Detecting algebraic categories with an internal Hom

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Introduction

Ever since Lawvere's thesis [Law63] was published in 1963, there has been an interest in incorporating techniques from category theory into the study of universal algebra. One of the main motivations for this is that if, for example, a group is defined without making use of elements, then the concept can be applied to categories other than the category of sets; this allows different notions such as topological groups, Lie groups and H-spaces to be studied from a unified perspective that deals with them simply as groups in different target categories.

However, the language of categories, functors and natural transformations was not always popular or widespread, and for this reason Freyd decides to study algebraic theories from a more equational approach. For him, an algebraic theory will be a family of abstract operators, each of them with an associated non-negative arity, together with a family of equations relating the operators to each other.

On a seemingly unrelated note, in his 1963 dissertation [Lin63], Linton defined an autonomous category as one possessing a forgetful functor

$$U : \mathcal{C} \to \mathbf{Set}$$

and an internal hom,

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$$

such that

$$U \,\mathrm{Hom}(X, Y) \simeq \mathcal{C}(X, Y)$$

In other words, this means we can look at every hom-set as an object of the category $\mathcal{C}$ itself.

In this talk, we will define algebraic theories from an equational point of view, and after introducing some basic concepts, show the following criterion for determining whether an algebraic category (i.e. the category of models for a given algebraic theory) is autonomous:

Theorem 1. Let $\mathbb{T}$ be an algebraic theory. Then its category of models $\mathbf{Set}^{\mathbb{T}}$ is an autonomous category if and only if $\mathbb{T}$ is a commutative theory.

We claim no original content; the ideas introduced and the main result can be found in [Fre66].

Algebraic theories and their algebras

An *algebraic theory* $\mathbb{T}$ is a family of operator symbols $\{f_i\}$, together with non-negative integers $\{v_i\}$ and equations relating the f_i's, each of which looks in the equations as if it were a function on v_i arguments.

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For a locally small category $\mathcal{A}$ with finite products, we say that $A \in \mathcal{A}$ is a $\mathbb{T}$-*algebra* if each operator $f_i \in \mathbb{T}$ has an interpretation $\bar{f}_i : \prod_{v_i} A \to A$ such that all equations are true when interpreted as maps in $\mathcal{A}$ (if so, these interpretations form the $\mathbb{T}$-algebra structure of A). Interpretations of these n-ary expressions follow two very reasonable rules:

1. Interpretation of the ith-projection is the ith-projection.
2. Interpretation of a composition is the composition of the interpretations.

We will denote by $\mathcal{A}^{\mathbb{T}}$ the category of $\mathbb{T}$-algebras and homomorphisms in $\mathcal{A}$. As usual, we obtain a notion of $\mathbb{T}$-co-algebra by dualizing the definition of $\mathbb{T}$-algebra.

Before moving forward I want to remark that these notions are equivalent to the ones introduced by Lawvere and mentioned previously by Evangelia.

Recall that a Lawvere theory is a small category L with finite products, together with a strict finite-product preserving identity-on-objects functor $I : \aleph_0^{\mathrm{op}} \to L$, and a model for the theory is a finite-product preserving functor $M : L \to C$. In this case, the operators f_i determine the non-basic operations in $L(v_i, 1)$ (i.e. maps not coming from the product structure), with the equations corresponding to composition in L. As to the algebras/models, an algebra $A \in \mathcal{A}$ with interpretations $\bar{f}_i$ corresponds to the finite-product preserving functor $M : L \to \mathcal{A}$ such that $M(1) = A$ and $M(f_i) = \bar{f}_i$.

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Back to our main result. Saying that the category $\mathbf{Set}^{\mathbb{T}}$ is autonomous is equivalent to saying we can see

$$\mathbf{Set}^{\mathbb{T}}(-,-) : (\mathbf{Set}^{\mathbb{T}})^{\mathrm{op}} \times \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$$

as algebra-valued. This, in turn, is the same as saying that the functor

$$\mathbf{Set}^{\mathbb{T}}(-, B) : (\mathbf{Set}^{\mathbb{T}})^{\mathrm{op}} \to \mathbf{Set}$$

is algebra valued for every algebra $B \in \mathbf{Set}^{\mathbb{T}}$ and that

$$\mathbf{Set}^{\mathbb{T}}(A, -) : (\mathbf{Set}^{\mathbb{T}}) \to \mathbf{Set}$$

is algebra valued for every coalgebra $A \in (\mathbf{Set}^{\mathbb{T}})^{\mathrm{op}}$.

