Introduction to Topology

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1 Topological spaces

A topology is a geometric structure defined on a set. Basically it is given by declaring which subsets are "open" sets. Thus the axioms are the abstraction of the properties that open sets have.

Definition 1.1 (§12 [Mun])**.** A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X such that

(T1) ϕ and X are in $\mathcal{T}$;

(T2) Any union of subsets in $\mathcal{T}$ is in $\mathcal{T}$;

(T3) The finite intersection of subsets in $\mathcal{T}$ is in $\mathcal{T}$.

A set X with a topology $\mathcal{T}$ is called a ***topological space***. An element of $\mathcal{T}$ is called an ***open set***.

Example 1.2. Example 1, 2, 3 on page 76,77 of [Mun]

Example 1.3. Let X be a set.

- (**Discrete topology**) The topology defined by $\mathcal{T} := \mathcal{P}(X)$ is called the ***discrete topology*** on X.
- (**Finite complement topology**) Define $\mathcal{T}$ to be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then $\mathcal{T}$ defines a topology on X, called ***finite complement topology*** of X.

1.1 Basis of a Topology

Once we define a structure on a set, often we try to understand what the minimum data you need to specify the structure. In many cases, this minimum data is called a basis and we say that the basis generate the structure. The notion of a basis of the structure will help us to describe examples more systematically.

Definition 1.4 (§13 [Mun])**.** Let X be a set. A ***basis of a topology*** on X is a collection $\mathcal{B}$ of subsets in X such that

(B1) For every $x \in X$, there is an element B in $\mathcal{B}$ such that $x \in U$.

(B2) If $x \in B_1 \cap B_2$ where B_1, B_2 are in $\mathcal{B}$, then there is B_3 in $\mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Lemma 1.5 (Generating of a topology)**.** *Let $\mathcal{B}$ be a basis of a topology on X. Define $\mathcal{T}_{\mathcal{B}}$ to be the collection of subsets $U \subset X$ satisfting*

(G1) For every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$.

Then $\mathcal{T}_{\mathcal{B}}$ defines a topology on X. Here we assume that $\varnothing$ trivially satisfies the condition, so that $\varnothing \in \mathcal{T}_{\mathcal{B}}$.

Proof. We need to check the three axioms:

(T1) $\varnothing \in \mathcal{T}_{\mathcal{B}}$ as we assumed. $X \in \mathcal{T}_{\mathcal{B}}$ by (B1).

(T2) Consider a collection of subsets $U_\alpha \in \mathcal{T}_{\mathcal{B}}, \alpha \in \mathsf{J}$. We need to show

$$U := \bigcup_{\alpha \in \mathsf{J}} U_\alpha \quad \in \mathcal{T}_{\mathcal{B}}.$$

By the definition of the union, for each $x \in U$, there is U_α such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}_{\mathcal{B}}$, there is $B \in \mathcal{B}$ such that $x \in B \subset U_\alpha$. Since $U_\alpha \subset U$, we found $B \in \mathcal{B}$ such that $x \in B \subset U$. Thus $U \in \mathcal{T}_{\mathcal{B}}$.

(T3) Consider a finite number of subsets $U_1, \cdots, U_n \in \mathcal{T}_{\mathcal{B}}$. We need to show that

$$U := \bigcap_{i=1}^{n} U_i \quad \in \mathcal{T}_{\mathcal{B}}.$$

- Let's just check for two subsets U_1, U_2 first. For each $x \in U_1 \cap U_2$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. This is because $U_1, U_2 \in \mathcal{T}_{\mathcal{B}}$ and $x \in U_1, x \in U_2$. By (B2), there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Now we found $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset U$.
- We can generalize the above proof to n subsets, but let's use ***induction*** to prove it. This is going to be the induction on the number of subsets.
  * When $n = 1$, the claim is trivial.
  * Suppose that the claim is true when we have $n-1$ subsets, i.e. $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{T}_{\mathcal{B}}$. Since

$$U = U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n$$

and regarding $U' := U_1 \cap \cdots \cap U_{n-1}$, we have two subsets case $U = U' \cap U_n$. By the first arguments, $U \in \mathcal{T}_{\mathcal{B}}$.

Definition 1.6. $\mathcal{T}_{\mathcal{B}}$ is called the ***topology generated by a basis*** $\mathcal{B}$. On the other hand, if $(X, \mathcal{T})$ is a topological space and $\mathcal{B}$ is a basis of a topology such that $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$, then we say $\mathcal{B}$ is a basis of $\mathcal{T}$. Note that $\mathcal{T}$ itself is a basis of the topology $\mathcal{T}$. So there is always a basis for a given topology.

Example 1.7.

- (**Standard Topology of** $\mathbb{R}$) Let $\mathbb{R}$ be the set of all real numbers. Let $\mathcal{B}$ be the collection of all open intervals:

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}.$$

Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the ***standard topology*** of $\mathbb{R}$.

- Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers, i.e. $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ (cartesian product). Let $\mathcal{B}$ be the collection of cartesian product of open intervals, $(a, b) \times (c, d)$. Then $\mathcal{B}$ is a basis of a topology and the topology generated by $\mathcal{B}$ is called the standard topology of $\mathbb{R}^2$.

- (**Lower limit topology** of $\mathbb{R}$) Consider the collection $\mathcal{B}$ of subsets in $\mathbb{R}$:

$$\mathcal{B} := \left\{ [a, b) := \{x \in \mathbb{R} \mid a \leq x < b\} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

This is a basis for a topology on $\mathbb{R}$. This topology is called the ***lower limit topology***.

The following two lemmata are useful to determine whehter a collection $\mathcal{B}$ of open sets in $\mathcal{T}$ is a basis for $\mathcal{T}$ or not.

Remark 1.8. Let $\mathcal{T}$ be a topology on X. If $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{B}$ satisfies (B1) and (B2), it is easy to see that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. This is just because of (G1). If $U \in \mathcal{T}_{\mathcal{B}}$, (G1) is satisfied for U so that $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. Therefore $U = \cup_{x \in U} B_x$. By (T2), $U \in \mathcal{T}$.

Lemma 1.9 (13.1 [Mun])**.** *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and only if $\mathcal{T}$ is the set of all unions of subsets in $\mathcal{B}$.*

Proof.

- ($\Rightarrow$) Let $\mathcal{T}'$ be the set of all unions of open sets in $\mathcal{B}$. If $U \in \mathcal{T}$, then U satisfies (G1), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$. Thus $U = \cup_{x \in U} B_x$. Therefore $U \in \mathcal{T}'$. We proved $\mathcal{T} \subset \mathcal{T}'$. It follows from (T2) that $\mathcal{T}' \subset \mathcal{T}$.

- ($\Leftarrow$) Since $X \in \mathcal{T}$, $X = \cup_\alpha B_\alpha$ some union of sets in $\mathcal{B}$. Thus $\forall x \in X, \exists B_\alpha$ s.t. $x \in B_\alpha$. This proves (B1) for $\mathcal{B}$. If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{T}$ by (T2). Thus $B_1 \cap B_2 = \cup_\alpha B_\alpha, B_\alpha \in \mathcal{B}$. So $\forall x \in B_1 \cap B_2, \exists B_\alpha \in B$ s.t. $x \in B_\alpha$. This B_α plays the role of B_3 in (B2). Thus $\mathcal{B}$ is a basis. Now it makes sense to consider $\mathcal{T}_{\mathcal{B}}$ and we need to show $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$. By the remark, we already know that $\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}$. On the other hand, if $U \in \mathcal{T}$, then $U = \cup_\alpha B_\alpha, B_\alpha \in \mathcal{B}$. Hence, $\forall x \in U$, $\exists B_\alpha$ such that $x \in B_\alpha \subset U$. Thus (G1) is satisfied for U. Thus $U \in \mathcal{T}_{\mathcal{B}}$. This proves $\mathcal{T}_{\mathcal{B}} \supset \mathcal{T}$.

$\square$

Lemma 1.10 (13.2 [Mun])**.** *Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis and $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ if and if any $U \in \mathcal{T}$ satisfies (G1), i.e. $\forall x \in U, \exists B_x \in \mathcal{B}$ s.t. $x \in B_x \subset U$.*

Proof.

$\Rightarrow$ Trivial by the definition of $\mathcal{T}_{\mathcal{B}}$.

$\Leftarrow$ X satisfies (G1) so $\mathcal{B}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B} \subset \mathcal{T}$. By (T3), $B_1 \cap B_2 \in \mathcal{T}$. Thus $B_1 \cap B_2$ satisfies (G1). This means (B2) holds for $\mathcal{B}$. Thus $\mathcal{B}$ is a basis. Now the assumption can be rephrased as $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. By the remark above, we already know $\mathcal{T} \supset \mathcal{T}_{\mathcal{B}}$. ◻

1.2 Comparing Topologies

Definition 1.11. Let $\mathcal{T}, \mathcal{T}'$ be two topologies for a set X. We say $\mathcal{T}'$ is ***finer*** than $\mathcal{T}$ or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if $\mathcal{T} \subset \mathcal{T}'$. The intuition for this notion is "$(X, \mathcal{T}')$ *has more open subsets to separate two points in X than* $(X, \mathcal{T})$".

Lemma 1.12 (13.3)**.** *Let $\mathcal{B}, \mathcal{B}'$ be bases of topologies $\mathcal{T}, \mathcal{T}'$ on X respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ $\Leftrightarrow$ $\forall B \in \mathcal{B}$ and $\forall x \in B$, $\exists B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.*

Proof. $\Rightarrow$ Since $\mathcal{B} \subset \mathcal{T} \subset \mathcal{T}'$, all subsets in $\mathcal{B}$ satisfies (G1) for $\mathcal{T}'$, which is exactly the statement we wanted to prove. $\Leftarrow$ The LHS says $\mathcal{B} \subset \mathcal{T}'$. We need to show that it implies that any $U \in \mathcal{T}$ satisfies (G1) for $\mathcal{T}'$ too.

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B \in \mathcal{B} \ s.t.\ x \in B \subset U$$

But

$$\forall B \in \mathcal{B}, \forall x \in B, \exists B' \in \mathcal{B}' \ s.t.\ x \in B' \subset B.$$

Combining those two,

$$\forall U \in \mathcal{T}, \forall x \in U, \exists B' \in \mathcal{B}' \ s.t.\ x \in B' \subset B \subset U.$$

◻

2 Product topology, Subspace topology, Closed sets, and Limit Points

This week, we explore various way to construct new topological spaces. And then we go on to study limit points. For this end, it is convenient to introduce closed sets and closure of a subset in a given topology.

2.1 The Product Topology on $X \times Y$

The cartesian product of two topological spaces has an induced topology called the product topology. There is also an induced basis for it. Here is the example to keep in mind:

Example 2.1. Recall that the standard topology of $\mathbb{R}^2$ is given by the basis

$$\mathcal{B} := \{(a, b) \times (c, d) \subset \mathbb{R}^2 \mid a < b, c < d\}.$$

Check the axioms (B1) and (B2)!

Definition 2.2 (§15 [Mun])**.** If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, then the collection $\mathcal{B}$ of subsets of the form $U \times V \subset X \times Y, U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ forms a basis of a topology. The topology generated by $\mathcal{B}$ is called ***product topology*** on $X \times Y$.

Proof.

(B1) Let $(x, y) \in X \times Y$ be an arbitrary element. We need to find a subset in $\mathcal{B}$ containing (x, y), but since $X \times Y \in \mathcal{B}$, it is obvious.

(B2) For any $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$, the intersection is $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$. So it is obvious again.

□

Example 2.3. The above definition gives a topology on $\mathbb{R}^2$. The following theorem identify this topology with the standard one!

Theorem 2.4 (15.1)**.** *If $\mathcal{B}_X$ is a basis of $(X, \mathcal{T}_X)$ and $\mathcal{B}_Y$ is a basis of $(Y, \mathcal{T}_Y)$, then*

$$\mathcal{B}_{X \times Y} := \{B \times C \mid B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$$

is a basis of the product topology on $X \times Y$.

Proof. To check $\mathcal{B}_{X \times Y}$, let's use Lemma 1.10 which state that $\mathcal{B}$ is a basis for $\mathcal{T}$ iff for any $U \in \mathcal{T}$ and any $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$. Let $W \in \mathcal{T}$ and $(x, y) \in W$. By the definition of product topology, there are $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x, y) \in U \times V \subset W$. Since $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases, there are $B \in \mathcal{B}_X$ and $C \in \mathcal{B}_Y$ such that $x \in B \subset U$ and $y \in C \subset V$. Thus we found $B \times C \in \mathcal{B}_{X \times Y}$ such that $(x, y) \in B \times C \subset W$. □

Example 2.5. The standard topology of $\mathbb{R}^2$ is the product topology of two copies of $\mathbb{R}$ with the standard topology.

Example 2.6. The standard topology of $\mathbb{R}^n$ is given by the basis

$$\mathcal{B} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n \mid a_i < b_i\}.$$

Example 2.7. For any $p = (x_0, y_0) \in \mathbb{R}^2$, let $B_{\epsilon,x}$ be the open disk of radius $\epsilon > 0$ centered at p. We can define a topology of $\mathbb{R}^2$ by

$$\mathcal{B}_D := \{D_{\epsilon,x} \mid x \in \mathbb{R}^2, \epsilon \in \mathbb{R}_{>0}\}.$$

The topology defined by $\mathcal{B}$ coincides with the standard topology on $\mathbb{R}^2$.

2.2 The Subspace Topology

A subset of a topological space has a naturally induced topology, called the subspace topology. In geometry, the subspace topology is the source of all funky topologies.

Definition 2.8. Let $(X, \mathcal{T})$ be a topological space. Let Y be a subset of X. The collection

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the ***subspace topology***.

Lemma 2.9. *If $\mathcal{B}$ is a basis for $\mathcal{T}$, then*

$$\mathcal{B}_Y := \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis of the subspace topology $\mathcal{T}_Y$ for Y.

Proof. Use Lemma 1.9. Let $V \in \mathcal{T}_Y$, i.e. $V = Y \cap U$ for some $U \in \mathcal{T}$. For every $x \in V$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$ since $\mathcal{B}$ is a basis of $\mathcal{T}$ (Lemma 1.9). Now we found $Y \cap B \in \mathcal{B}_Y$ such that $x \in Y \cap B \subset V$. □ □

2.3 Closed Sets, Closure, Interior, and limit points

Closed sets are nothing but complement of open sets. On the other hand, we can also say that open sets are nothing but complement of closed sets. Thus we can actually use closed sets to define topology, although mathematicians usually use open sets to define topology.

Definition 2.10. Let A be a subset of a topological space $(X, \mathcal{T})$.

- A is a ***closed set*** of X if $X - A$ is an open set.
- The ***closure*** $\bar{A}$ of A in X is the intersection of all closed sets of X, containing A.

$$\bar{A} = \bigcup_{\substack{C \supset A \\ closed}} C$$

- The ***interior*** $\operatorname{Int} A$ of A in X is the union of all open sets of X, contained in A.

$$\operatorname{Int} A = \bigcap_{\substack{O \subset A \\ open}} O$$

- $x \in X$ is a ***limit point*** of A if $x \in \overline{A - \{x\}}$.

Remark 2.11. It is not so difficult to see from the definition that

$$\bar{A} = A \Leftrightarrow A : \ closed, \quad \text{and} \quad \operatorname{Int} A = A \Leftrightarrow A : \ open.$$

Example 2.12.

- In the standard topology for $\mathbb{R}$, a set of a single element (we say a *point*) is a closed set, because $\mathbb{R} - \{a\}$ is an open set. Any finite set is also closed, since $X - \{a_1, \cdots, a_n\} = \cap_{i=1}^{n}(\mathbb{R} - \{a_i\})$ is a finite intersection of open sets.
- In the discrete topology of a set X, every point is a closed set but also an open set.
- In the lower limit topology, a point is a closed set.
- In the finite complement topology of any set X, a point is a closed set. But any infinite set is not closed by definition of finite complement topology, except X itself. For example, $\mathbb{Z}$ is a closed set in $\mathbb{R}$ in the standard topology but not in the finite complement topoloty.

Lemma 2.13 (Interior in terms of closure)**.**

$$\operatorname{Int} A = X - (\overline{X - A}).$$

Proof. We need to show that $X - \operatorname{Int} A = \overline{X - A}$. Let U denote open sets and C denote closed sets. By definition $\operatorname{Int} A = \bigcup_{U \subset A} U$. Therefore we can do the set theoretic computation:

$$X - \operatorname{Int} A = X - \bigcup_{U \subset A} U = \bigcap_{U \subset A} (X - U) = \bigcap_{C \supset (X - A)} C = \overline{X - A}.$$

□

Remark 2.14 (Defining topology by closed sets)**.** A topology on a set X is given by defining "open sets" of X. Since closed sets are just exactly complement of open sets, it is possible to define topology by giving a collection of closed sets. Let $\mathcal{K}$ be a collection of subsets of X satisfying

(C1) $\emptyset, X \in \mathcal{P}$.

(C2) Any intersection of subsets in $\mathcal{K}$ is also in $\mathcal{K}$.

(C3) Any finite union of subsets in $\mathcal{K}$ is also in $\mathcal{K}$.

Then define $\mathcal{T}$ by

$$\mathcal{T} := \{X - C \mid C \in \mathcal{K}\}$$

is a topology, i.e. it satisfies $(T1, 2, 3)$. On the other hand, if $\mathcal{T}$ is a topology, i.e. the collection of open sets, then

$$\mathcal{K} := \{X - U \mid U \in \mathcal{T}\}$$

satisfies $(C1, 2, 3)$.

Exercise 2.15. Prove the above claims.

Theorem 2.16. *Let A be a subset of the topological space $(X, \mathcal{T})$. Let $\mathcal{B}$ be a basis of $\mathcal{T}$.*

(a) $x \in \bar{A}$ if and only if every neighborhood U of x intersects with A non-trivially, i.e. $U \cap A \neq \emptyset$.

(b) $x \in \bar{A}$ if and only if every neighborhood $B \in \mathcal{B}$ of x intersects with A non-trivially.

Terminology: U is a ***neighborhood*** of x if $U \in \mathcal{T}$ and $x \in U$.

Proof. It is easier to prove the ***contrapositive statements*** of the theorem. We will prove

$$x \notin \bar{A} \Leftrightarrow_a \exists U \in \mathcal{T},\ s.t.\ x \in U \ \ and \ \ U \cap A = \emptyset \Leftrightarrow_b \exists B \in \mathcal{B},\ s.t.\ x \in B \ \ and \ \ B \cap A = \emptyset.$$

$\Rightarrow_a$ If $x \notin \bar{A}$, then $\exists C$ a closed set such that $C \supset A$ and $x \notin C$. Then $x \notin C$ implies that x is in $X - C$ which is an open set. $C \supset A$ implies that $(X - C) \cap A = \emptyset$. Let $U = X - C$ and we are done for the middle statement.

