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Final project
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On the cardinalities of the Fatou and Julia sets

Abstract

The aim of this expository paper is to briefly study possible cardinalities for the Fatou and Julia sets; namely, we will show that the Julia set of (almost every) holomorphic map from the Riemann sphere into itself must always be infinite, and that in contrast there exist maps with infinite and with empty Fatou sets.

1 Some basic notions

In this short paper, we will be interested in the behaviour of the iterates of holomorphic maps $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, where $\mathbb{C}_\infty$ denotes the one-dimensional Riemann sphere, i.e. the compactification of the complex plane. With this in mind, we begin by describing the structure of such maps.

Proposition 1.1. *Every non-constant holomorphic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function, that is, there exist polynomials P and Q without common factors and unique up to a multiplicative constant such that*

$$f(z) = \frac{P(z)}{Q(z)}$$

Proof. Since f is holomorphic and non-constant, the set $f^{-1}(\infty)$ must be a discrete subset in the compact space $\mathbb{C}_\infty$, and therefore finite. For the same reason, the set $f^{-1}(0)$ must also be finite. Let $z_1, \dots, z_h \in \mathbb{C}$ be the complex poles of f, and $w_1, \dots, w_k \in \mathbb{C}$ the complex zeroes of f, all listed as many times as their respective multiplicities. Then the function

$$g(z) = \frac{(z - z_1)\dots(z - z_h)}{(z - w_1)\dots(z - w_k)} f(z)$$

has neither poles nor zeroes in $\mathbb{C}$. If $g(\infty) \in \mathbb{C}$, then $g(\mathbb{C}_\infty)$ is a bounded subset of $\mathbb{C}$ and by Liouville's theorem g is a constant map, so f can be written as a rational function with $P(z) = c(z - w_1)\dots(z - w_k)$ and $Q(z) = (z - z_1)\dots(z - z_h)$.

If instead $g(\infty) = \infty$, then $1/g$ is bounded and hence constant, which implies g is constant too (note that $1/g$ is always holomorphic as a map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, as long as g is holomorphic). Thus $g = \infty$ and therefore $f = \infty$, contradicting the fact that f is non-constant.

Finally, the uniqueness statement is immediate from the fact that a polynomial is determined by its roots up to a multiplicative constant.

□

Due to this proposition, from now on we will be concerned with rational maps $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$.

Definition 1.2. The degree of a non-constant rational map $R = \dfrac{P}{Q}$ is given by $\deg R = \max\{\deg P, \deg Q\}$.

Definition 1.3. A family $\mathcal{F}$ of maps from (X_1, d_1) to (X_2, d_2) is equicontinuous at x if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $f \in \mathcal{F}$ and $y \in X_1$

$$d_1(x, y) < \delta \text{ implies } d_2(f(x), f(y)) < \epsilon$$

We will make use without proof of the following known result:

Theorem 1.4. *Let $\mathcal{F}$ be any family of maps from (X_1, d_1) to (X_2, d_2). Then there is a maximal open subset of X_1 on which $\mathcal{F}$ is equicontinuous. In particular, if f maps a space (X, d) into itself, there is a maximal open subset of X on which the family of iterates $\{f^n\}$ is equicontinuous.*

Thanks to this result, we can define the Fatou and Julia sets in the following way.

Definition 1.5. Let R be a non-constant rational map. The Fatou set of R is the maximal open subset of $\mathbb{C}_\infty$ on which $\{R^n\}$ is equicontinuous, and the Julia set is its complement in $\mathbb{C}_\infty$.

It should be noted that there are various definitions of the Fatou and Julia sets of a rational map (all of them equivalent); we choose this one simply because it is convenient, keeping in mind the results we intend to show.

We denote the Fatou set of a rational map R by $F(R)$ or simply F, and the Julia set by $J(R)$ or J. Note that by definition, $F(R)$ is open and $J(R)$ is compact.

It will also be useful to think about the Fatou and Julia sets in terms of normality, as follows.

Definition 1.6. A family $\mathcal{F}$ of maps from (X_1, d_1) to (X_2, d_2) is normal in X_1 if every infinite sequence of functions in $\mathcal{F}$ contains a subsequence that converges locally uniformly on X_1.

Recall the following statement of the Arzelá-Ascoli Theorem, which we state without proof:

Theorem 1.7. *Let $\mathcal{F}$ be a family of maps from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. Then $\mathcal{F}$ is equicontinuous if and only if it is a normal family.*

This naturally yields the following equivalent definition of the Fatou and Julia sets.

