∞-cosmoi and their homotopy 2-categories

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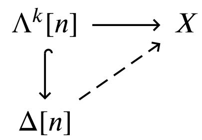
Notes for the first talk at the MIT Talbot Workshop "Model-independent theory of ∞-categories" .

Quasi-categories

Definition 1 (quasi-categories)**.** A *quasi-category* is a simplicial set X such that every inner horn has a filler. Explicitly, this means that for every $0 < k < n$ and horn $\Lambda^k[n] \to X$ there exists an extension along the inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$

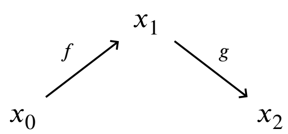


By the Yoneda lemma, the map $\Delta[n] \to X$ identifies an n-simplex in X whose faces agree with those specified by the horn.

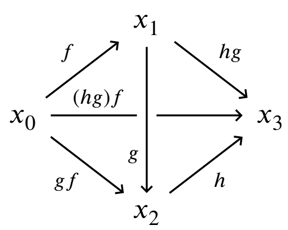
One of the most important examples come from categories themselves:

Example 2. For any category C, its nerve NC[1] is a quasicategory.

Note that a horn $\Lambda^1[2] \to NC_2$ can be represented by



where f and g are morphisms in C, and so asking for this horn to have a filler is the same as asking for the existence of a 1-cell in NC that acts as a composite of f and g; this cell will of course be gf. For an example of a higher dimension, a horn $\Lambda^1[3] \to NC_3$ can be represented by



and so asking for this horn to have a filler is the same as asking for $(hg)f = h(gf)$; similarly, all other fillers for horns of dimension $n \geq 3$ are given by the associativity in C of compositions of n maps.

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[1] Sorry Emily, I'm not ready to get rid of the N yet

Remark 3. We can see that, in this case, all fillers will be unique. In fact, the converse is also true: any quasi-category with unique fillers comes from the nerve of a category.

Remark 4. It's not hard to show that the nerve functor

$$\mathbf{Cat} \xrightarrow{N} \mathbf{sSet}$$

is full and faithful; this means we can study categories by looking at them as quasi-categories, and so quasi-categories are a generalization of categories via the nerve functor.

Another important example of quasi-categories comes from topological spaces:

Example 5. If X is a topological space, recall that its singular complex is

$$\mathrm{Sing}_n X = \mathbf{Top}(\Delta_n, X)$$

where Δ_n denotes the geometric n-simplex (i.e. the convex hull of the canonical basis in $\mathbb{R}^{n+1}$).

We know that the functor

$$\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$$

has a left adjoint given by geometric realization

$$|-| : \mathbf{sSet} \to \mathbf{Top}$$

which we can use to easily show that Sing X is a quasi-category: a diagram

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathrm{Sing}\, X \\ \downarrow & \nearrow & \\ \Delta[n] & & \end{array}$$

in **sSet** transposes to a diagram

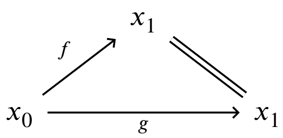
$$\begin{array}{ccc} |\Lambda^k[n]| & \longrightarrow & X \\ \downarrow & \nearrow & \\ |\Delta[n]| & & \end{array}$$

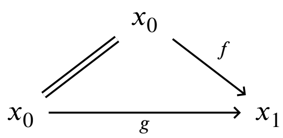
in **Top**. Then, since a topological (n, k)-horn is a deformation retract of the geometric n-simplex $\Delta_n = |\Delta[n]|$, this last lift always exists.

Remark 6. In this case, we don't necessarily have unique fillers, but using the same argument we find fillers for *all* horns, not just inner ones. Such a simplicial set is called *Kan complex*, and they play an important role in studying the homotopy theory of **sSet**.

As we now see, quasi-categories already come with a natural notion of homotopy.

