Math 2220 Prelim 2 Solutions Fall 2010

Problem 1: Calculate the integral of the function \( f(x, y) = \frac{1}{x} \) over the region bounded by the parabolas \( y = x^2 \) and \( y = 2x - x^2 \).

Solution: We can calculate that the two parabolas intersect at \( x = 0 \) and \( x = 1 \), and in this region \( 2x - x^2 \geq x^2 \). So we wish to integrate:

\[
\int_0^1 \int_{x^2}^{2x-x^2} \frac{1}{x} \, dy \, dx
\]

\[
= \int_0^1 \frac{1}{x} \cdot ((2x - x^2) - x^2) \, dx
\]

\[
= \int_0^1 2 - 2x \, dx = (2x - x^2)|_0^1 = 1 - 0 = 1
\]

The integral is improper since \( 1/x \) is not defined at the point \((0,0)\) which is on the boundary of the domain, however the above computation shows that the integral converges.

Problem 2: Evaluate the iterated integral

\[
\int_{-1}^1 \int_0^{1-|y|} \frac{y^2 e^x}{(1-x)^3} \, dx \, dy.
\]

Solution: Since the integral of \( \frac{e^x}{(1-x)^3} \, dx \) cannot be written in terms of elementary functions, we must change the order of integration. The domain of the integral looks like a triangle:
So we can change the order of integration:
\[
\int_{-1}^{1} \int_{0}^{1-|y|} \frac{y^2e^x}{(1-x)^3} \, dx \, dy
\]
\[
= \int_{0}^{1} \int_{y}^{1-x} \frac{y^2e^x}{(1-x)^3} \, dy \, dx = \int_{0}^{1} \left[ \frac{e^x}{(1-x)^3} \cdot \frac{y^3}{3} \right]_{x-1}^{1-x} \, dx
\]
\[
= \int_{0}^{1} \frac{e^x}{(1-x)^3} \cdot \frac{(x-1)^3 - (1-x)^3}{3} \, dx
\]
\[
= \int_{0}^{1} 2 \frac{e^x}{3} \, dx = \frac{2}{3} e^1 |_{0}^{1} = \frac{2}{3} (e - 1).
\]

**Problem 3:** Find the volume of the solid consisting of all points which satisfy the inequalities
\[
0 \leq x \leq 1 \quad 0 \leq y \leq 1 \quad 0 \leq z \leq x(y - x).
\]

**Solution:** Notice that the last inequality automatically that \(x(y - x) \geq 0\), thus we need to strengthen the second inequality to \(x \leq y \leq 1\).

Which makes our volume integral:
\[
\int_{0}^{1} \int_{x}^{1} \frac{1}{2} \, dz \, dy \, dx = \int_{0}^{1} \int_{x}^{1} x(y - x) \, dy \, dx
\]
\[
= \int_{0}^{1} \left[ \frac{xy^2}{2} - x^2y \right]_{x}^{1} \, dx = \int_{0}^{1} \left( \frac{x}{2} - x^2 \right) - \left( \frac{x^3}{2} - x^3 \right) \, dx
\]
\[
= \int_{0}^{1} \frac{x^3}{2} - x^2 + \frac{x}{2} \, dx = \frac{x^4}{8} - \frac{x^3}{3} + \frac{x^2}{4} \bigg|_{0}^{1} = \frac{1}{8} - \frac{1}{3} + \frac{1}{4} = \frac{1}{24}
\]

If one untgrates between 0 and 1 for \(y\) the resulting integral is equal to \(-1/12\) which is a sign that something is wrong, since the volume must be non-negative.

We can see that \(y \geq x\) by graphing the region:
**Problem 4:** Use cylindrical coordinates to find the center of mass (assuming uniform density) of the solid bounded by the paraboloids

\[ z = x^2 + y^2 \quad \text{and} \quad z = 3 - 2x^2 - 2y^2. \]

**Solution:** We will assume uniform density \( \delta = 1 \).

These equalities can be rewritten as \( z = r^2 \) and \( z = 3 - 2r^2 \) respectively, giving us the graph of the constraints and a cross section:

Notice that the region is rotationally symmetric, so its center of mass will be on the \( z \)-axis. It suffices then to only calculate the \( z \)-coordinate of the center of mass, which is given by the formula:

\[
\frac{M_z}{M} = \frac{\iiint_R z \, dV}{\iiint_R \, dV}
\]

\[
M_z = \int_0^{2\pi} \int_0^1 \int_{r^2}^{3-2r^2} rz \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \frac{r}{2} \left( z^{3-2r^2} \right) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \frac{1}{2} \left[ \frac{9}{2} r^2 - 3r^4 + \frac{1}{2} r^6 \right]_0^1 \, d\theta = \int_0^{2\pi} \frac{9}{4} - \frac{3}{2} + \frac{1}{4} \, d\theta = 2\pi \cdot 1 = 2\pi
\]

\[
M = \int_0^{2\pi} \int_0^1 \int_{r^2}^{3-2r^2} \, dr \, dz \, d\theta = \int_0^{2\pi} \int_0^1 (3 - 2r^2 - r^2) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 3r - 3r^3 \, dr \, d\theta = \int_0^{2\pi} \frac{3}{2} \left[ \frac{1}{2} \right] - \frac{3}{4} \left[ \frac{1}{4} \right] \, d\theta = 2\pi \cdot \frac{3}{4}
\]

So,

\[
\frac{M_z}{M} = \frac{2\pi}{2\pi \cdot \frac{3}{4}} = \frac{4}{3}
\]
**Problem 5:** Find the volume of the part of the ball \( x^2 + y^2 + z^2 \leq 4 \) which lies in the first octant and satisfies the inequality \( z^2 \leq x^2 + y^2 \).