Definition 1.8. Let R be a non-constant rational map. The Fatou set of R is the maximal open subset of $\mathbb{C}_\infty$ on which $\{R^n\}$ is normal, and the Julia set is its complement in $\mathbb{C}_\infty$.

We conclude this section with a simple fact that will be useful later on.

Remark 1.9. If P is a polynomial of degree at least two, then ∞ is in $F(P)$. This is easy if we use the last definition given for the Fatou set, since it is clear that there exists some neighborhood of ∞ on which $P^n \to \infty$ uniformly.

2 Completely invariant sets

In this section, we introduce the notion of completely invariant sets, and see how they relate to the Fatou and Julia sets of a rational map.

Definition 2.1. If f is a map from a set X into itself, a subset E of X is:

- forward invariant if $f(E) = E$,
- backward invariant if $f^{-1}(E) = E$,
- completely invariant if it is both forward and backward invariant.

Note that for surjective maps (like the rational maps that concern us), the concepts of backward invariance and complete invariance coincide.

Proposition 2.2. *Let R be a rational map of degree at least two, and suppose that a finite set E is completely invariant under R. Then E has at most two elements.*

Proof. Suppose E has k elements. Because E is finite and R maps E into E, R must act as a permutation of E and so for a suitable integer q, R^q is the identity map from E into itself. Now suppose that R^q has degree d. It follows that for every w in E, the equation $R^q(z) = w$ has d solutions, all equal to w, and so applying the Riemann-Hurwitz formula to R^q, we have

$$k(d-1) \leq 2d-2$$

As $d \geq 2$, we get $k \leq 2$ as required. □

For any given elements $x, y \in X$, we define the relation $\sim$ on X by $x \sim y$ if and only if there exist non-negative integers n and m with $f^n(x) = f^m(y)$. It is clear that $\sim$ is an equivalence relation on X, and it is easy to check that the class of any element $[x]$ is the smallest completely invariant set that contains x.

From this, we see that a set E is completely invariant if and only if it is a union of equivalence classes $[x]$, and if this is the case, then its complement must also be a union of equivalence classes and therefore completely invariant. This fact makes it easier to show the following result.

Theorem 2.3. *Let R be any rational map. Then the Fatou and Julia sets of R are completely invariant.*

Proof. From our previous discussions, it suffices to show that F is backwards invariant. First, take any $z_0 \in R^{-1}(F)$ and let $w_0 = R(z_0)$; thus $w_0 \in F$. It follows that given any positive ϵ there exists a positive δ such that if $d(w, w_0) < \delta$, then $d(R^n(w), R^n(w_0)) < \epsilon$. By continuity, there is also a positive ρ such that if $d(z, z_0) < \rho$, then $d(R(z), w_0) < \delta$, and hence $d(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon$. This shows that $\{R^{n+1} : n \geq 1\}$ is equicontinuous at z_0; thus, $\{R^n : n \geq 1\}$ is equicontinuous at z_0 and hence on $R^{-1}(F)$. Since $R^{-1}(F)$ is open, we deduce that $R^{-1}(F) \subset F$.

For the other inclusion, take any $z_0 \in F$ and let $w_0 = R(z_0)$. Because z_0 is in F, given any positive ϵ there is a positive δ such that for all n, if $d(z, z_0) < \delta$ then $d(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon$. The set of points satisfying $d(z, z_0) < \delta$ is an open neighborhood

U of z_0, and so $R(U)$ is an open neighborhood of w_0. If $w \in R(U)$, then $w = R(z)$ for some $z \in U$ and so

$$d(R^n(w), R^n(w_0)) = d(R^{n+1}(z), R^{n+1}(z_0)) < \epsilon$$

This shows that $w_0 \in F$, so $F \subset R^{-1}(F)$.

□

Definition 2.4. A point z is said to be exceptional for R when $[z]$ is finite, and the set of all such points is denoted by $E(R)$.

The following theorem justifies the terminology, showing that such points are indeed exceptional, and that they always belong to the Fatou set.