Definition 7 (homotopy relation on 1-simplices)**.** Given a parallel pair of 1-simplices f and g in a quasi-category X, we say that there is a *homotopy* from f to g if there exists a 2-simplex of either of the following forms:

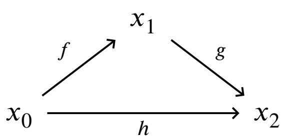




It's not hard (but probably enlightening if one is not used to working with quasi-categories) to show that the relation witnessed by any of the types of 2-simplex on display in this definition is an equivalence relation, and these equivalence relations coincide. We use this to define the following:

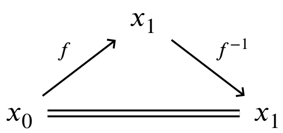
Definition 8 (homotopy category)**.** If X is a quasi-category, its *homotopy category* hX has

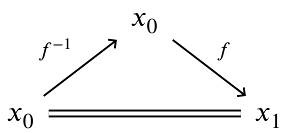
- as objects, the set X_0,
- as morphisms, the set of homotopy classes of 1-simplices in X_1,
- a composition relation $h = g \circ f$ if and only if, for any choices of 1-simplices representing these maps, there exists a 2-simplex



Remark 9. A very careful person might have realized it's not obvious that this definition actually works; we're taking a quotient of X_1 by something that at plain sight may not be an equivalence relation. However, the nice properties of quasi-categories make it so, and allow us to define composition very in a very descriptive way, without the need of phrases such as "the equivalence generated by this relation" or such.

Definition 10 (isomorphisms in a quasi-category)**.** A 1-simplex in a quasi-category is an *isomorphism* if it represents an isomorphism in the homotopy category. Explicitly, this means that $f : a \to b$ is an isomorphism if and only if there exist a 1-simplex $f^{-1} : b \to a$ together with 2-simplices





Remark 11. Just like an arrow in a quasi-category A is represented by a simplicial map $\mathbb{2} \to A$ from the nerve of the free-living arrow, an isomorphism in A is represented by a simplicial map $\mathbb{I} \to A$ from the nerve of the free-living isomorphism.

We now define some important classes of maps.

Definition 12 (isofibrations)**.** A simplicial map $f : X \to Y$ is an *isofibration* if it lifts against the inner horn inclusions, and against the inclusion of either vertex into the free standing isomorphism $\mathbb{I}$

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} 1 & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \mathbb{I} & \longrightarrow & Y \end{array}$$

Notation: $X \twoheadrightarrow Y$.

Remark 13. Note that X is a quasi-category if and only if the map $X \to *$ is an isofibration. This gives a characterization of quasi-categories by a right lifting property, which may come in handy later.

Definition 14 (equivalences between quasi-categories)**.** A map $f : A \to B$ between quasi-categories is an *equivalence* if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval; that is, if there exist maps $g : B \to A$, α and β such that

$$\begin{array}{ccc} & & A \\ & \nearrow\!\!= & \big\uparrow ev_0 \\ A & \xrightarrow{\alpha} & A^{\mathbb{I}} \\ & \searrow gf & \big\downarrow ev_1 \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} & & B \\ & \nearrow fg & \big\uparrow ev_0 \\ B & \xrightarrow{\beta} & B^{\mathbb{I}} \\ & \searrow\!\!= & \big\downarrow ev_1 \\ & & B \end{array}$$

Notation: $A \xrightarrow{\sim} B$.

Definition 15 (trivial fibrations)**.** A simplicial map $f : X \to Y$ is a *trivial fibration* if it lifts against all boundary inclusions

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

Notation: $A \overset{\sim}{\twoheadrightarrow} B$

Remark 16. If this last nomenclature reminds you of model categories, that is exactly right: there exists a model structure on simplicial sets (the Joyal model structure) whose fibrant objects are the quasi-categories. The fibrations, weak equivalences, and trivial fibrations between fibrant objects are precisely the classes of isofibrations, equivalences, and trivial fibrations, respectively. If you don't know what any of this means, that's totally fine; you can simply remember the following.

Proposition 17. *As suggested by the notation,*

$$\textit{Trivial fibration} = \textit{isofibration} + \textit{equivalence}$$

∞-cosmoi

Like Dom mentioned today, we are not going to define exactly what an ∞-category should be; rather, we will axiomatize the "universe" in which ∞-categories live, and give an idea of how they interact with each other via some special classes of maps, and as usual in category theory, these probings should give us some idea of what these things look like.