$\Rightarrow_b$ For any $U \in \mathcal{T}$ such that $x \in U$ and $U \cap A = \emptyset$, by the definition of a basis, there exists B such that $x \in B \subset U$ (G1). This B clearly satisfies the last statement.

$\Leftarrow_b$ If there is $B \in \mathcal{B}$ such that $x \in B$ and $B \cap A$, then this B also plays the role of U in the middle statement.

$\Leftarrow_a$ If U satisfies the middle statement, then $C := X - U$ is closed and $x \notin C$. Thus by definition of $\bar{A}$, $x \notin A$.

□

Remark 2.17.

If it is a bird, then it is an animal.

The ***contrapositive statement*** of the above statement is

If it is not an animal, then it is not a bird.

It is the theorem that those two statements are equivalent.

Example 2.18. The subset $A := \{1/n \mid n = 1, 2, 3, \cdots\} \subset \mathbb{R}$ is not closed in the standard topology. To see this, we can appy Theorem 2.16 and Remark 2.11. A is closed if and only if $\bar{A} = A$. So we will show $\bar{A} \neq A$. Observe $0 \notin A$ and let (a, b) be an arbitrary neighborhood of 0 where $a < 0 < b$. Then, no matter how small b is there is n such that $1/n < b$. Thus (a, b) intersects with A non-trivially. Therefore by the theorem, $0 \in \bar{A}$.

Exercise 2.19. The ***boundary*** ∂A of a subset A of a topological space X is defined by

$$\partial A := \overline{A} - \operatorname{Int} A.$$

From this definition, it follows that $\bar{A}$ is the ***disjoint union*** of ∂A and $\operatorname{Int} A$, i.e. $\bar{A} = \partial A \cup \operatorname{Int} A$ and $\partial \cap \operatorname{Int} A$.

(a) Find the boundary, the closure and the interior of $(0, 1]$ in $\mathbb{R}$ with the *standard topology*.

(b) Find the boundary, the closure and the interior of $(0, 1]$ in $\mathbb{R}$ with the *finite complement topology*.

(c) Find the boundary, the closure and the interior of of $\mathbb{Q}$ in $\mathbb{R}$ with the *standard topology*.

(d) Prove that $\partial A = \overline{A} \cap \overline{X - A}$.

Exercise 2.20. Consider standard topology $\mathcal{T}_{st}$, finite complement topology $\mathcal{T}_{f.c.}$ and the discrete topology $\mathcal{T}_{dsct}$ on $\mathbb{R}$. We have

$$\mathcal{T}_{f.c.} : \bar{\mathbb{Q}} = \mathbb{R}, \ \ \mathcal{T}_{st} : \bar{\mathbb{Q}} = \mathbb{R}, \ \ \mathcal{T}_{dsct} : \bar{\mathbb{Q}} = \mathbb{Q}.$$
$$\mathcal{T}_{f.c.} : \bar{\mathbb{Z}} = \mathbb{R}, \ \ \mathcal{T}_{st} : \bar{\mathbb{Z}} = \mathbb{Z}, \ \ \mathcal{T}_{dsct} : \bar{\mathbb{Z}} = \mathbb{Z}.$$

Observe that $\mathcal{T}_{f.c.} \subset \mathcal{T}_{st} \subset \mathcal{T}_{dsct}$. Now consider two topologies $\mathcal{T} \subset \mathcal{T}'$ on X, i.e. $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $A \subset X$ a subset. Let $\overline{A}^{\mathcal{T}}$ and $\overline{A}^{\mathcal{T}'}$ be the closures in the corresponding topologies. Prove that

$$\overline{A}^{\mathcal{T}} \supset \overline{A}^{\mathcal{T}'}.$$

2.4 Subspace topology and closed sets/closure

Theorem 2.21. *Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a* ***supspace*** *of $(X, \mathcal{T})$, i.e. a subset with the subspace topology. Then a subset A of Y is closed in Y if and only if A is an intersection of Y and a closed subset in X.*

Proof. Since an open set in Y is an intersection of Y and an open set in X by definition of subspace topology, this theorem is rather trivial in the perspective of Remark 2.14. Here is another way to prove:

$$\begin{aligned} A: \ \text{closed in } Y \quad &\Leftrightarrow \quad Y - A: \ \text{open in } Y \text{ by def of closed sets} \\ &\Leftrightarrow \quad Y - A = Y \cap U \ \ U \text{ is some open set in } X, \text{ by def of subspace} \\ &\Leftrightarrow \quad A = Y - (Y \cap U) = Y \cap (X - U) \\ &\Leftrightarrow \quad A = Y - C \ \ \ C \text{ is a closed set in } X \end{aligned}$$

□

Exercise 2.22. Let Y be a subspace of a topological space $(X, \mathcal{T})$. Prove that, if A is a closed subset of Y and Y is a closed subset in X, then A is a closed subset of X.

Theorem 2.23. *Let Y be a subspace of $(X, \mathcal{T})$ and let A be a subset of Y. The closure of A in Y is $\bar{A} \cap Y$ where $\bar{A}$ is the closure of A in X.*

Proof. Let $\bar{A}^X$ and $\bar{A}^Y$ be the closures of A in the corresponding spaces.

$$\bar{A}^Y =_1 \bigcap_{C_Y \supset A} C_Y =_2 \bigcap_{(Y \cap C_X) \supset A} (Y \cap C_X) =_3 \bigcap_{C_X \supset A} (Y \cap C_X) =_4 Y \cap \bigcap_{C_X \supset A} C_X = Y \cap \bar{A}^X.$$

$=_1$ is by def of closures in Y. $=_2$ is by the previous theorem. $=_3$ follows since $Y \supset A$. $=_4$ is just the set theoretic computation. $=_5$ is the definition of closures in X. □

3 Hausdorff Spaces, Continuous Functions and Quotient Topology

3.1 Hausdorff Spaces

Definition 3.1. A topological space $(X, \mathcal{T})$ is called a ***Hausdorff space*** if

$$(H1) \quad \forall x, y \in X \text{ such that } x \neq y, \ \ \exists U_x, U_y \in \mathcal{T} \text{ such that } x \in U_x, y \in U_y, \text{ and } U_x \cap U_y = \varnothing$$

i.e. for every pair of distinct points x, y in X, there are *disjoint* neighborhoods U_x and U_y of x and y respectively.

Example 3.2.

(a) $\mathbb{R}^n$ with the standard topology is a Hausdorff space.

(b) $\mathbb{R}$ with the finite complement topology is NOT a Hausdoff space. Suppose that there are disjoint neighborhoods U_x and U_y of distinct two points x and y. Observe that U_x must be an infinite set, since $\mathbb{R} - U_x$ is finite and $\mathbb{R}$ is an infinite set (see the definition of finite complement topoloty). It is the same for U_y. By the disjointness, $U_y \subset X - U_x$. Therefore it contradicts with the finiteness of $X - U_x$. Thus U_x and U_y can not be disjoint. This proves that $\mathbb{R}$ with the finite complement topology is not Hausdorff.

(c) Any *infinite* set X with the finite complement topology is not a Hausdorff. This is because every non-empty open sets intersect non-trivially. (b) is just a special case of (c).

Proof. Let $U_1 := X - F_1$ and $U_2 := X - F_2$ be open sets (F_i is finite). Then $U_1 \cap U_2 = X - (F_1 \cup F_2)$. Since X is infinite and $F_1 \cup F_2$ is finite, this can not be empty. □

The following exercise is a nice one to relate the Hausdorff condision and the product topology.

Exercise 3.3. Show that X is a Hausdorff space if and only if the ***diagonal*** $\Delta := \{(x, x) \mid x \in X\} \subset X \times X$ is closed with respect to the product topology.

Theorem 3.4. *Every finite set in a Hausdorff space X is closed.*

Proof. A point $\{x\} \subset X$ is closed set because we can show that $X - \{x\}$ is open: $\forall y \in X - \{x\}$, by the Hausdorff condition, there is an open set U_y such that $y \in U_y$ but $x \notin U_y$ (we are not using the whole condition) so that $U_y \subset X - \{x\}$. Now, since a finite set is a finite union of single points, it is closed. □

Remark 3.5. The opposite statement of Theorem 3.4 is not true. The counter example is Example 3.2 (b).

Definition 3.6. Let $\{x_n \mid n \in \mathbb{N}\}$ be a *sequence* of points in a topological space. The sequence $\{x_n \mid n \in \mathbb{N}\}$ ***converges to a point*** $x \in X$ if, for every neighborhood U of x, there is a positive integer $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In other words, U contains all but finitely many points of $\{x_n\}$.

Lemma 3.7. *Let $\mathcal{B}$ be a basis of a Hausdorff space X. Then $\{x_n\}$ converges to x iff every $B \in \mathcal{B}$ containing x contains all but finitely many points of $\{x_n\}$.*

Exercise 3.8. Find all points that the sequence $\{x_n = 1/n \mid \mathbb{Z}_{>0}\}$ converges to with respect to the following topology of $\mathbb{R}$. Justify your answer.

(a) Standard Topology

(b) Finite Complement Topology

(c) Discrete Topology

(d) Lower Limit Topology

Are (c) and (d) Hausdorff?

Theorem 3.9. *If X is a Hausdorff space, then every sequence of points in X converges to at most one point of X.*

Proof. We prove by deriving a contradiction. Suppose that $\{x_n\}$ converges to x and y and that $x \neq y$. Then by (H1), there are U_x and U_y in $\mathcal{T}_X$ such that $x \in U_x$ and $y \in U_y$ and $U_x \cap U_y = \emptyset$. Since U_x contains all but finitely many points of $\{x_n\}$, it is not possible that U_y contains all but finitely many points of $\{x_n\}$. Thus $\{x_n\}$ can not converges to y. □

3.2 Continuous Maps

Definition 3.10 (ε-δ continuity)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is ***ε-δ continuous at*** $x_0 \in \mathbb{R}$ when "$f(x)$ gets closer to $f(x_0)$ as x gets closer to x_0".More precisely,

$$(\epsilon\text{-}\delta\text{-cont at } x_0) \quad \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon), \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$

In terms of the basis of topology $\mathcal{B} := \{B_{x,\epsilon} := (x - \epsilon, x + \epsilon) \mid x \in \mathbb{R}, \epsilon \in \mathbb{R}_{>0}\}$,

$$(\epsilon\text{-}\delta\text{-cont at } x_0) \Leftrightarrow \forall B_{f(x_0),\epsilon}, \exists B_{x_0,\delta} \text{ such that } f(B_{x_0,\delta}) \subset B_{f(x_0),\delta}.$$

$$\Leftrightarrow (\text{Cont at } x_0) \quad \forall B_{f(x_0),\epsilon}, \exists B_{x_0,\delta} \text{ such that } B_{x_0,\delta} \subset f^{-1}(B_{f(x_0),\delta})$$

A map $f : \mathbb{R} \to \mathbb{R}$ is an ***ε-δ continuous function*** if f is ε-δ continuous at every $x \in \mathbb{R}$.

$$\Leftrightarrow (\text{Cont}) \quad \forall B_{f(x),\epsilon}, f^{-1}(B_{f(x),\epsilon}) \text{ is open .}$$

Using (G1)

$$\Leftrightarrow (\text{Cont}) \quad \text{for all open set } U \subset \mathbb{R}, f^{-1}(U) \text{ is open.}$$

Definition 3.11. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $\mathcal{B}_Y$ be a basis of $\mathcal{T}_Y$. A map $f : X \to Y$ is ***continuous at*** $x_0 \in X$ if

$$(\text{Cont at } x_0) \quad \forall V_{f(x_0)}\text{: a nbhd of } f(x_0) \text{ in } Y, \exists U_{x_0} \text{ a nbhd of } x_0 \text{ in } X \text{ such that } U_{x_0} \subset f^{-1}(V_{f(x_0)}).$$

By using (G1),

$$\Leftrightarrow (\text{Cont at } x_0) \quad \forall V_{f(x_0)} \in \mathcal{B}_Y\text{: a nbhd of } f(x_0) \text{ in } Y, \exists U_{x_0} \text{ a nbhd of } x_0 \text{ in } X \text{ such that } U_{x_0} \subset f^{-1}(V_{f(x_0)}).$$

A map $f : X \to Y$ is ***continuous*** if f is continuous at every point of X.

$$(\text{Cont}) \quad \forall V \in \mathcal{T}_Y, \quad f^{-1}(V) \in \mathcal{T}_X, \text{ i.e. the preimage of open sets are open.}$$

By using (G1), we can just check the condition for open sets in the basis:

$$f : X \to Y \text{ continuous} \quad \Leftrightarrow (\text{Cont}) \quad \forall V \in \mathcal{B}_Y, \quad f^{-1}(V) \in \mathcal{T}_X$$

By the way we get the definition of continuity of a map, we have

Theorem 3.12. *A map $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ relative to the standard topology if and only if f is ε-δ continuous at $x_0 \in \mathbb{R}$.*

Example 3.13. Here are some trivial example

- Let $f : X \to Y$ be a map of topological spaces. Show that f is always continuous, if X has the discrete topology.
- Let $\mathbb{R}_s$ and $\mathbb{R}_f$ be the set of all real numbers with the standard topology and the finite complement topology. If $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ be the identity map, i.e. $\mathrm{id}_{\mathbb{R}}(x) = x$. Then $\mathrm{id} : \mathbb{R}_s \to \mathbb{R}_f$ is continuous but $\mathrm{id} : \mathbb{R}_f \to \mathbb{R}_s$ is not continuous. This is because the standard topology is strictly finer than the finite complement topology, i.e. the standard topology has strictly more open sets than the finite complement topology. In general, we have

  $$\text{If } (X, \mathcal{T}) \text{ is finer than } (X, \mathcal{T}'), \text{ then } \mathrm{id}_X : (X, \mathcal{T}) \to (X, \mathcal{T}') \text{ is continuous.}$$

- (***Constant functions***) If $f : X \to Y$ maps all points of X to a single point $y_0 \in Y$, then f is a continuous function.

Exercise 3.14. We can formulate the continuity by using closed sets: show that a map $f : X \to Y$ of topological spaces is continuous if and only if for every closed set C_Y of Y, the preimage $f^{-1}(C_Y)$ is closed in X.

Exercise 3.15. Prove that $f : X \to Y$ is continuous if and only if for every subset A of X, we have $f(\overline{A}) \subset \overline{f(A)}$.

Exercise 3.16. Define a map $f : \mathbb{R} \to \mathbb{R}^2$ by $x \mapsto (\cos x, \sin x)$. Then f is continuous relative to the standard topologies.

Exercise 3.17. Define a map $f : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto \begin{cases} |x| & \text{if } x \text{ is rational} \\ -|x| & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is continuous at $x = 0$ but not continuous at other points.

Lemma 3.18. *Let $f : X \to Y$ be a continuous map and let $\{x_n \mid n \in \mathbb{Z}_{>0}\} \subset X$ be a sequence which converges to $x \in X$. Then the sequence $\{f(x_n)\} \subset Y$ converges to $f(x)$.*

Proof. Let V be any neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set containing x so it's a neighborhood of x. By the definition of convergence, there is N such that $x_n \in f^{-1}(V)$ for all $n > N$. This N satisfies that $f(x_n) \in V$ for all $n > N$ so that $\{f(x_n)\}$ converges to $f(x)$. □

3.3 Homeomorphisms

Definition 3.19. A map $f : X \to Y$ of topological spaces is a ***homeomorphism*** if f is bijective and both f and f^{-1} are continuous.

Remark 3.20. f is a homeomorphism if f is bijective and

- for any open set $U \subset X$, $f(U)$ is open in Y,
- for any open set $V \subset Y$, $f^{-1}(V)$ is open in X.

Indeed, if we have bases for topologies for X and Y, we just need to see the conditions above for basis elements.

Example 3.21.

- The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a homeomorphism. The inverse function $f^{-1}(y) = x^{\frac{1}{3}}$ is also continuous.
- $f(x) = x^2$ is not a homeomorphism because it is not a bijection. But we restrict f to some subset, it is a homeomorphism relative to the subspace topology. For example, $f : [0, \infty) \to [0, \infty)$, then f is a bijective continuous map and $f^{-1}(x) = x^{\frac{1}{2}}$ is also continuous.
- Consider he function $f : [0, 2\pi) \to S^1$ given by $t \mapsto (\cos t, \sin t)$ where S^1 is the unit circle in $\mathbb{R}^2$ with the subspace topology. It is a bijective continuous map but f^{-1} is not continuous. This is because, for example, the image of the open set $[0, 1)$ under f is not open in S^1.

  How to show $f([0, 1))$ is not closed in S^1: Let $\mathcal{B}_{ball}$ be the basis of standard topology of $\mathbb{R}^2$ given by open disk. The basis of the subspace topology on S^1 is $\mathcal{B} := \{S^1 \cap B \mid B \in \mathcal{B}_{ball}\}$. Let B_0 be an arbitrary open disk centered at $f(0)$. Then $S^1 \cap B_0$ can not be contained in $f([0, 1))$, since there is always a small ϵ such that $f(-\epsilon) \notin f([0, 1))$ and $f(-\epsilon) \in S^1 \cap B_0$. This shows that $f([0, 1))$ is not open in the subspace topology S^1

Exercise 3.22. Show that the open interval $(-\pi/2, \pi/2)$ of $\mathbb{R}$ with the subspace topology is homeomorphic to $\mathbb{R}$. Show that any open interval is homeomorphic to $\mathbb{R}$.

Definition 3.23. A map $f : X \to Y$ is a ***topological embedding*** if f is injective and $f : X \to f(X)$ is a homeomorphism where $f(X)$ has the subspace topology inherited from Y.

Exercise 3.24. Let X and Y be topological spaces and $A \subset X$ a subspace. Let $f : A \to Y$ be a continuous function and assume that Y is Hausdorff. Show that if f can be extend to a continuous function $\tilde{f} : \bar{A} \to Y$, i.e. if there is a continuous function $\tilde{f} : \bar{A} \to Y$ such that $\tilde{f}|_A = f$, then it is unique.

3.4 Properties of continuous functions

Lemma 3.25. *The composition $g \circ f$ of continuous functions $f : X \to Y$ and $g : Y \to Z$ is continuous.*

Proof. Let U be an open set in Z. Then $g^{-1}(U)$ is open in Y since g is continuous. Then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in X since f is continuous. □

Exercise 3.26. Suppose that X, Y, Z are topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be maps of sets. Prove or disprove the following statement:

(a) If $f : X \to Y$ is continuous and the composition map $g \circ f : X \to Z$ is continuous, then $g : Y \to Z$ is continuous.