Theorem 2.5. *A rational map R of degree at least two has at most two exceptional points, both of which lie in $F(R)$.*

Proof. Clearly $E(R)$ is completely invariant under R, and so by Proposition 2.2, R has at most two exceptional points. Thus, there are four possibilities to consider, namely:

1. $E(R) = \emptyset$,
2. $E(R) = \{\zeta\} = [\zeta]$,
3. $E(R) = \{\zeta_1, \zeta_2\},\ [\zeta_1] = \{\zeta_1\},\ [\zeta_2] = \{\zeta_2\}$,
4. $E(R) = \{\zeta_1, \zeta_2\} = [\zeta_1] = [\zeta_2]$.

We can conjugate R by a Moebius transformation in a suitable way, such that in the previous cases ζ corresponds to ∞, ζ_1 corresponds to 0 and ζ_2 corresponds to ∞. Then, if $S = hRh^{-1}$, the possibilities for $E(S)$ are

1. $E(S) = \emptyset$,
2. $E(S) = \{\infty\} = [\infty]$,
3. $E(S) = \{0, \infty\},\ [0] = \{0\},\ [\infty] = \{\infty\}$,
4. $E(S) = \{0, \infty\} = [0] = [\infty]$.

There is nothing to say about case 1. If 2 holds, then S has a pole at ∞ and nowhere else, so S must be a polynomial. Thus, by Remark 1.9 we have that $\infty \in F(S)$. If 3 holds, then for the same reason S must be a polynomial, but since $S(0) = 0$ it must be $S(z) = az^d$ for some positive integer d. Similarly, if 4 holds then $S(0) = \infty,\ S(\infty) = 0$ and S has all its zeros and poles in $\{0, \infty\}$, so it must be of the form $S(z) = az^d$ for some negative integer d. For both of these possible S's, it is clear that $\{0, \infty\} \subset F(S)$.

To conclude the proof, it suffices to show that $F(S) = h(F(R))$. But this comes from the fact that h satisfies a Lipschitz condition with respect to the metric on $\mathbb{C}_\infty$, and therefore the family $\{hR^k\}$ will be equicontinuous wherever $\{R^k\}$ is. Then, if $z \in F(R)$, $S^k(h(z)) = hR^k(z)$ and so $h(F(R)) \subset F(S)$. Applying the same reasoning to h^{-1}, we get $h^{-1}(F(S)) \subset F(R)$, and so $F(S) = h(F(R))$.

□

3 The interesting stuff

As advertised, we proceed to show some results on the cardinality of the Julia and Fatou sets of a rational map. We begin with the study of $J(R)$.

Our aim is to show that the Julia set of any rational map with degree at least two must have infinitely many elements. This is not hard to prove, making use of all the results we established in the previous section.

Proposition 3.1. *If* $\deg R \geq 2$*, then* $J(R)$ *is not empty.*

Proof. If J is empty, then the family $\{R^n\}$ is normal on the entire complex sphere, so there is some subsequence $\{R^{n_k}\}$ that converges uniformly on the complex sphere to a map Q. Then Q must also be a rational map (since it will be holomorphic), and so for all $k \gg 0$ we have $\deg R^{n_k} = \deg Q$. But $\deg R^{n_k} = (\deg R)^{n_k}$, which implies $\deg R = 1$, contrary to our assumption.

□

Theorem 3.2. *If* $\deg R \geq 2$*, then* $J(R)$ *is infinite.*

Proof. We know J is not empty, so it contains some point z_0. Now J is completely invariant from Theorem 2.3, so if J is finite, then z_0 must be an exceptional point. This is not possible though, since all exceptional points lie in F due to Theorem 2.5; thus J is infinite.

□

With some additional work, one can prove that in fact $J(R)$ is a perfect set (if $\deg(R) \geq 2$), and therefore $J(R)$ must be uncountable.

Finally, we turn our attention to the Fatou set of a rational map. As we will show, not only does $F(R)$ not need to be an infinite set, but it can even be empty. The first example of a rational map with empty Fatou set was given by Lattès in 1918; in [Lat18], he showed that the Julia set for the map

$$z \mapsto \frac{(z^2+1)^2}{4z(z^2-1)}$$

consists of the whole complex sphere. We now give a characterization for when such a thing can occur.