Algebra-valued representable functors

Let's take a look at the contravariant hom-set functors $\mathcal{A}(-, B) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ for a fixed $B \in \mathcal{A}$. If we set $\mathcal{A} = \mathbf{Set}$ and let $G \in \mathbf{Grp}$, we know that for any set X, the set of functions (X, B) admits a canonical group structure, induced by the product in G. Actually, there's nothing particular about groups in this fact, and so we have:

$\mathcal{A}(-, B) : \mathcal{A}^{op} \to \mathbf{Set}$ *(resp.* $\mathcal{A}(A, -)$*) is algebra valued if* B *is a* $\mathbb{T}$*-algebra (resp.* A *is a* $\mathbb{T}$*-coalgebra).*

To illustrate this, we define the $\mathbb{T}$-algebra structure of $\mathcal{A}(A,B)$: given an operator f_i in $\mathbb{T}$, we can take its interpretation to be the composition

$$\prod_{v_i} \mathcal{A}(A,B) \cong \mathcal{A}(A, \prod_{v_i} B) \xrightarrow{\bar{f}_i \circ -} \mathcal{A}(A,B)$$

where $\bar{f}_i : \prod_{v_i} B \to B$ is the interpretation determined by the algebra structure of B.

All equations of the theory will be preserved, since $\mathcal{A}(A,-)$ is a product-preserving functor.

Interestingly, the converse of the previous statement is also true! Algebra valued representable functors yield canonical (co)structures on their representatives:

To show this less evident converse, fix an operator f_i in $\mathbb{T}$; using the $\mathbb{T}$-algebra structure of $\mathcal{A}(A,B)$ we have, for every $A \in \mathcal{A}$, a map

$$\mathcal{A}(A, \prod B) \simeq \prod \mathcal{A}(A,B) \xrightarrow{\tilde{f}_i} \mathcal{A}(A,B)$$

It's possible to show that these maps can be put together to form a natural transformation

$$\mathcal{A}(-, \prod B) \Rightarrow \mathcal{A}(-,B)$$

Indeed, this amounts to showing that, for any $g : A \to A'$, the following commutes

$$\begin{array}{ccc} \prod \mathcal{A}(A',B) & \xrightarrow{\tilde{f}_i'} & \mathcal{A}(A',B) \\ \downarrow{\scriptstyle \prod -\circ g} & & \downarrow{\scriptstyle -\circ g} \\ \prod \mathcal{A}(A,B) & \xrightarrow{\tilde{f}_i} & \mathcal{A}(A,B) \end{array}$$

But $\mathcal{A}(-,B)$ is an algebra valued functor, so $\mathcal{A}(A',B) \to \mathcal{A}(A,B)$ must be a homomorphism, which by definition translates into that diagram being commutative.

Then, by Yoneda, that natural transformation must come from a map

$$\prod B \to B$$

which we take as the interpretation of f_i.

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Summing up, the fact that $\mathbb{T}$ is a theory such that $\mathbf{Set}^{\mathbb{T}}$ is autonomous means that each $B \in \mathbf{Set}^{\mathbb{T}}$ has a canonical $\mathbb{T}$-structure in $\mathbf{Set}^{\mathbb{T}}$ (and similarly for coalgebras).

This is equivalent to saying that every $\mathbb{T}$-algebra $B \in \mathbf{Set}^{\mathbb{T}}$ is actually a member of $(\mathbf{Set}^{\mathbb{T}})^{\mathbb{T}}$. But then every $\mathbb{T}$ operator must have an interpretation as a map in $\mathbf{Set}^{\mathbb{T}}$, and these are $\mathbb{T}$-algebra homomorphisms, so the theory $\mathbb{T}$ must be such that every $\mathbb{T}$ operator is a $\mathbb{T}$ homomorphism.

Explicitly: every pair of operators $f_i, g_j \in \mathbb{T}$ must satisfy

$$\begin{aligned} f_i\left(g_j(x_{11},\ldots,x_{1v_j}),\ldots,\ g_j(x_{v_i1},\ldots,x_{v_iv_j})\right) \\ =g_j\left(f_i(x_{11},\ldots,x_{v_i1}),\ldots,f_i(x_{1v_j},\ldots,x_{v_iv_j})\right), \end{aligned} \tag{1}$$

in other words, all operators in $\mathbb{T}$ commute, in the sense of (1).

Applying our theorem

As examples, we can easily see that

(a) R-**Mod** is not autonomous unless R is a commutative ring, since the operations "multiply by r" and "multiply by s" must commute for every $r, s \in R$.

(b) **Grp** is not autonomous, since the binary operation won't commute with itself.

(c) We can get around the problem in (b) by considering **Ab**, which is commutative and therefore autonomous.

(d) **Ring** is not autonomous since the product operation doesn't commute with itself.

(e) Interestingly, if we proceed as in (c) and consider **CommRing**, it doesn't solve the problem: the two constants would commute with each other, so

$$0 = 0.1 = 1.0 = 1$$

which would imply there is only one constant.

We're not saying these results are a novelty; for example, it's well known that **Grp** is not enriched over itself, and that R-**Mod** will only be so when R is commutative. However, even though it's easy to see that the naive approaches do not work, showing "by hand" that no approach will ever work requires some ingenuity, and the theorem we presented offers a systematic way to deal with this question.

References

[Fre66] P. Freyd. Algebra valued functors in general and tensor products in particular. XIV, 1966.

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