(b) If $g : Y \to Z$ is continuous and the composition map $g \circ f : X \to Z$ is continuous, then $f : X \to Y$ is continuous.

Lemma 3.27. *Suppose that X is a union of open sets $U_i, i \in \mathcal{I}$. Suppose we have continuous maps $f_i : U_i \to Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then there is a unique continuous map $f : X \to Y$ such that $f|_{U_i} = f_i$.*

Proof. There is a map $f : X \to Y$ such that $f|_{U_i} = f_i$ by saying for each $x \in X$, let $f(x) := f_i(U_i)$. This is well-defined since the choice of U_i doesn't change the map f. Furthermore, if $f, g : X \to Y$ are maps such that $f|_{U_i} = f_i$. Then for any $x \in X$, there is U_i such that $x \in U_i$ and $f(x) = f_i(x) = g(x)$ by the conditions. Thus $f = g$. So it's unique. To show that such f is continuous, let V be an open set in Y. Then

$$f^{-1}(V) = \bigcup_i f^{-1}(V) \cap U_i = \bigcup_i f_i^{-1}(V).$$

The first equality follows from the assumption that X is a union of U_i's. Thus $f^{-1}(V)$ is an open set since it is a union of open sets (each $f_i^{-1}(V)$ is an open set in U_i and it is an open in X since U_i is open in X.) □

Lemma 3.28. *A map $f : Z \to X \times Y$ is continuous if and only if $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continuous.*

Proof. Let $U_1 \times U_2$ be an open set in $X \times Y$. Then $f^{-1}(U_1 \times U_2)$ is an open set in Z. Let $U_2 = Y$. Then

$$f^{-1}(U_1 \times Y) = (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(Y) = (\pi_1 \circ f)^{-1}(U_1) \cap Z = (\pi_1 \circ f)^{-1}(U_1)$$

Thus $\pi_1 \circ f$ is continuous. Similarly for $\pi_2 \circ f$. On the other hand,

$$f^{-1}(U_1 \times U_2) = (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2)$$

implies that if U_1 and U_2 are open sets in X and Y respectively, then $f^{-1}(U_1 \times U_2)$ is an open set since $(\pi_1 \circ f)^{-1}(U_1)$ and $(\pi_2 \circ f)^{-1}(U_2)$ are open in Z. Since every open set in $X \times Y$ is of the form $U_1 \times U_2$, we can conclude that f is continuous. □

Theorem 3.29. *If $f : X \to Y$ is an injective continuous map and Y is Hausdorff, then X must be Hausdorff.*

Proof. Let $x_1, x_2 \in X$ are distinct points. Then $f(x_1)$ and $f(x_2)$ are distinct points in Y since f is injective. Therefore there are neighborhoods V_1 and V_2 of $f(x_1)$ and $f(x_2)$ respectively such that $V_1 \cap V_2 = \emptyset$. Since f is continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are open sets and in particular neighborhoods of x_1 and x_2. Since $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = f^{-1}(\emptyset) = \emptyset$ (see HW1), we found the disjoint neighborhoods of x_1 and x_2, thus X is Hausdorff. □

3.5 Quotient Topology

Definition 3.30. Let $\pi : X \to Y$ be a *surjective* map of topological spaces. The map π is a ***quotient map*** if

$$\text{a subset } U \text{ in } Y \text{ is open if and only if the preimage } \pi^{-1}(U) \text{ is open in } X.$$

Note that "only if" part is the continuity of π, so this condition is stronger than π being continuous.

Exercise 3.31. A map $f : X \to Y$ of topological spaces is called a ***open (closed) map*** if the image of every open (closed) set in X is again open (closed). Show that a continuous surjective map $f : X \to Y$ is a quotient map if it is either an open or closed map.

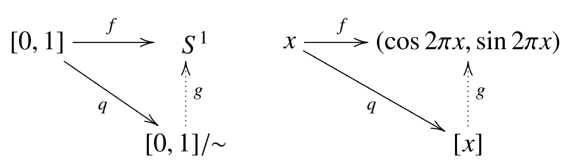
Example 3.32. Define a map $f : [0, 1] \to S^1$ by $x \mapsto (\cos 2\pi x, \sin 2\pi x)$ where S^1 is the unit circle in $\mathbb{R}^2$ with the induced topology. It is a continuous surjective map. It is not an open map but it is a closed map. Observe the images of $(1/2, 1]$ and $[1/2, 1]$. Thus, it is a quotient map.

Definition 3.33. Let X be a topological space and A a set. Let $f : X \to A$ be a surjective map. There is a unique topology on A which makes f to be a quotient map. It is called the ***quotient topology*** and the topological space A is called the ***quotient space***.

Proof. The definition of the quotient map actually determines the topology of A: $\mathcal{T}_A$ must be the set of all subsets $U \subset A$ such that $f^{-1}(U)$ is open in X. The axiom (T1) is obvious. The axioms (T2,3) follows from the fact that the preimage preserves the unions and the intersections. □

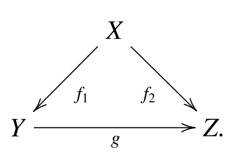
Remark 3.34. An equivalence relation on a set X defines a surjective map $f : X \to A$ where A is the set of all equivalence classes (see [Set]). The typical construction of a quotient space is given by identifying equivalent points. Let $\sim$ be an equivalence relation on X and then denote $X/\!\sim$ the set obtained by identifying equivalent points. Then we have the natural surjective map $q : X \to X/\!\sim, x \mapsto [x]$.

Example 3.35. Let $X := [0, 1]$ and identify 0 and 1, i.e. the equivalence relation is given by $a \sim a$ for all $a \in [0, 1]$ and $0 \sim 1$. The map f in Example 3.32 *factors through* $X/\!\sim$:



Then we can show that g is a homeomorphism, using the following useful lemma:

Lemma 3.36. *Consider the following commutative diagram, i.e.* $g \circ f_1 = f_2$*:*



If f_1 *and* f_2 *are quotient maps and* g *is a bijection, then* g *must be a homeomorphism.*

Proof. It suffices to show that g is continuous, since then g^{-1} is also continuous (the claim holds if we replace g by g^{-1}). Let U be an open set in Z. We want to show that $V := g^{-1}(U)$ is an open set in Y, but a set V in Y is an open set if and only if $f_1^{-1}(V)$ is an open set in X by the definition of the quotient topology. Since $f_1^{-1}(V) = f_1^{-1}(g^{-1}(U)) = f_2^{-1}(U)$ is an open set (the quotient topology of Z), V must be an open set. □

Example 3.37 (2-Sphere ver 1)**.** Let $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$. Identify all points on the boundary of the disk, i.e. $p \sim p$ for all $\operatorname{Int} X$ and $p \sim q$ if $p, q \in \partial D$. It can be shown to be homeomorphic to the unit 2 sphere which is defined by

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \quad \text{in } \mathbb{R}^3 \text{ with standard topology.}$$

Exercise 3.38 (2-sphere ver 2)**.** Consider the disjoint union $X := D_1 \sqcup D_2$ of two unit disk D_1 and D_2. Identify $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$ and $y_1 = y_2$ where $(x_1, y_1) \in D_1$ and $(x_2, y_2) \in D_2$. Then show that $X/\sim$ is homeomorphic to the unit 2-sphere S^2 in $\mathbb{R}^3$.

Example 3.39 (Torus)**.** Consider the unit square $X := [0, 1] \times [0, 1]$ in $\mathbb{R}^2$. Identify $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$ for all $x, y \in [0, 1]$. A ***torus*** T is defined by $X/\sim$. Show that it is homeomorphic to $S^1 \times S^1$.

Exercise 3.40. Prove the following:

(a) If $f : X \to Y$ and $g : Y \to Z$ are quotient maps, then $g \circ f : X \to Z$ is a quotient map.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. If f and $g \circ f$ are quotient maps, then g is a quotient map.

4 Compactness, Metric, and Universality

4.1 Definition of compactness and the compact subspaces of $\mathbb{R}$

Definition 4.1. Let X be a topological space. A collection of open sets $U_a, a \in \mathcal{A}$ is an ***open covering*** of X if $X = \cup_a U_a$.

X is ***compact*** if every open covering of X contains a finite subcollection that also covers X.

Example 4.2.

1. $\mathbb{R}$ is not compact: Consider the infinite covering $(n, n+2), n \in \mathbb{Z}$:

2. $X = \{0\} \cup \{1, 1/2, 1/3, \cdots\}$ is compact: every neighborhood of 0 contains all but finitely many points of X. Thus for any open covering of X, take one open set U containing 0 and choose other open sets that correspond to each of the finite many points not in U.

3. $(0, 1)$ is not compact since it is homeomorphic to $\mathbb{R}$.

Remark 4.3. The set of real numbers satisfy the following properties:

($\mathbb{R}$1) If a non-empty subset $A \subset \mathbb{R}$ is ***bounded above***, i.e. there exists $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$, then there is ***the least upper bound (or supremum) of*** A, i.e. there exists the *smallest* $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The least upper bound of A is denoted by $\sup A$.

($\mathbb{R}$2) If $x < y$, then there is a number $z \in \mathbb{R}$ such that $x < z < y$.

An order relation on a set satisfying these two properties is called a ***linear continuum***.

Theorem 4.4 (27.1 [Mun])**.** *Closed intervals of* $\mathbb{R}$ *are compact.*

Proof. Let $U_\alpha, \alpha \in \mathcal{A}$ be an open covering of $[a, b] \subset \mathbb{R}$.

(1) Let $x \in [a, b)$, then there is $y \in (x, b]$ such that $[x, y]$ is covered by one open set in $\mathcal{A} := \{U_\alpha\}$.

Each U_α is a union of open intervals. Let (h_1, h_2) be an open interval of U_α such that $h_1 < x < h_2$. By ($\mathbb{R}$2), there is y such that $x < y < h_2$. Now it is clear that $[x, y]$ is covered by (h_1, h_2) and so by U_α.

(2) Let C be the set of all $y \in (a, b]$ such that $[a, y]$ is covered by finitely many open sets in $\mathcal{A}$. Then C is not empty because applying (1) for $x = a$, then there is y such that $[a, y]$ is covered by one of U_α. Now C is bounded above since $C \subset (\infty, b]$. By ($\mathbb{R}$1), the least upper bound c which should satisfy $a < c \leq b$.

(3) c belongs to C, i.e. $[a, c]$ is covered by finitely many open sets in $\mathcal{A}$.

Suppose that $c \notin C$. Let (h_1, h_2) be one of open intervals of U_α, which contains c. Then there is $z \in C$ such that $z \in (h_1, c)$ because c is the smallest that bounds C. By the definition of C, $[a, z]$ is covered by finitely many open sets in $\mathcal{A}$. Since $[z, c] \subset U_\alpha$, $[a, z] \cup [z, c] = [a, c]$ is covered by finitely many open sets. Contradiction.

(4) c is actually b.

Suppose that $c < b$. Apply (1) to $x := c$. Then there is $y \in (c, b]$ such that $[c, y]$ is covered by one open set U_α. Since $c \in C$, $[a, c]$ is covered by finitely many open sets as we proved in (3). Thus $[a, c] \cup [c, y] = [a, y]$ is covered by finitely many open sets, i.e. $y \in C$. This contradict fact that c bounds C. Thus $c = b$.

Thus $[a, b]$ is covered by finitely open sets in $\mathcal{A}$. □

4.2 Useful Theorems about compactness

Theorem 4.5 (26.2 [Mun])**.** *Every closed subspace C of a compact space X is compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open covering of C. By the definition of subspace topology, each $U_\alpha = C \cap V_\alpha$ for some open set V_α in X. Then $\{V_\alpha\} \cup \{X - C\}$ is an open covering of X, therefore there is a finite subcollection $\{O_i, i = 1, \cdots, n\}$ that covers X, and so certainly $\{O_i \cap C, i = 1, \cdots, n\}$ covers C. If $X - C$ is in the subcollection, $(X - C) \cap C = \emptyset$, so we can assume that $\{O_i \cap C, i = 1, \cdots, n\}$ is a finite subcollection of $\{U_\alpha, \alpha \in \mathcal{A}\}$. Thus we have a finite subcollection that covers C. □

Theorem 4.6 (26.7 [Mun])**.** *The product of finitely many compact spaces is compact.*

Proof. If we just prove it for product of two compact spaces, then the claim follows from induction. Let X and Y be compact spaces. Let $\mathcal{T}_X, \mathcal{T}_Y$ be topologies of X, Y and let $\mathcal{T}_{X\times Y}$ be the product topology of $X \times Y$.

(1) $\forall x \in X$ and $\forall N \in \mathcal{T}_{X\times Y}$ such that $N \supset \{x\} \times Y$, $\exists W \in \mathcal{T}_X$ such that $x \in W$ and $N \supset W \times Y$.

An opet set N in $X \times Y$ containing $\{x\} \times Y$ is called a tube about $\{x\} \times Y$. This claim says, although a tube may not be of the form $W \times Y$, but for a given tube, we can find a smaller tube that is of the form $W \times Y$. To prove this, we need Y to be compact. The given N is of the form $\cup_\alpha U_\alpha \times V_\alpha$. Then $\{(U_\alpha \times V_\alpha) \cap (\{x\} \times V_\alpha)\}$ is an open covering of $\{x\} \times Y$ since $N \supset \{x\} \times Y$. Since $\{x\} \times Y \cong Y$, $\{x\} \times Y$ is compact and so there is a finite subcollection $\{U_{\alpha_1} \times V_{\alpha_1}, \cdots, U_{\alpha_n} \times V_{\alpha_n}\}$ that covers $\{x\} \times Y$. Then $W := \cap_{i=1}^n U_{\alpha_i}$ is an open set in X (since it is a **finite** intersection) and we can assume that $x \in W$ since if $x \notin U_{\alpha_i}$, we can get rid of $U_{\alpha_i} \times V_{\alpha_i}$ from the list. Now

$$N \supset \cup_{i=1}^n U_{\alpha_i} \times V_{\alpha_i} \supset W \times Y \supset \{x\} \times Y.$$

(2) Let $\{N_\alpha\}$ be an arbitrary open covering of $X \times Y$. For each $x \in X$, we have a subcollection $\{N_{\alpha_1,x}, \cdots, N_{\alpha_n,x}\}$ that covers $\{x\} \times Y$ since $\{x\} \times Y$ is compact. Then $N_x = \cup_{i=1}^n N_{\alpha_i,x} \supset \{x\} \times Y$. By (1), there is W_x such that $N_x \supset W_x \times Y \supset \{x\} \times Y$. Since X is compact and $\{W_x, x \in X\}$ is an open covering of X, we have a subcollection $\{W_{x_j}, j = 1, \cdots, m\}$. Now $\{N_{\alpha_i,x_j}, 1 \geq i \geq n, 1 \geq j \geq m\}$ gives a finite subcollection of $\{N_\alpha\}$ that covers $X \times Y$:

$$\cup_{i,j} N_{\alpha_i,x_j} = \cup_j N_{x_i} \supset \cup_j W_{x_j} \times Y \supset X \times Y.$$

□

Theorem 4.7 (26.5 [Mun])**.** *The image of a compact space under a continuous map is compact.*

Theorem 4.8 (26.3 [Mun])**.** *Every compact subspace Y of a Hausdorff space X is closed.*

Proof. We will show that $X - Y$ is open, i.e. for every $x \in X - Y$, there is a neighborhood of x contained in $X - Y$. For every $y \in Y$, there are disjoint neighborhoods $U_{x,y}$ of x and U_y of y (Hausdorff). By collecting such U_y's, we obtain an open covering of Y. Note that $U_{x,y}$ may be different for every y. Nevertheless by the compactness of Y, we can choose the finite subcollection $\{U_{y_1}, \cdots, U_{y_n}\}$ that covers Y. The finite intersection U_x of $U_{x,y_i}, i = 1, \cdots, n$ is again an open set and it is disjoint from Y. Therefore we found a neighborhood U_x contained in $X - Y$. □

Theorem 4.9 (26.6 [Mun])**.** *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f must be a homeomorphism.*

Proof. It suffices to show that f is a closed map since f is bijective. If A is closed in X, then A is compact by Theorem 4.5. Then by Theorem 4.7, $f(A)$ is compact. Thus by Theorem 4.8, $f(A)$ is closed. □

Example 4.10.

(1) The cartesian product of closed intervals (a box) in $\mathbb{R}^n$ is compact.

(2) Every closed subset of $\mathbb{R}^n$ contained in a box is compact.

(3) The unit n-sphere and the closed n-ball are compact.

(4) $\mathbb{R}$ with the finite complement topology is not Hausdorff. The subspace $\mathbb{Z}$ is compact: for any open covering of $\mathbb{Z}$, take an open set containing 0. With the finite complement topology, this open set misses only finitely many points. By choosing an open set for each missed point, we find the subcollection. Thus $\mathbb{Z}$ is compact. However $\mathbb{Z}$ is not closed.

(5) A closed interval in $\mathbb{R}$ with the finite complement topology is compact.

Theorem 4.11 (27.3 [Mun])**.** *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d, i.e. there is $M > 0$ such that $d(\vec{x}, \vec{y}) < M$ for all $\vec{x}, \vec{y} \in A$.*

Proof.

- ($\Leftarrow$) This direction is already hinted in the example above. We can show that it is bounded, then we can put it in a box or a closed n-ball. Let's fix a point $\vec{p}$ in A. Since A is bounded, so there is $M > 0$ such that $d(\vec{p}, \vec{x}) < M$ for all $\vec{x} \in A$. Hence A is a the closed subset of closed n-ball $\overline{B_d(\vec{p}, M)}$ which is compact. The claim follows from Theorem 4.5.

- ($\Rightarrow$) This direction is new, but it's easy. Let's cover A by open balls, namely, $A \subset \cup_{\vec{x} \in A} B_d(\vec{x}, \epsilon_x)$. Since A is compact, we can find a finite subcovering: $A \subset \cup_{i=1}^m B_d(\vec{x}_i, \epsilon_i)$. Now take arbitary $\vec{p}_1, \vec{p}_2 \in A$. There must be some i_1 and i_2 such that $\vec{p}_1 \in B_d(\vec{x}_{i_1}, \epsilon_{i_1})$ and $\vec{p}_2 \in B_d(\vec{x}_{i_2}, \epsilon_{i_2})$. By the triangle inequality of the metric, we have
$$d(\vec{p}_1, \vec{p}_2) \le d(\vec{p}_1, \vec{x}_{i_1}) + d(\vec{x}_{i_1}, \vec{x}_{i_2}) + d(\vec{x}_{i_2}, \vec{p}_2) \le d(\vec{x}_{i_1}, \vec{x}_{i_2}) + \epsilon_1 + \epsilon_2 \le M$$
where $M = \sum_{i,j=1}^m d(\vec{x}_i, \vec{x}_j) + \sum_{i=1}^m \epsilon_i$. Thus A is bounded. It is certainly closed since $\mathbb{R}^n$ is closed and by Theorem 4.8.

