mathrm{Hom}_{\mathbb{T}}(-, B) : \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$$

is algebra valued for every algebra $B \in \mathbf{Set}^{\mathbb{T}}$[2]

Algebra valued Hom functors

Let's take a look at the (contravariant) hom-set functors $\mathrm{Hom}_{\mathcal{C}}(-, B) : \mathcal{C} \to \mathbf{Set}$ for a fixed B in a category $\mathcal{C}$. If we set $\mathcal{C} = \mathbf{Set}$ and let G be a group, we know that for any set X, the set of functions $\mathrm{Hom}_{\mathbf{Set}}(X, G)$ admits a canonical (pointwise) group structure, induced by the product in G.

$$\mathrm{Hom}_{\mathcal{C}}(X, G) \times \mathrm{Hom}_{\mathcal{C}}(X, G) \cong \mathrm{Hom}_{\mathcal{C}}(X, G \times G) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, G)$$

Actually, there's nothing particular about groups in this fact, and so we have:

Proposition 10. $\mathrm{Hom}_{\mathcal{C}}(-, B) : \mathcal{C} \to \mathbf{Set}$ is algebra valued if B is a $\mathbb{T}$-algebra.

Just like in the case of groups, the $\mathbb{T}$-algebra structure of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is given by

$$\prod_{v_i} \mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, \prod_{v_i} B) \xrightarrow{\bar{f}_i \circ -} \mathrm{Hom}_{\mathcal{C}}(A, B)$$

[2] A very attentive reader might notice that this equivalence is not as trivial as it seems: our original question was whether $\mathrm{Hom}_{\mathbb{T}}(A, B)$ is an algebra or not, but asking the functor $\mathrm{Hom}_{\mathbb{T}}(-, B)$ to be algebra valued also imposes another condition; namely, that any $\mathbb{T}$-morphism $A \to A'$ induces a $\mathbb{T}$-morphism $\mathrm{Hom}_{\mathbb{T}}(A', B) \to \mathrm{Hom}_{\mathbb{T}}(A, B)$. However, if $\mathrm{Hom}_{\mathbb{T}}(A, B)$ is a $\mathbb{T}$-algebra, then it will be a subalgebra of $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ (which is always a $\mathbb{T}$-algebra with pointwise operations, as we saw before). Then, since any $\mathbb{T}$-morphism $A \to A'$ induces a $\mathbb{T}$-morphism $\mathrm{Hom}_{\mathbf{Set}}(A', B) \to \mathrm{Hom}_{\mathbf{Set}}(A, B)$, its restriction to the subalgebra $\mathrm{Hom}_{\mathbb{T}}(A', B)$ must also be a $\mathbb{T}$-morphism, and one can easily check that the image will be a subset of $\mathrm{Hom}_{\mathbb{T}}(A, B)$.

where $\bar{f}_i : \prod_{v_i} B \to B$ is the interpretation of the operator $f_i \in \mathbb{T}$ determined by the algebra structure of B.[3]

Interestingly, the converse of the previous statement is also true!

Proposition 11. If $\mathrm{Hom}_{\mathcal{C}}(-, B) : \mathcal{C} \to \mathbf{Set}$ is algebra valued, then B admits a canonical T-algebra structure.

To show this less evident converse, fix an operator f_i in $\mathbb{T}$; using the $\mathbb{T}$-algebra structure of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ we have, for every $A \in \mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(A, \textstyle\prod B) \simeq \prod \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{\tilde{f}_i} \mathrm{Hom}_{\mathcal{C}}(A, B)$$

It's possible to show that these maps can be put together to form a natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(-, \prod B) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, B)$$

Indeed, this amounts to showing that, for any $g : A \to A'$, the following commutes

$$\begin{tikzcd}
\prod \mathrm{Hom}_{\mathcal{C}}(A', B) \arrow[r, "\tilde{f}'_i"] \arrow[d, "\prod -\circ g"] & \mathrm{Hom}_{\mathcal{C}}(A', B) \arrow[d, "-\circ g"] \\
\prod \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[r, "\tilde{f}_i"] & \mathrm{Hom}_{\mathcal{C}}(A, B)
\end{tikzcd}$$

But $\mathrm{Hom}_{\mathcal{C}}(-, B)$ is an algebra valued functor, so $\mathrm{Hom}_{\mathcal{C}}(A', B) \to \mathrm{Hom}_{\mathcal{C}}(A, B)$ must be a homomorphism, which by definition translates into that diagram being commutative.