Theorem 3.3. *Let* R *be a rational map. Then* $J(R) = \mathbb{C}_\infty$ *if and only if there is some* z *whose forward orbit* $\{R^n(z)\}$ *is dense in the complex sphere.*

Proof. Let $\{B_n\}$ be a countable base for the topology on $\mathbb{C}_\infty$, and let D be the set of all z such that the forward orbit $O^+(z)$ is dense in $\mathbb{C}_\infty$; thus z is in D if and only if for all k there exists some n with $R^n(z) \in B_k$, and this implies that

$$D = \bigcap_{k\geq 1} \bigcup_{n \geq 1} R^{-n}(B_k)$$

Suppose now that $D = \emptyset$. We write $A_k = \mathbb{C}_\infty - B_k$ and

$$E_k = \bigcap_{n\geq 1} R^{-n}(A_k)$$

Then, as $D = \emptyset$, we have

$$\mathbb{C}_\infty = \bigcup_{k\geq 1} E_k$$

Now, by Baire's Theorem, $\mathbb{C}_\infty$ can't be a countable union of nowhere dense sets; thus, for some k, the closure of E_k has a non-empty interior, say W. However, E_k is closed, so W is a non-empty open subset of E_k. This means that for all n, $R^n(W) \subset A_k$; thus the functions R^n do not take values in B_k when applied to W, and so $W \subset F(R)$. This shows that if $J(R) = \mathbb{C}_\infty$, then $D \neq \emptyset$ so there is some z whose forward orbit is dense in $\mathbb{C}_\infty$.

For the converse, suppose that $J(R)$ is not the entire sphere, so $F(R) \neq \emptyset$, and also that there exists some z whose forward orbit is dense in the sphere. Note that $z \notin J(R)$, for if it were, then $O^+(z) \subset J(R)$ and since $J(R)$ is closed we would have $J(R) = \mathbb{C}_\infty$. It follows that z lies in some component Ω of the Fatou set $F(R)$, so we can consider the components

$$\Omega,\ R(\Omega),\ R^2(\Omega), \dots$$

of $F(R)$. As $O^+(z)$ is dense in $\mathbb{C}_\infty$, there must be some N such that $R^N(\Omega)$ meets (and so is) Ω, and we can assume that N is the minimal such integer. Then the decomposition into components is

$$F(R) = \Omega \cup R(\Omega) \cup R^2(\Omega) \cup \cdots \cup R^{N-1}(\Omega)$$

It follows that Ω is completely invariant under R^N, and that the set $\{R^{kN}(z) :\ k \geq 1\}$ is dense in Ω.

To show that this situation is not possible, we recall the following classification of forward invariant components of a Fatou set, the proof of which can be found in [Bea91, Chapter 7].

A forward invariant component Ω of the Fatou set $F(R)$ is of one of the following types:

(a) an attracting component, if it contains an attracting fixed point w of R,

(b) a super-attracting component, if it contains a super-attracting fixed point w of R,

(c) a parabolic component, if there is a rationally indifferent fixed point w of R on the boundary of Ω, and $R^n \to w$ on Ω,

(d) a Siegel disc, if $R : \Omega \to \Omega$ is analytically conjugate to an Euclidean rotation of the unit disc onto itself,

(e) a Herman ring, if $R : \Omega \to \Omega$ is analytically conjugate to an Euclidean rotation of some annulus onto itself.

Now, since $\{R^{kN}(z) :\ k \geq 1\}$ is dense in Ω, clearly R cannot have an attracting point, nor converge to a point in Ω, so cases (a), (b) and (c) are not possible. If Φ is a rotation as in cases (d) or (e) and $R = h^{-1}\Phi h$, then $R^{kN} = h^{-1}\Phi^{kN} h$ and so

$$|hR^{kN}(z)| = |\Phi^{kN} h(z)| = |h(z)| = r$$

is constant. Then, for every k, $R^{kN}(z)$ belongs to the preimage of a circle $h^{-1}(C_r)$, which cannot be dense. Therefore, none of the five cases are possible, which is a contradiction and concludes our proof.

□

We end this paper with an example of a rational map whose Fatou set is infinite.

Consider the polynomial $P : \mathbb{C}_\infty \to \mathbb{C}_\infty,\ P(z) = z^2 - 1$. Then $P^2(z) = z^2(z^2 - 2)$, so one can see that $0,\ -1$ and ∞ are all attracting fixed points of P^2. This clearly implies that $F(P)$ has at least three components, but it is possible to show (see [Bea91, Theorem 5.6.2]) that the Fatou set of a rational map can have either 0, 1, 2, or infinitely many components, so in this case $F(P)$ has infinitely many (non-empty) components, making $F(P)$ an infinite set.

References

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