□

Remark 4.12. We can not generalized the above theorem to arbitrary metric spaces. Here is a counter example. Let $\mathbb{R}$ be the metric space with Euclidean metric. It is easy to see that $\mathbb{R} - \{a\}$ has induced metric whose topology is the subspace topology of the standard topology. For a closed interval $[b, c]$ containing a, is a compact space, but $[b, c] - \{a\}$ is not compact. $[b, c] - \{a\}$ is closed in $\mathbb{R} - \{a\}$ and bounded in the induced metric. Thus the theorem fails for the metric space $\mathbb{R} - \{a\}$. This failure is related to the concept "*completeness*" of the metric. $\mathbb{R} - \{a\}$ is not a complete metric space but $\mathbb{R}$ is. See §45[Mun] to see more.

Exercise 4.13. Prove that every subspace of $\mathbb{R}$ with the finite complement topology is compact.

4.3 Metric

Definition 4.14 (§20 [Mun])**.** A ***metric*** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfing

1. $d(x, y) > 0$ for all $x, y \in X$ and the equality holds iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) + d(y, z) \ge d(x, z)$ for all $x, y, z \in X$.

$d(x, y)$ is often called the ***distance*** between x and y.

Definition 4.15. Let (X, d) be a set X with a metric d. For $x \in X$ and $\epsilon > 0$, the ***ϵ-ball*** $B_d(x, \epsilon)$ ***centered at*** x is the subset of X given by
$$B_d(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}.$$
The collection $\mathcal{B}_d := \{B_d(x, \epsilon) \mid x \in X, \epsilon \in \mathbb{R}_{>0}\}$ is a basis of a topology called the ***metric topology***. Check that $\mathcal{B}_d$ satisfies the axiom (B1) and (B2) in Section 1. Call it the ***open ball basis***. If $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ can be realized as a metric topology, then $(X, \mathcal{T})$ is called ***metrizable***.

Remark 4.16. A metrizable topological space is certainly a Hausdorff space. If $x \neq y$, then $d := d(x, y) > 0$ and so $d/2 > 0$. Now $B_d(x, d/2)$ and $B_d(y, d/2)$ separate x and y.

Definition 4.17 (The Euclidean metric on $\mathbb{R}^n$)**.** Let $\vec{x}, \vec{y} \in \mathbb{R}^n$.

- The ***inner product*** $\langle \vec{x}, \vec{y} \rangle := x_1 y_1 + \cdots + x_n y_n$.
- The ***norm*** $\|\vec{x}\| := \sqrt{\langle \vec{x}, \vec{x} \rangle}$.
- The ***Euclidean metric*** $d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\|$.

Remark 4.18. The standard topology of $\mathbb{R}^n$ coincides with the metric topology given by the Euclidean metric.

Remark 4.19. There is another metric on $\mathbb{R}^n$ called the ***square metric*** ρ given by

$$\rho(\vec{x}, \vec{y}) := \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}.$$

This is also a metric and its metric topology coincides with the standard topology.

Lemma 4.20. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ a subset. Let $x \in X$. If there is a sequence $\{x_n\} \subset A$ that converges to x, then $x \in \bar{A}$. If X is metrizable, the converse holds, i.e. if $x \in \bar{A}$, there is $\{x_n\}$ that converges to x.*

Proof. The first claim is trivial by the definition of convergence. Indeed, $\{x_n\} \to x$ if and only if $\forall U_x$, $\{x_n\} - U_x$ is finite. Thus $U_x \cap \{x_n\}$ is infinite, therefore $U_x \cap A \neq \emptyset$. For the second claim, choose a metric d so that $\mathcal{T}$ is its metric topology. Let $\mathcal{B} := \mathcal{B}_d$ be the open ball basis. Let $B_{x,\epsilon} := B_d(x, \epsilon)$. First apply Theorem 2.16 (b). We have

$$x \in \bar{A} \quad \Leftrightarrow \quad \forall B_{x,\epsilon} \in \mathcal{B}, B_{x,\epsilon} \cap A \neq \emptyset \quad \Leftrightarrow \quad \forall B_{x,1/n}, n \in \mathbb{Z}_{>0}, B_{x,1/n} \cap A \neq \emptyset$$

where the second equivalence follows from the fact that $\forall \epsilon > 0$, $\exists n$ such that $1/n < \epsilon$. Now pick $x_n \in B_{x,1/n} \cap A$. Then the sequence $\{x_n\} \subset A$ converges to x since $B_{x,1/n} \subset B_{x,1/m}$ if $n > m$. □

Theorem 4.21. *Let X, Y be topological spaces and $f : X \to Y$ a map. The continuity of f implies that*

$(S1)$ *for every convergent sequence $\{x_n\} \to x$ in X, the sequence $\{f(x_n)\}$ converges to $f(x)$.*

On the other hand, if X is metrizable, $(S1)$ implies the continuity of f.

Proof. The first claim is Lemma 3.18. To prove the second claim, we will use Lemma 4.20 and HW3 (3): $f : X \to Y$ continuous $\Leftrightarrow f(\bar{A}) \subset \overline{f(A)}, \forall A \subset X$. Let A be a subset of X and let $x \in \bar{A}$. By the second claim of Lemma 4.20, $x \in \bar{A}$ implies that there is a sequence $\{x_n\} \to x$. By the assumption, $\{f(x_n)\} \to f(x)$. Then by the first claim of Lemma 4.20, $f(x) \in \overline{f(A)}$. Therefore $f(\bar{A}) \subset \overline{f(A)}$. □

4.4 Limit Point Compactness and sequentially compact §28 [Mun]

Definition 4.22. Let X be a topological space and A a subset. Recall the following definitions:

- $x \in X$ is a limit(accumulation, cluster) point of A if $x \in \overline{A - \{x\}}$.
- A sequence $\{x_n \in X \mid n \in \mathbb{Z}_{>0}\}$ converges to $x \in X$ if for every neighborhood U_x of x, there is $N \in \mathbb{Z}_{>0}$ such that $x_n \in U_x$ for all $n > N$.

Definition 4.23. Let X be a topological space.

- X is ***limit point compact*** if, for any infinite subset A of X, there is a cluster point of A in X.
- X is ***sequentially compact*** if every sequence $\{x_n\}$ in X contains a subsequence that converges to a point x in X.

Theorem 4.24 (28.1 [Mun])**.** *Compactness implies limit point compactness. Converse is not true in general.*

Proof. Let X be a compact space. Suppose that X is not limit point compact. Let A be an infinite subset such that there is no cluster point of A in X, i.e. $\forall x \in X$, $x \notin \overline{A - \{x\}}$. By Theorem 2.16, there is an open set U_x of such that $x \in U_x$ and $U_x \cap (A - \{x\}) = \emptyset$. This implies that for every $x \in X - A$, there is U_x such that $x \in U_x \subset X - A$ so that A is closed. Now consider the following open covering of X: $\{U_a\}_{a \in A} \cup \{X - A\}$ where U_a satisfies $U_a \cap (A - \{a\}) = \emptyset$, i.e the element of A contained in U_a is only just a. Since X is compact, there is a finite subcovering $\{U_{a_1}, \cdots, U_{a_m}, X - A\}$. However the union of these can contain only finitely many elements of A which contradict to the assumption that A is an infinite subset. Thus there must be an cluster point of A in X. □

Theorem 4.25 (28.2 [Mun])**.** *If X is a metrizable topological space, then all three compactness of X are equivalent.*

Proof.

(i) **Compact ⇒ Limit point compact**: the previous theorem.

(ii) **Limit point compact ⇒ Sequentially compact**: Let $x_n, n \in \mathbb{Z}_{>0}$ be a sequence of points in X. If $A := \{x_n\}$ is a finite subset of X, then there are infinitely many x_n's that are the same point $x \in X$. Then the subsequence $\{x_n \mid x = x_n\} \subset \{x_n\}$ trivially converges to x. If A is an infinite subset, there is a cluster point x of A in X, i.e. $x \in \overline{A - \{x\}}$. By Theorem 2.16, every neighborhood of x intersects with $A - \{x\}$. Now consider the sequence of open balls around x, namely $B_d(x, 1) \supset B_d(x, 1/2) \supset B_d(x, 1/3) \supset \cdots$. Each open ball must contain an element of $A - \{x\}$, say x_{n_i}. The subsequence $\{x_{n_i}, i = 1, 2, \cdots\} \subset \{x_n\}$ converges to x apparently.

(iii) **Sequentially compact ⇒ Compact**:

(1) Given $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

Proof by contradiction: suppose that there is $\epsilon > 0$ such that there is no finite covering by ϵ-balls. We will find a sequence that doesn't have a convergent subsequence. Choose $x_1 \in X$. $B_d(x_1, \epsilon)$ doesn't cover X, so we can find $x_2 \in X - B_d(x_1, \epsilon)$. Then we can still find $x_3 \in X - B_d(x_1, \epsilon) - B_d(x_2, \epsilon)$, and in general, we can find $x_{n+1} \in X - (\cup_{i=1}^{n} B_d(x_n, \epsilon))$, the finite collection $\{B_d(x_1, \epsilon), \cdots, B_d(x_n, \epsilon)\}$ never covers X. Thus we have a sequence $\{x_n, n \in \mathbb{Z}_{>0}\}$. This sequence doesn't have a convergent subsequence because any $\epsilon/2$-ball can contain at most one of x_n's.

(2) Let $\mathcal{A} := \{U_\alpha\}$ be an open covering of X. Then by the Lebesgue number theorem, there is $\delta > 0$ such that each open set of diameter less than δ is contained in one of open sets in $\mathcal{A}$. Let $\epsilon := \delta/3$. By (1), we can cover X by finitely many ϵ-balls $B_1, \cdots, B_m$. Each open ball B_i has diameter $2\delta/3$ so it is less than δ, therefore there is an open set U_i in $\mathcal{A}$. Thus the finite subcollection $\{U_1, \cdots, U_m\}$ covers X.

□

Lemma 4.26 (§27 [Mun] Lebesgue number lemma)**.** *Let X is a sequentially compact metric space with metric d. Then for any open covering $\mathcal{A} := \{U_\alpha\}$ of X, there is a positive number $\delta > 0$ such that*

$\star$ *for each subset A of X with $d(A) < \delta$, there is an open set $U_\alpha \in \mathcal{A}$ such that $A \subset U$.*

*The **diameter** $d(A)$ of a subset A is defined by the least upper bound of the set $\{d(a_1, a_2) \mid a_1, a_2 \in A\} \subset A$.*

Proof. Proof by contradiction. Let $\{U_\alpha\}$ be an open covering of X such that there is no $\delta > 0$ which satisfy $\star$. So let C_n be a subset of diameter less than $1/n$ that is not contained in any of $\{U_\alpha\}$. Let $\{x_n, x_n \in C_n\}$ be a sequence. Since X is sequentially compact, there is a subsequence $\{x_{n_i}\}$ which converges to a point $a \in X$. Since $\{U_\alpha\}$ is an open cover, there is U_α containing a and there is $B_d(a, \epsilon)$ such that $a \in B_d(a, \epsilon) \subset U_\alpha$. Since $\{x_{n_i}\}$ converges to a, we can find a large n_i such that $x_{n_i} \in C_{n_i} \subset B_d(x_{n_i}, \epsilon/2) \subset B_d(a, \epsilon) \subset U_\alpha$ which contradict the assumption that C_{n_i} is not in any of $\{U_\alpha\}$. □

4.5 Completeness of metric and Theorem 4.11

Definition 4.27. Let (X, d) be a metric space. A sequence $(x_n, n \in \mathbb{Z}_{>0})$ is a ***Cauchy sequence*** in (X, d) if

$$\text{(Cauchy)} \qquad \forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0} \text{ such that } d(x_n, x_m) < \epsilon, \forall n, m > N.$$

A metric space (X, d) is ***complete*** if every Cauchy sequence in X converges.

Note that in a metric space, every convergent sequence must be Cauchy because of the definition of a convergent sequence (Definition 3.6).

Definition 4.28. A metric space (X, d) is ***totally bounded*** if $\forall \epsilon > 0$, there is a finite covering of X by ϵ-balls.

Theorem 4.29 (45.1, [Mun])**.** *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Corollary 4.30. *Let (X, d) be a complete metric space. A subset A is compact if and only if A is closed and A is totally bounded.*

Proof. If A is compact, then since X is Hausdoff, A is closed. It is bounded by the same argument as in the proof of Theorem 4.11. On the other hand, if A is closed, then the induced metric is complete by Lemma 4.31. Thus by Theorem 4.29. ◻

Lemma 4.31 (p.269 [Mun])**.** *Let (X, d) be a complete metric space. The induced metric on a closed set A is complete.*

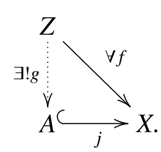
Example 4.32.

- Let A be a subset of $\mathbb{R}^n$ which is bounded w.r.t. the Euclidean metrix d. Then (A, d) is a metric space (with the restriction of d) and it is totally bounded. Let $M > 0$ be the number which bounds A, i.e. $d(x, y) < M$ for all $x, y \in A$. Then there is a box $[a, b]^n$ where $b - a = M$ that contains A. Take arbitrary $\epsilon > 0$. We can divide this box into a finite number of small boxes such that each can be contained in an epsilon ball. Thus A can be covered by finitely many ϵ-balls.
- The standard metric on $\mathbb{R}^n$ is complete (Theorem 43.2 [Mun]).

4.6 Universal properties of maps and induced topologies

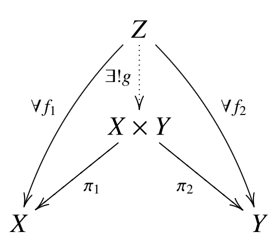
Theorem 4.33. *The induced topologies have universal properties:*

- *Let $j : A \to X$ be an injective map of sets. For every map $f : Z \to X$ such that $\operatorname{Im} f \subset \operatorname{Im} j$, there is the unique map $g : Z \to A$ such that $j \circ g = f$:*



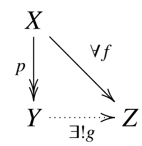
  If X and Z are topological spaces and f is continuous, then the subspace topology on A makes g continuous.

- *For every pair of maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ of sets, there exists the unique map $g : Z \to X \times Y$ such that $\pi_1 \circ g = f_1$ and $\pi_2 \circ g = f_2$ where π_1 and π_2 are the projections to the first and the second factors:*



If X, Y, Z are topological spaces and f_1, f_2 continuous, then the product topology on $X \times Y$ makes g continuous.

- *Let $p : X \to Y$ be a surjective map of sets. For every map $f : X \to Z$ that is constant on $p^{-1}(y)$ for each $y \in Y$, there is the unique map $g : Y \to Z$ such that $g \circ p = f$:*



If X, Z are topological spaces and f is continuous, then the quotient topology on Y makes g contituous.

5 Connected spaces

5.1 Connected spaces: the concept

Definition 5.1. A topological space X is ***connected*** if X is *not* a disjoint union of non-empty open sets.

Lemma 5.2. *The following are equivalent*

(0) X is not a disjoint union of non-empty open sets

(1) X is not a disjoint union of non-empty closed sets

(2) The only subsets of X that are both open and closed are $\varnothing$ and X.

Proof.

(0)$\Rightarrow$(2) If U_1 is open and closed but not $\varnothing$ and not X, then $U_2 := X - U_1$ is open and closed and not $\varnothing$. Thus we find the disjoint union $X = U_1 \sqcup U_2$ by open sets.

(0)$\Leftarrow$(2) If $X = U_1 \sqcup U_2$ with $U_1, U_2 \neq \varnothing$, i.e. not connected, then $U_1 = X - U_2$ is closed. Thus U_1 is open and closed which is not $\varnothing$ and not X since $U_2 \neq \varnothing$.

(0)$\Leftrightarrow$(1) If $X = U_1 \sqcup U_2$ and U_1 and U_2 are non-empty and open, then U_1 are U_2 are nonempty and closed. Similarly $X = C_1 \sqcup C_2$ and C_1 and C_2 are nonempty and closed, then C_1 are C_2 are nonempty open. □

Example 5.3.

1. A set X with more than one element is not connected if we put the discrete topology.

2. $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology induced from the standard topology is not connected as follows. Let $\alpha \in \mathbb{R}$ be an irrational number. Then (α, ∞) and $(-\infty, \alpha)$ are open sets of $\mathbb{R}$ and so $U_1 := \mathbb{Q} \cap (\alpha, \infty)$ and $U_2 := \mathbb{Q} \cap (-\infty, \alpha)$ are open sets of $\mathbb{Q}$. Since $\alpha \notin \mathbb{Q}$, $\mathbb{Q} = U_1 \sqcup U_2$.

3. $\mathbb{Z} \subset \mathbb{R}$ with the subspace topology induced from the finite complement topology is connected! We can not have that $\mathbb{Z} = C_1 \sqcup C_2$ where C_1 and C_2 are nonempty closed sets, since then C_1 or C_2 must be an infinite set which can not be so (all closed sets are finite by definition). Likewise, $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology induced from the finite complement topology is connected!

4. We can generalize 3 to an infinite set with f.c topology.

5.2 Connected spaces: basic propeties

Again, it is not so easy to show some space is connected, like compact spaces. We need to find basic facts and theorems with which we can show a space is connected.