Then, by the Yoneda Lemma, that natural transformation must come from a map

$$\prod B \to B$$

which we take as the interpretation of f_i.

Summing up

The previous result says that $\mathbb{T}$ will be a theory such that $\mathrm{Hom}_{\mathcal{C}}(-, B)$ is algebra valued precisely when every B in $\mathcal{C}$ admits a $\mathbb{T}$-algebra structure; that is, when every B is in $\mathcal{C}^{\mathbb{T}}$. Taking $\mathcal{C} = \mathbf{Set}^{\mathbb{T}}$, we see that $\mathbb{T}$ will be a theory such that $\mathrm{Hom}_{\mathbf{Set}^{\mathbb{T}}}(-, B)$ is algebra valued precisely when every B in $\mathbf{Set}^{\mathbb{T}}$ admits a $\mathbb{T}$-algebra structure; that is, when every B in $\mathbf{Set}^{\mathbb{T}}$ is actually a member of $(\mathbf{Set}^{\mathbb{T}})^{\mathbb{T}}$.

But then every $\mathbb{T}$ operator must have an interpretation as a map in $\mathbf{Set}^{\mathbb{T}}$, and these are $\mathbb{T}$-algebra homomorphisms, so the theory $\mathbb{T}$ must be such that every $\mathbb{T}$ operator is a $\mathbb{T}$ homomorphism.

Explicitly: every pair of operators $f_i, g_j \in \mathbb{T}$ must satisfy

$$\begin{aligned} f_i\left(g_j(x_{11}, \ldots, x_{1v_j}), \ldots,\ g_j(x_{v_i 1}, \ldots, x_{v_i v_j})\right) \\ = g_j\left(f_i(x_{11}, \ldots, x_{v_i 1}), \ldots, f_i(x_{1v_j}, \ldots, x_{v_i v_j})\right), \end{aligned} \tag{1}$$

in other words, all operators in $\mathbb{T}$ commute, in the sense of (1).

[3] All equations of the theory will be preserved, since $\mathrm{Hom}_{\mathcal{C}}(A, -)$ is a product-preserving functor.

Applying our theorem

As examples, we can easily see that

(a) R-**Mod** won't have hom-sets that are modules unless R is a commutative ring, since the operations "multiply by r" and "multiply by s" must commute for every $r, s \in R$.

(b) **Grp**, our motivating example, won't either , since the binary operation doesn't commute with itself.

(c) We can get around the problem in (b) by considering **Ab**, which is commutative and therefore has hom-sets that are abelian groups.

(d) **Ring** is not autonomous since the product operation doesn't commute with itself.

(e) Interestingly, if we proceed as in (c) and consider **CommRing**, it doesn't solve the problem: the two constants would commute with each other, so

$$0 = 0.1 = 1.0 = 1$$

which would imply there is only one constant.

We're not saying these results are a novelty; for example, it's well known that **Grp** is not enriched over itself, and that R-**Mod** will only be so when R is commutative. However, even though it's easy to see that the naive approaches do not work, showing "by hand" that no approach will ever work requires some ingenuity, and the theorem we presented offers a systematic way to deal with this question.

A last remark

I want to finish by turning the attention back to the role of category theory in all of this. Even though this algebraic result is really neat and memorable, my goal was to highlight the use of category theory in solving this problem.

Note that, at first, there is a rewriting involved in the passage from the original question about hom-sets, to its interpretation as a question about contravariant Hom functors taking values in some category. It then becomes clear that the next step to take is to study these functors, and try to characterize the situations in which they will be algebra valued. In doing this, the Yoneda Lemma, an abstract result in category theory, turns out to be instrumental. Finally, it is simply a matter of interpreting the abstract, general result that we get in any particular algebraic category that we might care about to obtain the answer we were looking for.

References

[Fre66] P. Freyd. Algebra valued functors in general and tensor products in particular. XIV, 1966.