**Solution:** Here is a picture of the region

We can express the region in spherical coordinates:

\[
0 \leq \rho \leq 2 \quad 0 \leq \theta \leq \frac{\pi}{2} \quad \text{and} \quad \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2},
\]

so we can express the volume of the region as:

\[
\int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{\pi}{2} \int_{0}^{\frac{\pi}{4}} \sin(\phi) \, d\phi \cdot \int_{0}^{2} \rho^2 \, d\rho
\]

\[
= \frac{\pi}{2} \left( -\cos(\phi) \right) \bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left( \frac{\rho^3}{3} \right) \bigg|_{0}^{2} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{8}{\sqrt{2}} \cdot \frac{2\sqrt{2}\pi}{3} = \frac{2\sqrt{2}\pi}{3}
\]

**Problem 6:** Let \( D \) denote the cone bounded by the plane \( z = -1 \) and the conic surface \( z^2 = x^2 + y^2 \). Set up, but DO NOT evaluate, iterated integrals which evaluate \( \iiint_{D} f \, dV \)

- in Cartesian coordinates in the orders \( dx \, dy \, dz \) and \( dz \, dy \, dx \);
- in cylindrical coordinates in the orders \( d\theta \, dr \, dz \) and \( dz \, dr \, d\theta \);
- Evaluate ONLY ONE of the iterated integrals above when \( f \) is the function \( f(x, y, z) = (x^2 + y^2)^{-\frac{1}{2}} \).

**Hint:** In part c) pick the integral which seems to be the easiest to compute. If you have done everything correctly you will not need any trigonometric substitutions nor integration by parts.

**Solution:** The constraints define a region in the lower half of the double-cone:
a)
\[ \int_{-1}^{0} \int_{z}^{\sqrt{z^2-y^2}} f \, dx \, dy \, dz \]
\[ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-1}^{1} f \, dz \, dy \, dx \]

b)
\[ \int_{-1}^{0} \int_{0}^{2\pi} fr \, d\theta \, dr \, dz \]
\[ \int_{0}^{2\pi} \int_{0}^{1} \int_{-r}^{r} fr \, dz \, dr \, d\theta \]

c) We will do the last integral. Note that \( f = \frac{1}{r} \).
\[ \int_{0}^{2\pi} \int_{0}^{1} \int_{-r}^{r} \frac{1}{r} r \, dz \, dr \, d\theta = 2\pi \int_{0}^{1} (1 - r) \, dr = 2\pi \left( r - \frac{r^2}{2} \right) \bigg|_{0}^{1} = \pi. \]

**Problem 7:** Evaluate the integral \( \iint_{D} xy \, dA \), where \( D \) is the domain bounded by the curves
\[ xy = 1 \quad y = x^2 \quad xy = 27 \quad y = 8x^2. \]

**Hint:** Find a change of variables which will simplify the domain \( D \).

**Solution:** Here is a picture of the domain bounded by these curves:
As we can tell by looking at the graph of this region, it would require 3 separate integrals to cover the region, so let us apply the following change of variables: $u = xy, v = yx^{-2}$. Now our region is the nice rectangle: $1 \leq u \leq 27$ and $1 \leq v \leq 8$. Let us calculate the Jacobian of this transformation:

$$\left| \frac{\partial (x, y)}{\partial (u, v)} \right|^{-1} = \left| \det \left( \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right) \right|^{-1} = \left| \det \left( \begin{array}{cc} y & x \\ -2yx^{-3} & x^{-2} \end{array} \right) \right|^{-1}$$

$$= |3yx^{-2}|^{-1} = \frac{x^2}{3y} = \frac{1}{3v}$$

Alternatively we can express $x$ and $y$ in terms of $u$ and $v$, i.e., $x = (u/v)^{1/3}$ and $y = (u^2v)^{1/3}$ and then compute the Jacobian directly.

After changing the variables the integral becomes

$$\int_1^{27} \int_1^{8} \frac{1}{3}uv^{-1} \, dv \, du = \frac{1}{3} \int_1^{27} u \, du \cdot \int_1^{8} v^{-1} \, dv$$

$$= \frac{1}{3} \left[ \frac{u^2}{2} \right]_1^{27} \quad \left[ \ln(v) \right]_1^8 = \frac{1}{3} \left[ \frac{127^2}{2} - 1 \right] \left( \ln(8) - 0 \right) = 364 \ln(2)$$