Theorem 5.4 (23.5 [Mun])**.** *The image of a connected space under a continuous map is connected. In particular, if $X \cong Y$ and X is connected, then Y is also connected.*

Proof. Let $f : X \to Y$ be a continuous map and X is connected. Since $f : X \to f(X)$ is cotinuous, we can assume f is surjective without loss of generality. Suppose Y is not connected, i.e. $Y = U_1 \sqcup U_2$ and $U_1, U_2 \not\varnothing$. Then $X = f^{-1}(Y) = f^{-1}(U_1) \sqcup f^{-1}(U_2)$. Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open and non-empty. □

Theorem 5.5 (23.2 [Mun])**.** *If $X = U_1 \sqcup U_2$ for some non-empty open sets and Y is a connected subspace, then $Y \subset U_1$ or $Y \subset U_2$.*

Proof. If not, $Y \cap U_1$ and $Y \cap U_2$ are both non-empty. Thus $Y = (Y \cap U_1) \sqcup (Y \cap U_2)$. Each $Y \cap U_i$ is open and non-empty, so we have a contradiction to the fact that Y is connected. □

Theorem 5.6 (23.3 [Mun])**.** *Arbitrary union of connected subspaces that have a common point of a topological space X is connected.*

Proof. Let $\{A_\alpha\}$ is a collection of connected subspaces of X. Let $a \in \bigcap_\alpha A_\alpha$ be a common point. Suppose $A := \bigcup_\alpha A_\alpha$ is not connected, i.e. $A = U_1 \sqcup U_2$ where U_1, U_2 are non-empty open in A. If $a \in U_1$, $A_\alpha \subset U_1$ for all α by Theorem 5.5. Thus $A \subset U_1$ which contradict the assumption that U_2 is non-empty. The case $a \in U_2$ is similar. So A must be connected. □

Theorem 5.7. *A finite cartesian product of connected spaces is connected.*

Proof. Let X and Y be connected and let $(a, b) \in X \times Y$. We have $X \times \{b\} \cong X$ and $\{a\} \times Y \cong Y$, so by Theorem 5.4, they are connected. Now consider, for each $x \in X$,

$$C_x := (X \times \{b\}) \cup (\{x\} \times Y).$$

C_x is connected because it is a union of connected subspaces that have a common point $x \times b$ by Theorem 5.6. Then $X \times Y = \cup_x C_x$ and $(a, b) \in C_x$ for all $x \in X$. So again by Theorem 5.6, $X \times Y$ is connected. □

Theorem 5.8. *If A is a connected subspace of a topological space X, then the closure $\bar{A}$ is also a connected subspace. Moreover, any subset B such that $A \subset B \subset \bar{A}$ is a connected subsapce.*

Proof. Suppose B is not connected, i.e. $B = U_1 \sqcup U_2$ for some non-empty open sets U_1, U_2. By Theorem 5.5, $A \subset U_1$ or $A \subset U_2$. We can assume $A \subset U_1$ without loss of generality. Since U_2 is non-empty, $\exists x \in U_2 \subset \bar{A}$. By Theorem 2.16, $x \in \bar{A}$ iff $\forall U_x$, $U_x \cap A = \emptyset$. Since $x \in U_2$ and $U_2 \cap A = \emptyset$, we have a contradiction. □

Exercise 5.9. Let $p : X \to Y$ be a quotient map. Show that, if $p^{-1}(y)$ is connected for each $y \in Y$ and Y is connected, then X is connected.

5.3 $\mathbb{R}$ is connected

Theorem 5.10. $\mathbb{R}$ *is connected.*

Proof. Suppose that $\mathbb{R} = A \sqcup B$ where A and B are non-empty open sets. Let $a \in A$ and $b \in B$. Consider

$$[a, b] = ([a, b] \cap A) \sqcup ([a, b] \cap B) =: A_0 \sqcup B_0.$$

Since A_0 is bounded by b, there is $c := \sup A_0$.

- Case $c \in B_0$. Since B_0 is open in $[a, b]$, there is $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset B_0$ if $c < b$ or $(c - \epsilon, b] \subset B_0$ if $c = b$. In either case, since c bounds A_0, $(c - \epsilon, b] \subset B_0$ which contradict that c is the smallest bounding A_0.

- Case $c \in A_0$. Since A_0 is open in $[a, b]$, there is $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset A_0$ if $a < c$ or $[a, c + \epsilon) \subset A_0$ if $a = c$. This contradicts with c bounding A_0.

□

Example 5.11. $(a, b) \subset \mathbb{R}$ is connected. $[a, b], (a, b], [a, b)$ are all connected. $\mathbb{R}^n$ is connected. All open/closed boxes are connected. S^1 is connected. A ***torus*** $S^1 \times S^1$ is connected.

Proof. These follows directly from the same arguments in Theorem 5.10, but we can also derive them from the basic properties of connected spaces. (a, b) is connected since it is homeomorphic to $\mathbb{R}$. Then it follows from Theorem 5.8 that $(a, b), (a, b], [a, b)$ are all connected. $\mathbb{R}^n$ and all open/closed boxes are connected by Theorem 5.7. S^1 is connected since it is the image of a continuous map $f : [0, 2\pi] \to S^1 (x \mapsto (\cos x, \sin x))$. By Theorem 5.7, $S^1 \times S^1$ is connected. □

5.4 Path connected

Definition 5.12. A topological space X is ***path-connected*** if for each $x, y \in X$, there is a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. This map f is called a ***path*** from x to y.

Theorem 5.13. *If a topological space X is path-connected, then it is connected.*

Proof. Suppose that X is not connected, i.e. $X = U_1 \sqcup U_2$ for non-empty open sets U_1, U_2. Since $[0,1]$ is connected by Example 5.11, if there is a path $f : [0,1] \to X$, then $[0,1] \subset C$ or $[0,1] \subset D$. So if we take $x \in C$ and $y \in D$, then there is no path from x to y, i.e. X is not path-connected. ◻

Example 5.14 (disks)**.** The unit n-disk D^n in $\mathbb{R}^n$ by the equation

$$D^n := \{\vec{x} := (x_1, \cdots, x_n) \in \mathbb{R}^n \mid \|\vec{x}\| := \sqrt{x_1^2 + \cdots + x_n^2} \leq 1\}.$$

Then D^n is path-connected and so by Theorem 5.13.

Proof. For given $\vec{x}, \vec{y} \in D^n$, there is a path $f : [0,1] \to \mathbb{R}^n$ defined by

$$f(t) := (1-t)\vec{x} + t\vec{y}. \tag{5.1}$$

We can show that this path stays inside of D^n:

$$\|f(t)\| = \|(1-t)\vec{x} + t\vec{y}\,\| \leq \|(1-t)\vec{x}\,\| + \|\,t\vec{y}\,\| = (1-t)\|\,\vec{x}\,\| + t\|\,\vec{y}\,\| \leq (1-t)1 + t \cdot 1 = 1$$

The inequality follows from the ***triangle inequality*** of the standard metric $\mathbb{R}^n$. ◻

Example 5.15 (open balls)**.** The unit open n-balls B^n is given by

$$B^n := \{\vec{x} := (x_1, \cdots, x_n) \in \mathbb{R}^n \mid \|\vec{x}\| := \sqrt{x_1^2 + \cdots + x_n^2} < 1\}$$

It is path-connected and so it is connected. Since $\overline{B^n} = D^n$, this also implies that D^n is connected.

Proof. The proof is similar as in the case of n-disks. Define a path from $\vec{x}$ to $\vec{y}$ by

$$f(t) := (1-t)\vec{x} + t\vec{y}.$$

Then

$$\|f(t)\| = \|(1-t)\vec{x} + t\vec{y}\,\| \leq \|(1-t)\vec{x}\,\| + \|\,t\vec{y}\,\| = (1-t)\|\,\vec{x}\,\| + t\|\,\vec{y}\,\| < (1-t)1 + t \cdot 1 = 1.$$

◻

Remark 5.16. The closure of a connected space is connected, but the closure of a path connected space may not be path connected (Example 5.18). So showing open balls are path connected (Example 5.15) is not sufficient to show n-disks are path connected.

Lemma 5.17 (Exercise)**.** *If $f : X \to Y$ is a continuous map and X is path-connected, then* Im f *is path-connected.*

Example 5.18 (Topologist's sine curve)**.** We give an example that is connected but not path-connected. Consider the following subset of $\mathbb{R}^n$:

$$S := \{(s, \sin(1/s)) \in \mathbb{R}^2 \mid 0 < s \leq 1\}.$$

It is connected because it is the image of the connected space $(0,1]$ under a continuous map. It is not so difficult to show that the closure $\bar{S}$ is the union of S and $I := \{0\} \times [-1,1]$. By Theorem 5.8, $\bar{S}$ is connected. Below we show that $\bar{S}$ is not path-connected.

- Assume that there is a path $f : [0,1] \to \bar{S}$ which connects the origin o and some point $p \in S$. Since $f^{-1}(I)$ is a closed set in $[0,1]$, we find $b := \sup f^{-1}(I) \in f^{-1}(I)$ by Lemma 5.19 and the restriction $f|_{[b,1]} : [b,1] \to \bar{S}$ is continuous and satisfies $f(b) = o$ and $f(t) \subset S$ for all $t \in (b,1]$. Since $[b,1] \cong [0,1]$, we have a path $g : [0,1] \to \bar{S}$ such that $g(0) = o$ and $g(t) \subset S$ for $t \in (0,1]$.

- Denote $g(t) = (x(t), y(t))$ where $y(t) = \sin(1/x(t))$. For a given n, we can find u such that $0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$: Consider the sequence of points on S

$$\left(\frac{1}{\frac{\pi}{2} + n\pi + 2m\pi}, \sin\left(\frac{\pi}{2} + n\pi + 2m\pi\right)\right) = \left(\frac{1}{\frac{\pi}{2} + n + 2m\pi}, (-1)^n\right), n, m \in \mathbb{Z}_{>0}$$

By taking m large enough, $u_n := u_{n,m} = \frac{1}{\frac{\pi}{2}+n+2m\pi}$ is less that $x(1/n)$ since $0 < x(1/n)$ (use the fact that $u_{n,m}, m \in \mathbb{Z}_{>0}$ converges to 0).

- By $x : [0,1] \to \mathbb{R}$ is continuous, the intermediate value theorem implies that there is t_n such that $0 < t_n < 1/n$ and $x(t_n) = u_n$. Then t_n converges to 0.

- However $(f(t_n), \sin(1/f(t_n))) = (u_n, \sin(1/u_n)) = (u_n, (-1)^n)$ does not converge to the origin. This contradicts with the continuity of g by Lemma 3.18.

Lemma 5.19. *Every closed set A of $[0,1]$ contains $b := \sup A$, i.e. $b \in A$ such that $a \leq b$ for all $a \in A$.*

Proof. Since A is bounded above by 1, there is $b := \sup A$. If $b \notin A$, then there is a small open interval $I_b := [b - \epsilon, b + \epsilon]$ such that $A \cap I_b$ by the fact that $A = \bar{A}$ and Theorem 2.16. Then $b - (\epsilon/2)$ also bounds A, so it contradict with that fact that b is the supremum of A. □

Example 5.20 (unit n-sphere)**.** Define the ***unit n-sphere*** S^n in $\mathbb{R}^{n+1}$ by

$$S^n := \{\vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| = 1\}$$

If $n \geq 1$, it is path-connected. To show the path-connectedness, consider the continuous surjective map

$$g : \mathbb{R}^{n+1} - \{\vec{0}\}, \quad g(\vec{x}) := \frac{\vec{x}}{\|\vec{x}\|}.$$

Observe that $\mathbb{R}^{n+1} - \{\vec{0}\}$ is path connected: for every two points $\vec{x}, \vec{y} \in \mathbb{R}^{n+1} - \{0\}$, there is always a third point $\vec{z} \in \mathbb{R}^{n+1} - \{0\}$ such that the straight lines (defined by the equation (5.1)) from $\vec{x}$ to $\vec{z}$ and from $\vec{z}$ to $\vec{y}$ that don't go through the origin. Concatenating these lines, we have a path from $\vec{x}$ to $\vec{y}$. Now the path-connectedness of S^n follows from 5.17.

5.5 Components, locally connected and locally path-connected

Definition 5.21. Let X be a topological space. A ***connected component*** of X is an equivalence class of the equivalence relation $\sim$ given by

$$x \sim y \quad \text{if there is a connected subspace containing } x \text{ and } y.$$

A ***path-connected component*** of X is an equivalence class of the equivalence relation $\sim$ given by

$$x \sim y \quad \text{if there is a path connecting } x \text{ and } y.$$

Example 5.22. Let S be the topologist's sine curve defined in Example 5.18. $\bar{S}$ is connected but not path-connected. So the connected component is all of $\bar{S}$ but the path-connected components are $\{0\} \times [-1,1]$ and S.

Definition 5.23. Let X be a topological space.

- X is ***locally connected*** if for every point $x \in X$ and every neighborhood U of x, there is a connected neighborhood V of x contained in U.
- X is ***locally path-connected*** if for every point $x \in X$ and every neighborhood U of x, there is a path-connected neighborhood V of x contained in U.

Example 5.24. Here are examples which show that (path-)connectedness and local (path-)connectedness don't imply each other.

(a) $\mathbb{R} - \{0\}$ is not connected but locally connected. It is not path-connected but locally path-connected.

(b) The topologists sine curve is connected but not locally connected. Take an open ball B of radius less than 1 centered at the origin. Every open set in B that contains the origin is not connected.

(c) Consider the subspace $X := \{(x, 1/n) \in \mathbb{R}^2 \mid x \in \mathbb{R}, n \in \mathbb{Z}_{>0}\} \cup (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ of $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. It is also connected but not locally connected.

6 Topological manifolds and embedding into $\mathbb{R}^N$

6.1 Topological Manifolds §7, 30, 32, 33, 36 [Mun]

In this section, we introduce the nice topological spaces which underlies the most of the geometry, called the topological manifolds. The algebraic topology we study has a well-developed theory on the topological manifolds. The main goal of this section is to prove that any topological manifold can be topologically embedded in $\mathbb{R}^N$ for some large N. In the proof, the partition of unity plays a key role.

Definition 6.1 (§7,30)**.** A topological space X has a ***countable basis*** if there is a basis $\mathcal{B}$ for the topology that has only countably many open sets, i.e. $\mathcal{B} = \{U_n \mid n \in \mathbb{Z}_{>0}\}$.

Definition 6.2 (§36)**.** A **topological m-manifold** is a Hausdorff space X with a countable basis such that each point $x \in X$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^m$.

Definition 6.3 (§36)**.**

- For a function $\varphi : X \to \mathbb{R}$, the ***support*** of φ is defined by the closure of the preimage of $\mathbb{R} - \{0\}$:

$$\mathrm{Supp}(\varphi) := \overline{\varphi^{-1}(\mathbb{R} - \{0\})}.$$

- Let $\{U_1, \cdots, U_n\}$ be a finite open covering of a topological space X. Then the collection of functions

$$\varphi_i : X \to [0, 1], \quad i = 1, \cdots, n$$

  is a ***partition of unity*** associated to the covering $\{U_i\}$ if

  (i) $\mathrm{Supp}(\varphi_i) \subset U_i$ for all $i = 1, \cdots, n$.

  (ii) $\sum_{i=1}^n \varphi_i(x) = 1$ for each $x \in X$.

Lemma 6.4 (§32)**.** *A compact Hausdorff space X satisfies the following condition (***normality***):*

For every disjoint closed subsets A and B, there are open sets $U_A \supset A$ and $U_B \supset B$ such that $U_A \cap U_B = \emptyset$.

Proof. By Theorem 4.5, A and B are compact. For every $a \in A$, there are open sets U_a containing a and V_a containing B: for each b take disjoint open sets U_b containing a and V_b containing b, then $\{V_b\}$ covers B so take finite subcollection $\{V_{b_i}\}_{i=1,\cdots,n}$. Then the union $V_a := \cup_{i=1}^n V_{b_i}$ containing B and the intersection $U_a := \cap_{i=1}^n U_{b_i}$ are the desired open sets. Now collect U_a's to form an open covering of A. We can make it into a finite collection $\{U_{a_i}\}_{i=1,\cdots,m}$. Then $\cup_{i=1}^m U_{a_i}$ is an open set containing A which is disjoint from the open set $\cap_{i=1}^m V_{a_i}$ which contains B. □

Lemma 6.5 (Urysohn Lemma §33)**.** *Let X be a topological space that satisfies the normality. Let A and B are disjoint closed subsets. Then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Theorem 6.6. *Let X be a compact Hausdorff space and $\{U_i\}_{i=1,\cdots,n}$ a finite open covering. Then there is a partition of unity associated to $\{U_i\}$.*

Proof. (1) There is an open covering $\{V_i\}_{i=1,\cdots,n}$ of X such that $\bar{V}_i \subset U_i$: first, apply the normality of X to the disjoint closed subsets

$$A := X - (U_2 \cup \cdots \cup U_n), \quad B := X - U_1.$$

Let U_A and U_B be the open sets separating A and B. Let $V_1 := U_A$. Then $\{V_1, U_2, \cdots, U_n\}$ covers X and $\bar{V}_1 \subset U_2$. The next, apply the normality again for

$$A := X - V_1 - (U_3 \cup \cdots \cup U_n), \quad B := X - U_2.$$

Similary let $V_2 := U_A$. We have a cover $\{V_1, V_2, U_3, \cdots, U_n\}$ such that $\bar{V}_1 \subset U_1$ and $\bar{V}_2 \subset U_2$. Similarly for V_3, apply the normality for

$$A := X - (V_1 \cup V_2) - (U_4 \cup \cdots \cup U_n), \quad B := X - U_3.$$

We can keep doing these steps to replace all U_i's by desired V_i's.

(2) Apply (1) to $\{V_i\}$ again to obtain another open covering $\{W_i\}$ such that $\bar{W}_i \subset V_i$. Using the Urysohn's lemma, we find functions

$$\psi_i : X \to [0,1] \quad \text{such that } \psi(\bar{W}_i) = \{1\} \quad \text{and} \quad \psi(X - V_i) = \{0\}.$$

Observe that

$$\mathrm{Supp}(\psi_i) \subset \bar{V}_i \subset U_i.$$

Since $\{W_i\}$ is a covering, $\Psi(x) := \sum_{i=1}^n \psi_i(x) > 0$ for all $x \in X$. Define

$$\varphi_i(x) := \frac{\psi_i(x)}{\Psi(x)}.$$

Then

$$\sum_{i=1}^n \varphi_i(x) = \frac{1}{\Psi(x)} \sum_{i=1}^n \psi_i(x) = 1$$

and

$$\mathrm{Supp}(\varphi_i) = \mathrm{Supp}(\psi_i) \subset \bar{V}_i \subset U_i.$$

□

Theorem 6.7. *If X is a compact topological m-manifold, then X can be topologically embedded in $\mathbb{R}^N$ for some positive integer N.*

Proof.