Definition 18 (∞-cosmoi)**.** An ∞-*cosmos* $\mathcal{K}$ is a category enriched over quasi-categories, meaning that it has

- objects A, B, that we call ∞-categories, and
- its morphisms define the vertices of functor-spaces $\mathrm{Fun}(A, B)$, which are quasi-categories,

that is also equipped with a specified class of maps that we call isofibrations and denote by "$\twoheadrightarrow$".

From these classes, we define a map $f : A \to B$ to be an equivalence if and only the induced map $f_* : \mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ on functor-spaces is an equivalence of quasi-categories for all $X \in \mathcal{K}$, and we define f to be a trivial fibration just when f is both an isofibration and an equivalence; these classes are denoted by $\xrightarrow{\sim}$ and $\overset{\sim}{\twoheadrightarrow}$ respectively.

These classes must satisfy the following three axioms:

(i) (completeness) $\mathcal{K}$ has a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with all simplicial sets, each of these limit notions satisfying a universal property that is enriched over simplicial sets.

(ii) (isofibrations) The class of isofibrations contains all isomorphisms and any map whose codomain is the terminal object; is closed under composition, product, pullback, forming inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and has the property that if $f : A \to B$ is an isofibration and X is any object then $\mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ is an isofibration of quasi-categories.

(iii) (cofibrancy)[2] Every trivial fibration admits a section

$$\begin{array}{ccc} & & E \\ & \nearrow & \big\downarrow\wr \\ B & = & B \end{array}$$

[2]Actually, one of the first exercises today was to show that this can be deduced from axioms (i) and (ii)

I like to pack some of this information as "isofibrations behave like fibrations, and everything is fibrant and cofibrant".

At this point you're probably wishing for some examples, but bear with me; Joseph will take care of that in the next talk.

As a consequence of the axioms in definition 18, we see that the class of trivial fibrations enjoys the same stability properties as the class of fibrations.

Lemma 19. *If you replace "isofibrations" by "trivial fibrations" in axiom (ii), everything is still true.*

Another thing that works just like in quasi-categories is that we can characterize equivalences as "homotopy equivalences".

Lemma 20 (equivalences are homotopy equivalences)**.** *A map $f : A \to B$ in an ∞-cosmos $\mathcal{K}$ is an equivalence if and only if it extends to the data of a "homotopy equivalence", that is, if there exist maps $g : B \to A, \alpha$ and β*[3] *such that*

$$\begin{array}{ccc} & & A \\ & \nearrow\!\!\!\!= & \uparrow{\scriptstyle ev_0} \\ A & \xrightarrow{\alpha} & A^{\mathbb{I}} \\ & \searrow{\scriptstyle gf} & \downarrow{\scriptstyle ev_1} \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} & & B \\ & \nearrow{\scriptstyle fg} & \uparrow{\scriptstyle ev_0} \\ B & \xrightarrow{\beta} & B^{\mathbb{I}} \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle ev_1} \\ & & B \end{array}$$

The homotopy 2-category

In future talks, a lot of the definitions and constructions will be given not in an ∞-cosmos, but in a more tractable 2-category that we now define.

Definition 21 (homotopy 2-category)**.** The *homotopy 2-category* of an ∞-cosmos $\mathcal{K}$ is the strict 2-category $h\mathcal{K}$ whose

- objects are the objects of $\mathcal{K}$, i.e. the ∞-categories
- 1-cells $f : A \to B$ are the 0-arrows in the simplicial set $\mathrm{Fun}(A, B)$, i.e. the ∞-functors
- 2-cells $A \underset{g}{\overset{f}{\Downarrow \alpha}} B$ are homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$, which we call ∞-natural transformations.