- From the definition of manifolds and the compactness, we can find a finite open cover $\{U_i\}_{i=1}^n$ together with imbeddings $g_i : U_i \hookrightarrow \mathbb{R}^m$ (homeomorphism to the image).
- Since X is compact and Hausdorff, it satisfies the normality condition and hence we find a partition of unity $\{\varphi_i : X \to [0,1]\}$ associted to $\{U_i\}$. Let $A_i := \mathrm{Supp}(\varphi_i)$.
- Define functions $h_i : X \to \mathbb{R}^m$ by

$$h_i(x) = \begin{cases} \varphi_i(x) \cdot g_i(x) & \text{if } x \in U_i \\ (0, \cdots, 0) & \text{if } x \in X - A_i. \end{cases}$$

  It is well-defined continuous functions (See Lemma 3.27).
- The embedding of X into some $\mathbb{R}^N$ is

$$F : X \to \underbrace{\mathbb{R} \times \cdot \times \mathbb{R}}_{n\ times} \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{n\ times}$$

  given by

$$F(x) := (\varphi_1(x), \cdots, \varphi_n(x), h_1(x), \cdots, h_n(x)).$$

  It is continuous by Lemma 3.28. If F is injective, then $F : X \to \mathrm{Im}\, F$ is a continuous bijection. Therefore by Theorem 4.9, it must be a homeomorphism.

- Suppose $F(x) = F(y)$ so that $\varphi_i(x) = \varphi_i(y)$ and $h_i(x) = h_i(y)$ for all i. Since $\sum_i \varphi_i(x) = 1$, $\varphi_i(x) > 0$ for some i which also implies $\varphi_i(y) > 0$. Thus $x, y \in U_i$. Now dividing $h_i(x) = h_j(y)$ by the positive number $\varphi_i(x) = \varphi_i(y)$, we get $g_i(x) = g_i(y)$. Since g_i is injective, we have $x = y$.

□

Remark 6.8. The claim still holds even if a manifold X is not compact (p.225 [Mun]). If a manifold X is not compact, we can't use Lemma 6.4. But to find the partition of unity, we need to show X is normal. We can show that a manifold X satisfies the *regularity*, i.e. open sets can separate a point and a closed set. Then together with the second-countability, i.e. there is countably basis, we can prove that X satisfies normality (Theorem 32.1 [Mun]). This explains why we include the second-countability condition in the definition of manifolds.

7 Group theory

Groups are very simple algebraic objects. It is a set with binary operations with very symmetric structures. Well, it is the algebraic structure to keep track of symmetry of some geometric spaces after all.

7.1 Definition of groups and homomorphisms

Definition 7.1. A ***group*** $(\mathsf{G}, \cdot)$ is a set G together with a map $\mathsf{G}\times\mathsf{G} \to \mathsf{G}, (a, b) \mapsto a\cdot b$, called the *multiplication*, satisfying

- (Associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathsf{G}$.
- (Identity) There is an element $e \in \mathsf{G}$ such that $a \cdot e = e \cdot a = a$ for all $a \in \mathsf{G}$.
- (Inverses) For each $a \in \mathsf{G}$, there an element $a^{-1} \in \mathsf{G}$ such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

A ***subgroup*** H of a group $(\mathsf{G}, \cdot)$ is a subset H of G such that the restriction $\mathsf{H} \times \mathsf{H} \to \mathsf{H}, (a, b) \mapsto a \cdot b$ makes H a group. we write $\mathsf{H} \leq \mathsf{G}$. A group G is called ***Abelian*** if the multiplication is *commutative*, i.e. $a \cdot b = b \cdot a$ for all $a, b \in \mathsf{G}$.

Definition 7.2. A map $\phi : \mathsf{G} \to \mathsf{H}$ between groups is a ***group homomorphism*** if

$$\phi(ab) = \phi(a)\phi(b).$$

If it is bijective, then the inverse map is automatically a group homomorphism (unlike the continuous maps!). We need to show that $\phi^{-1}(x)\phi^{-1}(y) = \phi^{-1}(xy)$:

$$\phi(\phi^{-1}(x)\phi^{-1}(y)) = \phi(\phi^{-1}(x))\phi(\phi^{-1}(y)) = xy = \phi(\phi^{-1}(xy)).$$

Since ϕ is injective, it follows that $\phi^{-1}(x)\phi^{-1}(y) = \phi^{-1}(xy)$.

7.2 Examples

Example 7.3. $(\mathbb{R}, +)$ and $(\mathbb{R}^\times := \mathbb{R} - \{0\}, \times)$ are groups. Then $\mathbb{Z}, \mathbb{Q} \leq (\mathbb{R}, +)$ and $\mathbb{Q}^\times \leq (\mathbb{R}^\times, \times)$. However, $\mathbb{Z} - \{0\}$ is not a subgroup of $\mathbb{R}^\times$ because the only elements of $\mathbb{Z} - \{0\}$ that are invertible are ± 1.

Example 7.4. Let X be a finite set. The set S_X of all bijections $X \to X$ is a finite group where the multiplication is given by the composition of maps. It is called a permutation group. If $X := \{1, \cdots, n\}$, the permutation group in this case is often denoted by S_n. If the cardinality of X is n, then $S_X \cong S_n$.

Example 7.5. Let X be a topological space. Then the set $\mathrm{Aut}(X)$ of all homeomorphisms from X to X itself is a group where the multiplication is given by compositions.

Example 7.6. Consider $U(1) = \{e^{i\theta} = \cos\theta + i \sin\theta \in \mathbb{C} \mid 0 \leq \theta < 2\pi\} \subset \mathbb{C}$. Then $U(1)$ is a subgroup of $(\mathbb{C}^\times, \times)$ with respect to the multiplication. Indeed, $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$. We can actaully see that there is a group homomorphism

$$\exp : (\mathbb{R}, +) \to U(1), \quad x \mapsto e^{ix}.$$

This is surjective but not injective. Moreover, the subset $\{e^{2\pi i\cdot\frac{k}{n}} \mid k = 0, 1, \cdots, n-1\}$ is a finite subgroup of $U(1)$. It is called a ***cyclic group of order*** n.

Example 7.7. Let $\mathrm{Mat}(n, \mathbb{R})$ be the set of all $n \times n$ matrices. It is not a group with the matrix multiplication. But inside of $\mathrm{Mat}(n, \mathbb{R})$, there are a bunch of groups:

$$\begin{aligned}
\mathrm{GL}(n, \mathbb{R}) &= \{M \mid \det M \neq 0\} \\
\mathrm{SL}(n, \mathbb{R}) &= \{M \mid \det M = 1\} \\
\mathrm{O}(n, \mathbb{R}) &= \{M \mid M \cdot M^t = I_n\} \\
\mathrm{SO}(n, \mathbb{R}) &= \{M \mid M \cdot M^t = I_n, \det M = 1\}
\end{aligned}$$

Similarly $\mathrm{Mat}(n,\mathbb{C})$ is not a group but inside there are a bunch:

$$\begin{aligned}\mathrm{GL}(n,\mathbb{C}) &= \{M \mid \det M \neq 0\}\\ \mathrm{SL}(n,\mathbb{C}) &= \{M \mid \det M = 1\}\\ \mathrm{U}(n) &= \{M \mid M\cdot \overline{M}^t = I_n\}\\ \mathrm{SU}(n) &= \{M \mid M\cdot \overline{M}^t = I_n, \det M = 1\}\end{aligned}$$

Example 7.8. A vector space V is a group with respect to the sum. So we can say a vector space is a group with more structures. The linear map $V \to W$ is a group homomorphism in this sense. Again, it has more structures. The general linear group $\mathrm{GL}(n,\mathbb{R})$ is exactly the set of all invertible linear maps from $\mathbb{R}^n \to \mathbb{R}^n$.

7.3 More definitions

The most basic concept is the concept of the normal subgroup. It is defined as the subgroups such that the quotient is naturally a group again. It is the important fact that the pre-image of the identity, called the kernel of a homomorphism, is normal.

Definition 7.9. Let H be a subgroup of a group G. Let G/H be the quotient of G be the following equivalence relation: $x \sim y$ if $y = xh$ for some h. For every $x \in \mathsf{G}$, the equivalence class of x is $x\mathsf{H} := \{xh \mid h \in \mathsf{H}\}$ and there is a bijection $H \to xH$ sending h to xh.

Lemma 7.10. *G/H has the induced group structure if $x\mathsf{H} \subset \mathsf{H}x$ for every $x \in \mathsf{G}$.*

Proof. The natural multiplication on $\mathsf{G}/\mathsf{H} = \{x\mathsf{H} \mid x \in \mathsf{G}\}$ is

$$x\mathsf{H}\cdot y\mathsf{H} = xy\mathsf{H}.$$

However, since $x' \in x\mathsf{H}$ implies that $x'\mathsf{H} = x\mathsf{H}$, we have to make sure that

$$xy\mathsf{H} = x'y'\mathsf{H} \quad \text{if} \quad x' \in x\mathsf{H} \quad \text{and} \quad y' \in y\mathsf{H}.$$

Since $x' = xh$ and $y' = yk$ for some $h, k \in \mathsf{H}$, the right hand side is

$$x'y'\mathsf{H} = xhyk\mathsf{H} = xhy\mathsf{H}$$

since $k\mathsf{H} = \mathsf{H}$. Now if $x\mathsf{H} \subset \mathsf{H}x, \forall x \in \mathsf{G}$, then for every $h \in \mathsf{H}$, $xh = h'x$ for some $h' \in \mathsf{H}$. So $hy = yh'$ for some $h' \in \mathsf{H}$. Thus

$$xhy\mathsf{H} = xyh'\mathsf{H} = xy\mathsf{H}.$$

This proves that the multiplication is well-defined. Now the identity is obviously $1\mathsf{H}$ and the inverse of $x\mathsf{H}$ is $x^{-1}\mathsf{H}$. Thus G/H is naturally a group. □

Definition 7.11. A ***normal subgroup*** of a group G is a subgroup N such that $x\mathsf{N} \subset \mathsf{N}x, \forall x \in \mathsf{G}$. If N is a normal subgroup, then G/N is a group and the quotient map $\mathsf{G} \to \mathsf{G}/\mathsf{N}$ is a group homomorphism.

Lemma 7.12. *Let $\phi : \mathsf{G} \to \mathsf{H}$ be a group homomorphism. The subset $\phi^{-1}(1_{\mathsf{H}}) \subset \mathsf{G}$ is a normal subgroup of G.*

Proof. It is a subgroup since $\phi(1_{\mathsf{G}}) = 1_{\mathsf{H}}$ and $\phi(g_1g_2) = \phi(g_1)\phi(g_2) = 1_{\mathsf{H}}1_{\mathsf{H}} = 1_{\mathsf{H}}$ for all $g_1, g_2 \in \phi^{-1}(1_{\mathsf{H}})$. To prove it is normal, we compute

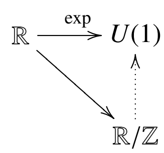
$$\phi(g\phi^{-1}(1_{\mathsf{H}})g^{-1}) = \phi(g)\phi(\phi^{-1}(1_{\mathsf{H}}))\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = 1_{\mathsf{H}}, \forall g \in \mathsf{G}.$$

This means that $g\phi^{-1}(1_{\mathsf{H}})g^{-1} \subset \phi^{-1}(1_{\mathsf{H}})$ for all $g \in \mathsf{G}$, which exactly means that $g\phi^{-1}(1_{\mathsf{H}}) \subset \phi^{-1}(1_{\mathsf{H}})g$ for all $g \in \mathsf{G}$. □

Definition 7.13. The pre-image of a group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ is called the ***kernel*** of ϕ, denoted by $\ker \phi := \phi^{-1}(1_{\mathsf{H}})$.

Example 7.14. If G is an abelian group, then every subgroup is a normal subgroup.

- $\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$ with respect to $+$. The quotient $\mathbb{R}/\mathbb{Z}$ is isomorphic to $U(1)$. To see that, consider the diagram:



  The exponential map in Example 7.6 factors through $\mathbb{R}/\mathbb{Z}$ and the resulting map f is a bijective homomorphism. So it is an isormophism as discussed in Definition 7.2.

- For a positive integer $p \in \mathbb{Z}_{>0}$, $p\mathbb{Z} := \{pn \mid n \in \mathbb{Z}\}$ is a normal group of $\mathbb{Z}$. The quotient $\mathbb{Z}/p\mathbb{Z} = \{i + p\mathbb{Z} \mid i = 0, 1, \cdots, p-1\}$ is a well-defined group. It is isomorphic to the cyclic group of order p defined in Example 7.6. Namely there is an isomorphism

$$\mathbb{Z}/p\mathbb{Z} \to \{e^{2\pi i \frac{k}{p}} \mid k = 1, \cdots, p-1\}, \quad k + p\mathbb{Z} \mapsto e^{2\pi i \frac{k}{p}}.$$

  We can check that it is well-defined (i doesn't depend of the choice of representative k) and it is a bijective homomorphism.

Example 7.15. A vector space is an abelian group with respect to the sum. A linear map is then a group homomorphism. Let $\phi : V \to W$ be a linear map. Then $\ker V$ is a normal subgroup of V and $V/\ker V$ is an abelian group again. Of course!

In the group theory, it is easy to check the injectivity of a homomorphism by the next lemma.

Lemma 7.16. *Let $\phi : \mathsf{G} \to \mathsf{H}$ be a group homomorphism. Then ϕ is injective if and only if* $\ker \phi$ *is trivial, i.e.* $\ker \phi = \{1_{\mathsf{G}}\}$.

Proof. If ϕ is injective, it is obvious that $\ker \phi = \{1_{\mathsf{G}}\}$. Suppose that $\ker \phi = \{1_{\mathsf{G}}\}$. Let $g_1, g_2 \in \mathsf{G}$ such that $\phi(g_1) = \phi(g_2)$. Then $\phi(g_1 g_2^{-1}) = 1_{\mathsf{H}}$, so $g_1 g_2^{-1} = 1_{\mathsf{G}}$, which implies $g_1 = g_2$ by multiplying g_2 from both sides. □

Corollary 7.17. *If $\phi : \mathsf{G} \to \mathsf{H}$ is a surjective homomorphism, then* $\mathsf{G}/\ker \mathsf{G}$ *is isomorphic to* H.

Proof. The homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ naturally factors through $\mathsf{G}/\ker \mathsf{G} \to \mathsf{H}$. This induced map has trivial kernel. Thus it is injective. Since ϕ is surjective, the induced map is a bijective homomorphism. Thus it is an isomorphism. □

7.4 Groups given by generators and relations

Definition 7.18. Let $\{g_1, \cdots, g_n\}$ be a finite set. The ***free group*** $\langle g_1, \cdots, g_n \rangle$ generated by $\{g_1, \cdots, g_n\}$ is the collection of *reduced* finite words in $\{g_1, \cdots, g_n, g_1^{-1}, \cdots, g_n^{-1}\}$, including the empty word denoted by 1. "reduced" means no element in word sits next to it's inverse.

Example 7.19. A free group generated by one element is isomorphic to $\mathbb{Z}$.

Definition 7.20. Let $\mathsf{G} := \langle g_1, \cdots, g_n \rangle$ be a free group. The list of relations in the free group is just a list of reduced words W:=$\{w_1, \cdots, w_k\}$ and we can form a normal subgroup by setting

$$\mathsf{N} := \text{ the smallest normal subgroup containing } W.$$

Then G/N is the group generated by $\{g_1, \cdots, g_n\}$ with the relations $\{w_1, \cdots, w_k\}$. We denote this group as

$$\langle g_1, \cdots, g_n \mid w_1 = w_2 = \cdots = w_k = 1 \rangle.$$

Example 7.21. The group

$$\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$$

is isomorphic to $\mathbb{Z}^2$.

7.5 Topological groups

Definition 7.22. A ***topological group*** G is a group G together with a Hausdorff topology such that

(i) $m : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, $(g, h) \mapsto gh$ is continuous.

(ii) $i : \mathsf{G} \to \mathsf{G}$, $g \mapsto g^{-1}$ is continuous.

Remark 7.23 (p.145 p.146 [Mun])**.** It is actually enough to assume that $\{1_{\mathsf{G}}\}$ is closed. We can prove from this assumption that G is Hausdorff. See Proposition 5.7, 5.8 [P].

Example 7.24. Every group can be considered as a topological group with a discrete topology.

Example 7.25. $(\mathbb{R}, +), (\mathbb{C}, +), (\mathbb{R}_{>0}, \times), (\mathbb{R}^\times, \times), (\mathbb{C}^\times, \times)$ are topological groups with the topology induced from the standard topology.

Example 7.26. $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are topological groups with respect to the subspace topology induced from the standard topologies of $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$. All the groups listed in Example 7.7 are topological groups with respect to the subspace topologies, by the following lemma.

Lemma 7.27. *Let* G *be a topological group. Every subgroup* $\mathsf{H} \leq \mathsf{G}$ *is a topological group with the subspace topology.*

Example 7.28. $(\mathbb{Z}, +), (\mathbb{Q}, +)$ are topological groups with respect to the group structure restricted from $(\mathbb{R}, +)$ and the subspace topologies. In particular, $\mathbb{Z}$ is a discrete group. $U(1)$ is a subgroup of $(\mathbb{C}^\times, \times)$ and so it is a topological group with respect to the subspace topology (it is homeomorphic to S^1.)

7.6 Group actions on topological spaces

Definition 7.29 (Ex.31.8 [Mun])**.** A (continuous) ***action*** of a topological group G on a topological space X is a continuous map $\rho : \mathsf{G} \times X \to X$, denoted by $\rho(g)x := g \cdot x$, such that

(i) $1_{\mathsf{G}} \cdot x = x, \forall x \in X$.

(ii) $(g_1 \cdot g_2) \cdot x = g_1 \cdot (g_2 \cdot x), \forall g_1, g_1 \in \mathsf{G}, \forall x \in X$.

An orbit of the G-action on X is a subset $O := \{g \cdot x \mid g \in \mathsf{G}\} \subset X$. The relation $\sim$ defined by

$$x \sim y \quad \text{if} \quad g \cdot x = y \quad \text{for some } g \in \mathsf{G}$$

is an equivalence relation and an equivalence class is nothing but an orbit. The quotient of X by the group G-action is the quotient space X/G of X defined by this relation.

Definition 7.30. Let $\mathrm{Homeo}(X)$ be the set of all homeomorphism $f : X \to X$. Then $f \cdot g := f \circ g$ defines a group structure on $\mathrm{Homeo}(X)$. Namely, the composition is a homeomorphism again. The identity is the identify map and the inverse of f in this group is the inverse as a map. If there is an action of a topological group G on X, then $\rho(g) : X \to X, x \mapsto gx$ is a homeomorphism and $\mathsf{G} \to \mathrm{Homeo}(X), g \mapsto \rho(g)$ is a group homomorphism.

Example 7.31. If G is a topological group and H a subgroup. There are two actions of H on G defined by $(h, g) \mapsto hg$ or $(h, g) \mapsto gh^{-1}$. The quotient spaces $\mathsf{H}\backslash\mathsf{G}$ and G/H are called a homogeneous spaces. If N is a closed and normal subgroup, then $\mathsf{G}/\mathsf{N} = \mathsf{N}\backslash\mathsf{G}$ and it has an induced group structure which makes G/N a topological group. See also Ex 5 p.146 [Mun].

Example 7.32. $\mathbb{Z}$ acts on $\mathbb{R}$ by $\mathbb{Z}\times\mathbb{R}\to\mathbb{R}, (n,x)\mapsto x+n$. The quotient $\mathbb{R}/\mathbb{Z}$ is homeomorphic to a circle S^1. $\mathbb{Z}\times\mathbb{Z}$ acts on $\mathbb{R}\times\mathbb{R}$ by $((n,m),(x,y))\mapsto(x+n,y+m)$ and the quotient $\mathbb{R}\times\mathbb{R}/\mathbb{Z}\times\mathbb{Z}$ is homeomorphic to a torus $S^1\times S^1$.

Example 7.33. $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}-\{\vec{0}\}$ by $\lambda\cdot(z_0,\cdots,z_n) := (\lambda z_0,\cdots,\lambda z_n)$. The quotient $\mathbb{C}^{n+1}-\{\vec{0}\}/\mathbb{C}^\times$ is the complex projective space denoted by $\mathbb{CP}^n$. Consider $S^{2n+1}\subset\mathbb{C}^{n+1}-\{\vec{0}\}$ which is the set of unit vectors in $\mathbb{C}^{n+1}$. Then the restriction of the quotient map to S^{2n+1} is surjective. There is an induced action of the subgroup $U(1)\subset\mathbb{C}^\times$ on S^{2n+1} and the quotient $S^{2n+1}/U(1)$ is exactly $\mathbb{CP}^n$.

Example 7.34. $\mathbb{R}^\times$ acts on $\mathbb{R}^{n+1}-\{\vec{0}\}$ by $\lambda\cdot(x_0,\cdots,x_n) := (\lambda x_0,\cdots,\lambda x_n)$. THe quotient $\mathbb{R}^{n+1}-\{\vec{0}\}/\mathbb{R}^\times$ is the real projective space denoted by $\mathbb{RP}^n$. Consider $S^n\subset\mathbb{R}^{n+1}-\{\vec{0}\}$ which is the set of unit vectors. Then the restriction of the quotient map to S^n is surjective. There is an induced action of the subgroup $\{1,-1\}\subset\mathbb{R}^\times$ on S^n and its quotient $S^n/\{1,-1\}$ is exactly $\mathbb{RP}^n$.

Example 7.35. There is an action of $\mathbb{R}^\times$ on $\mathbb{R}^2$ by $\lambda\cdot(x_1,x_2) := (x_1,x_2)$. The quotient $\mathbb{R}^2/\mathbb{R}^\times$ is not Hausdorff.

Example 7.36. Let G be $\mathrm{GL}(n,\mathbb{C})$. Let B be the closed subgroup of all upper triangular matrices in G. Consider the action of B on G defined by the left multiplication. The quotient space G/B is called the flag manifolds.

8 Homotopy of Paths §51 [Mun]

From this week, we venture into *algebraic topology*. "Algebraic topology is a branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence" ~ Wikipedia. The most fundamental algebraic object we extract from a space is a "group of loops". Here is how we should start: Let X be a path-connected space and fix a point $x_0 \in X$. Let $L(X, x_0)$ be the set of all paths $f : [0, 1] \to X$ from x_0 to x_0, i.e. *loops based at* x_0. Define a multiplication in $L(X, x_0)$ as follows: for loops f_1, f_2, define $f_1 \cdot f_2$ to be

$$(f_1 \cdot f_2)(x) := \begin{cases} f_1(2x) & x \in [0, 1/2] \\ f_2(2x-1) & x \in [1/2, 1] \end{cases}$$

This operation doesn't make $L(X, x_0)$ a group. For example, the associativity of the product fails. To produce a reasonable algebraic object out of this operation, we must pass it to the ***(path-)homotopy*** class. We actually start by defined this operation in the collection of all paths.

8.1 Homotopy and Path Homotopy equivalence

Definition 8.1 (p323 [Mun]). Let $f, g : X \to Y$ be continuous maps. f is ***homotopic*** to g, denoted by $f \cong g$, if there is a continuous map $F : X \times [0, 1] \to Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x).$$

This map F is called a ***homotopy*** between f and g.

Definition 8.2 (p.323 [Mun]). Let $f, g : \mathsf{I} \to X$ be paths from x to y where $\mathsf{I} := [0, 1]$. f is ***path-homotopic*** to g, denoted by $f \cong_p g$, if there is a homotopy $F : I \times [0, 1] \to X$ such that

$$F(0, s) = x \quad \text{and} \quad F(1, s) = y.$$

Remark 8.3. Let $A \subset X$ be a subset. A homotopy $F : X \times [0, 1] \to Y$ between some maps is said to be ***relative to*** A if $F(a, t)$ is independent of $t \in [0, 1]$ for each $a \in A$. A path-homotopy between paths f and g is nothing but a homotopy relative to $\partial\mathsf{I} = \{0, 1\}$.

Lemma 8.4 (51.1 [Mun]). *$\cong$ and $\cong_p$ are equivalence relations.*

Proof. Let $f, g, h : X \to Y$ be continuous maps. Let's use a temporary notation: $F : f \Rightarrow g$ is a homotopy $F : X \times [0, 1] \to Y$ from f to g, i.e. $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$

- (Reflexibity) The homotopy $F(x, t) := f(x)$ makes $F : f \Rightarrow f$.
- (Symmetry) If $F : f \Rightarrow g$, then $G(x, t) := F(x, 1-t)$ makes $G : g \Rightarrow h$.
- (Transitivity) If $F_1 : f \Rightarrow g$ and $F_2 : g \Rightarrow h$, then define

$$G(x, t) := \begin{cases} F(x, 2t) & t \in [0, 1/2] \\ F(x, 2t-1) & t \in [1/2, 1] \end{cases}$$

  By the pasting lemma, this is a well-defined $X \times [0, 1] \to Y$ and such that $G(x, 0) = f(x)$ and $G(x, 1) = h(x)$, so this makes $G : f \to h$.

These construction of homotopies for the axioms of an equivalence relation works for path homotopies, i.e. the construction preserves the relativeness of the homotopies. Thus $\cong_p$ is also an equivalence relation. □

Lemma 8.5 (Pasting lemma 18.3 [Mun]). *Let A and B be closed subsets of X and let $f : A \to Y$ and $g : B \to Y$ be continuous maps such that $f|_{A\cap B} = g|_{A\cap B}$. Then the map $h : A \cup B \to Y$ defined by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B$ is a wel-defined continuous map.*

The following are basic facts about homotopy and path homotopy of maps and paths into $\mathbb{R}^n$.

Lemma 8.6.

(1) Any two continuous maps $f, g : X \to \mathbb{R}^n$ are homotopic.

(2) A subspace $A \subset \mathbb{R}^n$ is ***convex*** *if the straight line segment between any $\vec{a}$ and $\vec{b}$ in A is contained in A. Any two path f, g in A from $\vec{x}$ to $\vec{y}$ are path homotopic.*

Proof.

(1) Define $F : X \times [0, 1] \to \mathbb{R}^n$ by

$$F(x, t) := (1 - t)f(x) + tg(x).$$

This is a continuous map because it is a composition of the following maps.

$$X \times [0,1] \xrightarrow[(f,g,\mathrm{id})]{} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow[(1-t)\vec{x}_1 + t\vec{x}_2]{} \mathbb{R}^n.$$

Since $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, it is a homotopy from f to g.

(2) The same homotopy defined in (1) works as a path homotopy. Define $F : \mathsf{I} \times [0, 1] \to \mathbb{R}^n$ by

$$F(s, t) := (1 - t)f(s) + tg(s).$$

For each $s = a$, $\{F(a, t) | t \in [0, 1]\}$ is a line segment from $f(a)$ to $g(a)$ so that $F(s, t) \in A$ for all $s, t \in \mathsf{I} \times [0, 1]$. Thus $F : \mathsf{I} \times [0, 1] \to A$ is a homotopy from f to g. By definition $F(s, 0) = \vec{x}$ and $F(s, 1) = \vec{y}$, it is indeed a path homotopy from f to g.

□

8.2 Definition of a product among paths and the homotopy invariance

Definition 8.7 (p.326 [Mun])**.** Let $f : \mathsf{I} \to X$ be a path from x_0 to x_1 and $g : \mathsf{I} \to X$ a path from x_1 to x_2. Define the product $f * g$ to be the path from x_0 to x_2 given by

$$(f * g)(s) := \begin{cases} f(2s) & \text{for } s \in [0, 1/2] \\ g(2s - 1) & \text{for } s \in [1/2, 1] \end{cases}.$$

By the pasting lemma, $f * g$ is a well-defined continuous function from $[0, 1]$ to X such that $(f * g)(0) = x_0$ and $(f * g)(1) = x_2$. So it is a path from x_0 to x_2.

Theorem 8.8 (51.2 [Mun])**.** *Let f, g, h be paths in X. Let $[f], [g], [h]$ be path-homotopy classes.*

(0) $*$ induces a well-defined product on path-homotopy classes of paths.

(1) (Associativity) $[f] * ([g] * [h]) = ([f] * [g]) * [h]$.

(2) (Identities) Let $e_x : [0, 1] \to X$ be a constant path. Then for every path $f : [0, 1] \to X$ from x to y, we have

$$[f] * [e_y] = [f] \quad \text{and} \quad [e_x] * [f] = [f].$$

(3) (Inverse) For every path $f : [0, 1] \to X$ from x to y, let $\bar{f}$ be a path given by $\bar{f}(s) := f(1 - s)$. Then

$$[f] * [\bar{f}] = [e_x] \quad \text{and} \quad [\bar{f}] * [f] = [e_y].$$

Proof.

(0) We need to show if $f \cong_p f'$ and $g \cong_p g'$, then $f * g \cong_p f' * g'$, i.e. $[f] * [g] = [f * g] = [f' * g'] = [f'] * [g']$ so that the product doesn't depend of the choice of the representatives.

For path-homotopies $F : f \Rightarrow_p f'$ and $G : g \Rightarrow_p g'$, define

$$H(s,t) := \begin{cases} F(2s,t) & s \in [0,1/2] \\ G(2s-1,t) & s \in [1/2,1] \end{cases}$$

Then $H : \mathsf{I} \times [0,1] \to X$ is a well-defined continuous map by the pasting lemma and $H(s,0) = (f * g)(s)$ and $H(s,1) = (f' * g')(s)$. Thus $H : f * g \Rightarrow f' * g'$.

(2) Consider paths in I: $e_0 : \mathsf{I} \to \mathsf{I}, e_0(s) = 0$ and $i : \mathsf{I} \to \mathsf{I}, i(s) = s$. Then i is path homotopic to $e_0 * i$ since I is convex (Lemma 8.6 (2)). By Lemma 8.9 (1), $f \circ i$ and $f \circ (e_0 * i)$ are path homotopic. Since,

$$f = f \circ i, \quad \text{and} \quad e_x * f = (f \circ e_0) * (f \circ i) = f \circ (e_0 * f) \quad \text{by Lemma 8.9 (1)},$$

We have $f * \bar{f} \cong_p e_x$. It follows from the similar argument that $\bar{f} * f \cong_p e_y$. we have $f \cong_p e_x * f$. It follows from the same argument that $f \cong_p f * e_y$.

(3) We use the same paths e_0, i in I. $\bar{i}(s) = i(s)$. We have $i * \bar{i} \cong_p e_0$ because I is convex (Lemma 8.6 (2)). Thus by Lemma 8.9 (1), $f \circ (i * \bar{i}) \cong_p f \circ (e_0)$ are path homotopic. Since

$$f \circ (i * \bar{i}) = (f \circ i) * (f \circ \bar{i}) = f * \bar{f} \quad \text{by by Lemma 8.9 (1) , and} \quad f \circ e_0 = e_x,$$

(1) For every $a, b \in [0,1]$ such that $0 < a < b < 1$, we define a triple product $(f * g * h)_{a,b} : \mathsf{I} \to X$ of paths $f, g, h : \mathsf{I} \to X$ such that $f(1) = g(0)$ and $g(1) = h(0)$ as follows

$$(f * g * h)_{a,b}(s) := \begin{cases} f(\frac{s}{a}) & s \in [0,a] \\ g(\frac{s-a}{b-a}) & s \in [a,b] \\ h(\frac{s-b}{1-b}) & s \in [b,1] \end{cases}$$

This is a well-defined continuous map by the pasting lemma again. We can check $(f * g) * h = (f * g * h)_{\frac{1}{4},\frac{1}{2}}$ and $f * (g * h) = (f * g * h)_{\frac{1}{2},\frac{3}{4}}$. Thus we are done if we show $(f * g * h)_{a,b} \cong_p (f * g * h)_{c,d}$ for every pairs $a < b$ and $c < d$.

Consider a path $p : \mathsf{I} \to \mathsf{I}$ whose graph is given by the three line segments $(0,0)$—(a,c), (a,c)—(b,d), (b,d)—$(1,1)$. Then p is path-homotopic to $i : \mathsf{I} \to \mathsf{I}, s \mapsto s$ since I is convex (Lemma 8.6 (2)). Let $F : \mathsf{I} \times [0,1] \to \mathsf{I}$ be the path-homotopy from p to i. By Lemma 8.9 (1), $(f * g * h)_{c,d} \circ F$ is a path-homotopy from $(f * g * h)_{c,d} \circ p$ to $(f * g * h)_{c,d} \circ i$. Since $(f * g * h)_{c,d} \circ p = (f * g * h)_{a,b}$ and $(f * g * h)_{c,d} \circ i = (f * g * h)_{c,d}$, we have $(f * g * h)_{a,b} \cong (f * g * h)_{c,d}$.

□

Lemma 8.9. *Let $f, g : \mathsf{I} \to X$ be paths and let $k : X \to Y$ be a continuous map.*

(1) If $F : f \Rightarrow_p g$ is a path homotopy, then $k \circ F : k \circ f \Rightarrow_p k \circ g$ is a path homotopy.

(2) If $f(1) = g(0)$, then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

9 Fundamental Group and Covering Spaces

Definition 9.1 (p.331 [Mun])**.** A loop bases at $x_0 \in X$ is a path from x_0 to x_0. Let $\pi_1(X, x_0)$ be the set of path-homotopy classes of loops based at x_0. It is a group by Theorem 8.8. It is called the **fundamental group** of X based at x_0.

> Since all the path defining a class in $\pi_1(X, x_0)$ has x_0 as the initial and terminal points, the product $*$ is defined for all pairs of classes. The associativity $([f] * [g]) * [h] = [f] * ([g] * [h])$ follows from Theorem 8.8 (1). The identity is $[e_{x_0}]$ and for $[f] \in \pi_1(X, x_0)$, the inverse $[f]^{-1}$ is $[\bar{f}]$.

Lemma 9.2. *If a subspace $A \subset \mathbb{R}^n$ is convex, then $\pi_1(A, a_0)$ is trivial for any $a_0 \in A$.*

Proof. By Lemma 8.6 (2), any loop f based at a_0 is path-homotopic to the constant loop e_{a_0}. □

9.1 Non-canonical uniqueness of the fundamental group of a path-connected space.

Theorem 9.3 (52.1 [Mun])**.** *Let $x_0, x_1 \in X$ and $\alpha : [0, 1] \to X$ a path from x_0 to x_1. Define a map*

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [f] \mapsto [\bar{\alpha}] * [f] * [\alpha].$$

Then $\hat{\alpha}$ is a group isomorphism.

Proof.

1. $\hat{\alpha}$ is a group homomrphism:

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] = [\bar{\alpha}] * [f] * [g] * [\alpha] = \hat{\alpha}([f] * [g]).$$

   The second equality follows from Theorem 8.8 (3).

2. To show that $\hat{\alpha}$ is an isomorphism, we show that there is an inverse homomorphism. Let $\beta := \bar{\alpha}$, then $\hat{\beta}$ is $\hat{\alpha}^{-1}$:

$$\hat{\beta}([f]) = [\bar{\beta}] * [f] * [\beta] = [\alpha] * [f] * [\bar{\alpha}].$$
$$\hat{\alpha}(\hat{\beta}([f])) = [\bar{\alpha}] * [\alpha] * [f] * [\bar{\alpha}] * [\alpha] = [f].$$

   Similarly $\hat{\beta} \circ \hat{\alpha}([f]) = [f]$.

□

Remark 9.4. If X is path-connected, then fundamental groups based at all points are isomorphic. But there is no natural isomorphism between way. The isomorphism depends on the path-homotopy classes of the chosen path from x_0 to x_1.

Definition 9.5. A space X is **simply-connected** if it is path-connected and $\pi_1(X, x_0)$ is a trivial group $\{1\}$.

Example 9.6. Any convex set $A \subset \mathbb{R}^n$ is path-connected. Also $\pi_1(A, a_0)$ is trivial by Lemma 8.6 which states that all paths with the same initial and terminal points are path-homotopic (so any loop at a_0 is path-homotopic to the constant look at a_0)

Lemma 9.7. *If X is simply-connected, then for any $x, y \in X$, all paths from x to y are path-homotopic.*

Proof. Let f and g be paths from x to y. Then $f * \bar{g}$ is a loop based at x. Thus by the assumption and by Theorem 9.3,

$$[f * \bar{g}] \in \pi(X, x) = \{1\}.$$

Thus $[e_x] = [f * \bar{g}] = [f] * [\bar{g}]$. Multiply $[g]$ from right, we get $[g] = [f]$ by Theorem 8.8 (2), (3). □

9.2 Functoriality of fundamental groups

Theorem 9.8 (Functor from based spaces to groups)**.** *Let $h : X \to Y$ be a continuous map. Then there is a group homomorphism*

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$$

defined by

$$h_*([f]) := [h \circ f].$$

Proof.

1. (Well-defined) If $f \cong_p f'$, then $h \circ f \cong_p h \circ f'$ by Lemma 8.9 (1).
2. (Group Homo) $h_*([f]) * h_*([g]) = [h \circ f] * [h \circ g] = [(h \circ f) * (h \circ g)] = [h \circ (f * g)] = h_*([f] * [g])$ where the third equality follows from Lemma 8.9 (2).