In other words, $h\mathcal{K}$ is the 2-category with the same objects as $\mathcal{K}$ and with hom-categories defined by

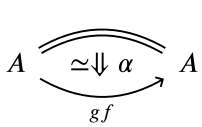
$$h\,\mathrm{Fun}(A, B) = h(\mathrm{Fun}(A, B))$$

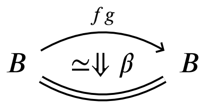
[3] Recall that in the definition for quasi-categories, $A^{\mathbb{I}}$ was an exponential; in general ∞-cosmos this notation means cotensor instead

Like any 2-category, $h\mathcal{K}$ comes equipped with a notion of equivalence.

Definition 22 (equivalence in a 2-category)**.** An *equivalence in a 2-category* is given by

- two objects A and B,
- two 1-cells $f : A \to B$ and $g : B \to A$,
- two invertible 2-cells





However, we also have a notion of equivalence in $h\mathcal{K}$ from the fact that $\mathcal{K}$ is an ∞-cosmos: that of 1-cells $f : A \xrightarrow{\sim} B$ inducing an equivalence of quasi-categories $f_* : \mathrm{Fun}(X, A) \xrightarrow{\sim} \mathrm{Fun}(X, B)$ for any $X \in \mathcal{K}$.

One of the reasons why the approach that we will be using (of working in $h\mathcal{K}$ instead of $\mathcal{K}$) actually works is that these two notions of equivalence coincide. All the constructions that we will introduce, and the universal properties that we will define, in the context of $h\mathcal{K}$, will of course be invariant under 2-categorical equivalence, and since these agree with the equivalences we have in the ∞-cosmos $\mathcal{K}$, they will be homotopically correct.

In simpler words, the things that our constructions in $h\mathcal{K}$ won't be able to tell apart are precisely the things that we do not wish to distinguish in $\mathcal{K}$ to begin with.

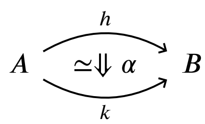
Theorem 23 (equivalences are equivalences)**.** *A functor $f : A \to B$ between ∞-categories defines an equivalence in the ∞-cosmos $\mathcal{K}$ if and only if it defines an equivalence in the 2-category $h\mathcal{K}$.*

Proof. Given an equivalence $f : A \xrightarrow{\sim} B$ in $\mathcal{K}$, lemma 20 stated that this is equivalent to the existence of an inverse equivalence $g : B \xrightarrow{\sim} A$ and homotopies $\alpha : A \to A^{\mathbb{I}}$ and $\beta : B \to B^{\mathbb{I}}$ in $\mathcal{K}$. But recall that cotensors are defined by the universal property

$$\mathrm{Fun}(-, A^{\mathbb{I}}) \simeq \mathrm{Fun}(-, A)^{\mathbb{I}}$$

so the 0-cells α and β give 0-cells $\hat{\alpha} : \mathbb{I} \times \Delta[0] \simeq \mathbb{I} \to \mathrm{Fun}(A, A)$ and $\hat{\beta} : \mathbb{I} \to \mathrm{Fun}(B, B)$ which specify an equivalence in $h\mathcal{K}$.

For the converse[4], we claim that if two parallel 1-cells $h, k : A \to B$ in the homotopy 2-category are connected by an invertible 2-cell



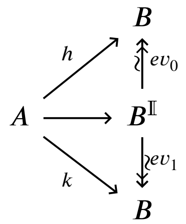
[4] I'm leaving this here because it's cool and someone might want to read it, but I didn't get to show this.

then h is an equivalence in the ∞-cosmos $\mathcal{K}$ if and only if k is. Using this, we see that the existence of invertible 2-cells

implies that gf and fg are equivalences, and then the fact that equivalences satisfy the 2-out-of-6 property means that f and g must be equivalences too.

So, finally, why is the claim true? First, note that the evaluation maps $ev_0, ev_1 : B^{\mathbb{I}} \to B$ present in a homotopy equivalence are always trivial fibrations, which can be easily deduced by applying the Leibniz cotensor property of lemma 19 to the isofibration $B \twoheadrightarrow *$ and the simplicial inclusion $\mathbb{1} \hookrightarrow \mathbb{I}$.

Then, the invertible 2-cell from h to k can be represented by a map $\mathbb{I} \to \mathrm{Fun}(A, B)$, which in turn (by the universal property of the cotensor) corresponds to a map $A \to B^{\mathbb{I}}$ in $\mathcal{K}$ that fits in the following diagram



Since equivalences satisfy the 2-out-of-3 property, we deduce our claim. □

Appendix: unpacking the limit conditions

Definition 24 (cotensor)**.** Let $\mathcal{A}$ be a simplicial category. The *cotensor* of an object $A \in \mathcal{A}$ by a simplicial set U is characterized by an isomorphism of simplicial sets

$$\mathcal{A}(X, A^U) \simeq \mathcal{A}(X, A)^U$$

natural in $X \in \mathcal{A}$. Assuming such objects exist, the simplicial cotensor defines a bifunctor

$$\mathbf{sSet}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{A}$$
$$(U, A) \mapsto A^U$$

in a unique way making the isomorphism natural in U and A as well.

Example 25. Cotensors of simplicial sets are exponentials (which supports the abuse of notation).

Definition 26 (enriched limits)**.** Enriched limits, when they exist, correspond to the usual limits in the underlying category, but the usual universal property is strengthened. Applying the covariant representable functor

$$\mathcal{A}(X, -) : \mathcal{A}_0 \to \mathbf{sSet}$$

to a limit cone $(\lim_{j\in J} A_j \to A_j)_{j\in J}$ in $\mathcal{A}_0$, there is natural comparison map

$$\mathcal{A}(X, \lim_{j\in J} A_j) \to \lim_{j\in J} \mathcal{A}(X, A_j)$$

and we say that $\lim_{j\in J} A_j$ defines a *simplicially enriched limit* when this is an isomorphism (of simplicial sets) for all $X \in \mathcal{A}$.

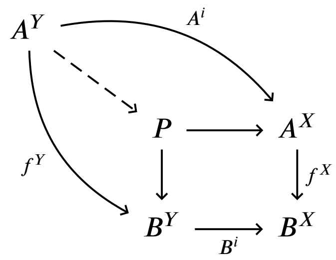
Definition 27 (towers)**.** A *tower* is a diagram of the shape of the poset of natural numbers

$$\cdots \to X_2 \to X_1 \to X_0$$

A limit over this type of diagram is sometimes called an inverse limit, or a directed limit, or sequential limit.

A *tower of isofibrations* is a special instance of a tower where all the maps involved are isofibrations.

Definition 28 (Leibniz cotensors)**.** Given an ∞-functor $f : A \twoheadrightarrow B$ and a simplicial map $i : X \to Y$, the *Leibniz cotensor* map is the induced map to the pullback



In the case where $i : X \hookrightarrow Y$ is an inclusion of simplicial sets, this pullback exists; we show this by proving that the map $f^X : A^X \to B^X$ is an isofibration, and then appealing to the completeness axiom 18(i).

For this, consider the special case $i : \emptyset \to X$; the diagram reduces to

$$\begin{array}{ccc} & & A^{\emptyset} \\ & & \downarrow{\scriptstyle f^{\emptyset}} \\ B^X & \xrightarrow[B^i]{} & B^{\emptyset} \end{array} \quad = \quad \begin{array}{ccc} & & * \\ & & \| \\ B^X & \longrightarrow & * \end{array}$$

and the pullback of the latter always exists, since it is given by

$$\begin{array}{ccc} B^X & \longrightarrow & * \\ \| & & \| \\ B^X & \longrightarrow & * \end{array}$$

Now, since the pullback exists, this axiom ensures that the induced map $A^X \to B^X$ is an isofibration, which concludes the explanation.

References

[RV1] Riehl, Emily and Verity, Dominic. Fibrations and Yoneda's lemma in an ∞-cosmos, (2015) *to appear in J. Pure Appl. Algebra*.

[RV2] Riehl, Emily and Verity, Dominic. ∞-category theory from scratch, (2015), `http://www.math.jhu.edu/~eriehl/scratch.pdf`

[RV3] Riehl, Emily and Verity, Dominic. Elements of ∞-category theory, *preliminary draft version available from* `http://www.math.jhu.edu/~eriehl/elements.pdf`