□

Remark 9.9. The group homomorphism h_* induced from $h : X \to Y$ depends on the choice of base points $x_0 \in X$ and $f(x_0) \in Y$. To make this association clear, we should introduce a map beween pairs: for a subspace $A \subset X$ and $B \subset Y$, a map $h : (X, A) \to (Y, B)$ is a continuous map $h : X \to Y$ such that $h(A) \subset B$. In this way, we have an associate without ambiguity:

$$h : (X, x_0) \to (Y, y_0) \qquad \Rightarrow \qquad h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

Theorem 9.10 (Functoriality)**.**

1. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$, we have $(k \circ h)_* = k_* \circ h_*$.*
2. *If $\mathrm{id}_X : (X, x_0) \to (X, x_0)$ is the identity map, then $(\mathrm{id}_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity homomorphism.*

Proof.

$$(k \circ h)_*([f]) = [k \circ h \circ f] = k_*([h \circ f]) = k_* \circ h_*([f])$$

$$(\mathrm{id}_X)_*([f]) = [\mathrm{id}_X \circ f] = [f].$$

□

Corollary 9.11. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Let h^{-1} be the inverse of h. Then $h \circ h^{-1} = \mathrm{id}_Y$ and $h^{-1} \circ h = \mathrm{id}_X$ imply that

$$h_* \circ (h^{-1})_* = \mathrm{id}_{\pi_1(Y, y_0)}, \qquad (h^{-1})_* \circ h_* = \mathrm{id}_{\pi_1(X, x_0)}.$$

Thus $(h_*)^{-1} = (h^{-1})_*$ and h_* is an isomorphism. □

Definition 9.12. A ***category*** $\mathcal{C}$ consists of a collection $\mathrm{Ob}(\mathcal{C})$ of ***objects*** and, for each objects A, B, a collection $\mathrm{Mor}_{\mathcal{C}}(A, B)$ of morphisms from A to B. The following axioms must be satisfied:

1. (Composition of morphisms) For $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$, there is a unique $g \circ f \in \mathrm{Mor}_{\mathcal{C}}(A, C)$.
2. (Associativity) $k \circ (g \circ f) = (k \circ g) \circ f$.
3. (Identity) For every object $A \in \mathrm{Ob}(\mathcal{C})$, there is the ***identity*** morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ such that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f$ for every $g \in \mathrm{Mor}_{\mathcal{C}}(A, B)$.

A ***(covariant) functor*** $\mathcal{F}$ from a category $\mathcal{C}$ to $\mathcal{D}$ is an association

1. $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
2. $\mathcal{F} : \mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{D}}(A, B)$ for all $A, B \in \mathrm{Ob}(\mathcal{C})$.

satisfying $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ and $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$.

Example 9.13. Let $\mathcal{T}op^b$ be the category that consists of a topological space with a based point (X, x_0) and continuous maps $f : (X.x_0) \to (Y, y_0)$. Let $\mathcal{G}rp$ be the category that consists of groups and homomorphisms. Then the association

1. (On Objects) (X, x_0) to $\pi_1(X, x_0)$
2. (On Morphisms) $f : (X, x_0) \to (Y, y_0)$ to $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

defines a functor $\mathcal{T}op^b$ to $\mathcal{G}rp$.

9.3 Covering spaces and the example

Definition 9.14. A surjective continuous map $p : E \to B$ is a ***covering map*** if it satisfies

For each $b \in B$, there is an open nbd U_b such that $\underline{p^{-1}(U_b) = \sqcup_\alpha V_\alpha \text{ where } p|_{V_\alpha} : V_\alpha \cong U_b}$

We say, U_b ***is evenly covered by*** p for the underlined condition. E is called a ***covering space*** of B.

Lemma 9.15 (p.336 [Mun]). *If $p : E \to B$ is a covering map, then*

1. *For each $b \in B$, $p^{-1}(b) \subset E$ has the discrete topology.*
2. *p is an open map. In particular, it is a quotient map.*

Proof.
1. Let U_b be a nbhd of b evenly covered by p, i.e. $\pi^{-1}(U_b) = \sqcup_\alpha V_\alpha$ and $p : V_\alpha \cong U$. Then each V_α contains exactly one element of $p^{-1}(b)$, thus it defines the discrete topology on $p^{-1}(b)$.
2. Let O be an open set in E. We need to show that $p(O)$ is open. Let $x \in p(O)$ and let U_x be a nbhd of x evenly covered by p, $\pi^{-1}(U_x) = \sqcup V_\alpha$. Since $V_\alpha \cap O$ is open and $p : V_\alpha \cong U_x$, $p(V_\alpha \cap O)$ is open and $x \in p(V_\alpha \cap O) \subset p(O)$.

□

Theorem 9.16 (53.1 [Mun]). *The map $p : \mathbb{R} \to S^1$ defined by $p(x) := (\cos 2\pi x, \sin 2\pi x)$ is a covering map.*

Proof.
1. It is a continuous and surjective map.
2. Consider an open covering $U_1, \cdots, U_4$ where U_1, U_3 are right and left circles and U_2, U_4 are upper and lower half circles. We show that each U_i is evenly covered by p. Then since U_i covered S^1, p is a covering map.

   We show it for U_1. Other U_i's are similarly proved. First
   $$p^{-1}(U_1) = \bigsqcup_{n\in\mathbb{Z}} (n - \frac{1}{4}, n + \frac{1}{4}).$$
   Let $V_n := (n - \frac{1}{4}, n + \frac{1}{4})$. $p|_{\overline{V_n}} : \overline{V_n} \to \overline{U_1}$ is obviously injective and surjective. Since $\overline{V_n}$ is compact and $\overline{U_1}$ is Hausdorff, $p|_{\overline{V_n}}$ is a homeomorphism by Theorem 4.9. Thus $p|_{V_n} : V_n \to U_1$ is a homeomorphism too.

□

Theorem 9.17 (53.3). *If $p : E \to B$ and $p : E' \to B'$ are covering maps, then $(p, p') : E \times E' \to B' \times B$ is a covering map.*

Proof. It is obvious that if $U \subset B$ is evenly covered by p and $U' \subset B'$ is evenly covered by p', then $U \times U'$ is evenly covered by (p, p'). □

Example 9.18. Consider the covering map $p : \mathbb{R} \to S^1$ from Theorem 9.16. The above theorem says

$$\mathbf{p} := (p, p) : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$

is a covering map.

Theorem 9.19 (53.2 [Mun])**.** *Let $p : E \to B$ be a covering map. Let $B_0 \subset B$ be a subspace, then let $E_0 := p^{-1}(B_0)$. Then $p|_{E_0} : E_0 \to B_0$ is a covering map.*

Proof. If $U \subset B$ is evenly covered by p, then $U \cap B_0 \subset B_0$ is evenly covered by $p|_{E_0}$. □

Remark 9.20. If you restrict $p : E \to B$ to a subspace $E_0 \subset E$ and define $B_0 := p(E_0)$, then $p|_{E_0} : E_0 \to B_0$ may fail to be a covering map. For example, consider $E_0 := (0, \infty) \subset \mathbb{R}$ in Theorem 9.16. $p|_{E_0} : E_0 \to S^1$ is a surjective continuous map. However, for any open set U around $(1, 0) \in S^1$, $p^{-1}(U) = (0, \epsilon) \sqcup (1 - \epsilon, 1 + \epsilon) \sqcup \cdots$ and $p|_{(0,\epsilon)} : (0, \epsilon) \to U$ can never be a homeomorphism.

Example 9.21. Consider $\mathbf{p} : \mathbb{R}^2 \to S^1 \times S^1$ from Example 9.18 and let $b_0 := p(0) \in S^1$. Let $B_0 := S^1 \times \{b_0\} \cup \{b_0\} \times S^1$. This B_0 is called the ***figure-eight***. By Theorem 9.19, if $E_0 := \mathbf{p}^{-1}(B_0)$, then $\mathbf{p}|_{E_0} : E_0 \to B_0$ is a covering map. E_0 is the "infinite grid" given by

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}).$$

This is one covering space for the figure-eight and we will see others later.

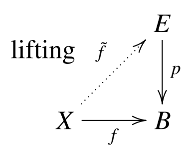
10 Fundamental Groups and Covering Spaces, §54 [Mun]

In this section, we study the crucial connection between the concepts of fundamental groups and covering spaces. This connection allows us to compute the fundamental group of spaces. The key concept to connect those two is a **lifting** *of a map along another map. As a first application, we compute the fundamental group of* S^1.

- Step 1. Let $p : E \to B$ be a covering. Let $[f] \in \pi_1(B, b_0)$. The loop f at b_0 is uniquely lifted to a path $\tilde{f}$ in E once we choose where $\tilde{f}$ starts.
- Step 2. If $[f] = [f']$, the unique lifts $\tilde{f}, \tilde{f}'$ starting from the same point have the same ending. Thus there is a map $[f] \mapsto \tilde{f}(1)$.
- Step 3. Apply it to $p : \mathbb{R} \to S^1$. We have a map $\pi_1(S^1, b_0) \to \pi^{-1}(b_0) \cong \mathbb{Z}$. The simple connectedness $\mathbb{R}$ implies that this map is bijective. Moreover, we can show that this map is a group homomorphism.

10.1 A lifting of a map along another map and liftings of paths along a covering map

Definition 10.1. Let $p : E \to B$ be a continuous map. ***A lifting*** of a continuous map $f : X \to B$ is a continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$:



Example 10.2. Consider $p : \mathbb{R} \to S^1, x \mapsto (\cos 2\pi x, \sin 2\pi x)$. Let $f : [0, 1] \to S^1, s \mapsto (\cos \pi x, \sin \pi x)$. Then $\tilde{f} : [0, 1] \to \mathbb{R}, s \mapsto s/2$ is a lifting of f. Also $\tilde{f} : [0, 1] \to \mathbb{R}, s \mapsto s/2 + 2\pi$ or in general $s \mapsto s/2 + 2\pi n$ where n is a fixed integer, is a lifting. Observe that in this case, the lifting is determined by the initial point, i.e by $\tilde{f}(0) \in 2\pi\mathbb{Z}$.

Lemma 10.3 (Unique Path Lifting for Covering 54.1 [Mun])**.** *If $p : E \to B$ is a covering map and $f : [0, 1] \to B$ is a path with the initial point $b_0 := f(0)$, then for each $e_0 \in p^{-1}(b_0)$, there is a unique lifting $\tilde{f} : [0, 1] \to E$ such that $\tilde{f}(0) = e_0$.*

Remark 10.4. $p^{-1}(b_0)$ is called the fiber of p at b_0. The above lemma says, each path in B with initial point b_0 can be lifted uniquely to a path in E once we choose a point e_0 in the fiber of b_0 where the lifted path should start from.

Proof.

1. Cover the image $f(\mathsf{I})$ of the path by $\cup_{b \in f(\mathsf{I})} U_b$ where U_b is evenly covered by p. Since f is continuous, $\{f^{-1}(U_b)\}$ is an open cover of I. By Lebesgue Measure Lemma (note I is compact), there is $\delta > 0$ such that any subset with max distance less than δ is contained in one of $f^{-1}(U_b)$. Therefore if we devide I into $\cup_{i=1}^{n} [s_i, s_{i+1}]$ in such a way that $s_{i+1} - s_i < \delta$, then each $[s_i, s_{i+1}]$ is contained in one of $f^{-1}(U_b)$, i.e. $f([s_i, s_{i+1}]) \subset U_b$.
2. We construct $\tilde{f}$ by induction on $i = 1, \cdots, n$:
   (a) Lifting $f|_{[0,s_1]}$. Let $f([0, s_1]) \subset U_{b_1}$. Since U_{b_1} is evenly covered, let $p^{-1}(U_{b_1}) = \sqcup V_\alpha$. Let $e_0 \in V_{\alpha_1}$. Since $p|_{V_{\alpha_1}} : V_{\alpha_1} \to U_{b_1}$ is a homeomorphism, the inverse $(p|_{V_{\alpha_1}})^{-1}$ is continuous, therefore define
   $$\tilde{f}|_{[0,s_1]}(s) := (p|_{V_{\alpha_1}})^{-1} \circ f(s).$$
   It is easy to see $\tilde{f}|_{[0,s_1]}(0) = (p|_{V_{\alpha_1}})^{-1} \circ f(0) = (p|_{V_{\alpha_1}})^{-1}(b_0) = e_0$ and $p \circ \tilde{f}|_{[0,s_1]} = f$.

(b) Suppose we have a desired lifting $\tilde{f}|_{[0,s_i]}$ of $f|_{[0,s_i]}$. Let $f([s_i, s_{i+1}]) \subset U_{b_{i+1}}$. Let $p^{-1}(U_{b_{i+1}}) = \sqcup V_\alpha$. Since $f(s_i) = p \circ \tilde{f}|_{[0,s_i]}(s_i) \in U_{b_{i+1}}$, there is V_{α_i} such that $\tilde{f}|_{[0,s_i]}(s_i) \in V_{\alpha_i}$. Define

$$\tilde{f}|_{[0,s_{i+1}]} := \begin{cases} \tilde{f}|_{[0,s_i]}(s) & s \in [0, s_i] \\ (p|_{V_{\alpha_i}})^{-1}(f(s)) & s \in [s_i, s_{i+1}] \end{cases}$$

By the argument in (a), the second map is a well-defined continuous map and by the way we chose V_α, those two functions agree at $s = s_i$. Thus by the pasting lemma, $\tilde{f}|_{[0,s_{i+1}]}$ is a well-defined continuous map. The conditions for lifting are obviously satisfied.

3. Uniqueness of lifting. Let $\tilde{f}'$ be another lifting. Induction on i.

   (a) $\tilde{f}|_{[0,s_1]} = \tilde{f}'_{[0,s_1]}$: Since $\tilde{f}(0) = \tilde{f}'(0) = e_0$, the images of both are in the same V_{α_1} since $[0, s_1]$ is connected so that the images must lie entirely in the connected component containing e_0. Thus

$$\tilde{f}|_{[0,s_1]}(s) = (p|_{V_\alpha})^{-1} \circ f(s) = \tilde{f}'|_{[0,s_1]}(s).$$

   (b) Suppose that $\tilde{f}|_{[0,s_i]} = \tilde{f}'_{[0,s_i]}$ so that $\tilde{f}(s_i) = \tilde{f}'(s_i)$. Then the images of $[s_i, s_{i+1}]$ under $\tilde{f}$ and $\tilde{f}'$ are both in $V_{\alpha_{i+1}}$ because of the connectedness of $[s_i, s_{i+1}]$ as in (a). Therefore similarly to (a), we have $\tilde{f}|_{[s_i,s_{i+1}]}(s) = \tilde{f}'|_{[s_i,s_{i+1}]}(s)$. Thus together with the assumption, $\tilde{f}|_{[0,s_{i+1}]} = \tilde{f}'|_{[0,s_{i+1}]}$.

□

10.2 Lifting Path-Homotopy

Theorem 10.5 (Unique Homotopy Lifting, 54.2, 54.3 [Mun])**.**

- *If $p : E \to B$ is a covering map and $F : \mathsf{I} \times [0,1] \to B$ be a continuous map with $F(0,0) = b_0$. For each $e_0 \in p^{-1}(b_0)$, there is a unique lifting $\tilde{F} : \mathsf{I} \times [0,1] \to E$ such that $\tilde{F}(0,0) = e_0$.*
- *If F is a path-homotopy from f to g, then $\tilde{F}$ is a path-homotopy from the lifting $\tilde{f}$ of f at e_0 to the lifting $\tilde{g}$ of g at e_0. In particular, $\tilde{f}(1) = \tilde{g}(1)$.*

Proof.

- The argument to show there is a unique lifting is essentially the same as Theorem 10.3.
  1. We can divide $\mathsf{I} \times [0,1]$ into small rectangles $[s_i, s_{i+1}] \times [t_j, t_{j+1}]$, $1 \le i \le n, 1 \le j \le m$, so that the image of each under F is contained in an evenly covered open set.
  2. Number those rectangles by $k = 1, \cdots, nm$ in such a way that consecutively numbered rectangles share some points in the image under F. Then construct $\tilde{F}$ inductively on k.
  3. Let $\tilde{F}'$ be another lifting such that $\tilde{F}(0,0) = \tilde{F}'(0,0) = e_0$. Then show the uniqueness also by induction on k.
- We have $F(s,0) = f(s)$, $F(s,1) = g(s)$, $F(0,t) = f(0) = g(0) = b_0$ and $F(1,t) = f(1) = g(1) = b_1$.
  1. Regard $F(0,t)$ as a constant path at b_0. Then $\tilde{F}(0,t)$ is a lifting at e_0 which must be a constant path at e_0 by the uniqueness of path lifting. Similarly regarding $F(1,t)$ as a constant path at b_1, $\tilde{F}(1,t)$ is a constant path at some point in $p^{-1}(b_1)$. Thus $\tilde{F}$ must be a path-homotopy.
  2. Since $\tilde{F}(s,0)$ and $\tilde{F}(s,1)$ are liftings of f and g at e_0, by the uniqueness, $\tilde{F}$ is a path-homotopy from $\tilde{f}$ to $\tilde{g}$.

□

Corollary 10.6. *Let $p : E \to B$ be a covering. Fix b_0 and $e_0 \in p^{-1}(e_0)$. Define the following map of sets*

$$\Phi_{e_0} : \{ \textit{ loops } f \textit{ at } b_0 \textit{ in } B \} \to \pi^{-1}(b_0), \quad f \mapsto \tilde{f}(1).$$

where $\tilde{f}$ is the unique lifting of f at e_0. This map factors through

$$\phi_{e_0} : \pi_1(B, b_0) \to \pi^{-1}(b_0), \quad [f] \mapsto \tilde{f}(1)$$

since if $f \cong_p f'$, then $\tilde{f}(1) = \tilde{f}'(1)$ from the above theorem.

10.3 Surjectivity and bijectivity of ϕ_{e_0}

Theorem 10.7. *Let $p : E \to B$ be a covering map. Let $e_0 \in p^{-1}(b_0)$. (1) If E is path-connected, then ϕ_{e_0} is surjective. (2) If E is simply connected, i.e. path-connected and π_1 is trivial, then ϕ_{e_0} is bijective.*

Proof.

1. Let $e_1 \in p^{-1}(b_0)$. Since E is path-connected, there is a path $\tilde{f}$ from e_0 to e_1. Composing with p, we have a loop $f := p \circ \tilde{f}$ at b_0.

2. We need to show the map is injective if E is simply-connected. Let $[f], [g] \in \pi_1(B, b_0)$ such that $\phi_{e_0}[f] = \phi_{e_0}[g]$, i.e. if $\tilde{f}, \tilde{g}$ are lifts of f, g beginning at e_0, then $\tilde{f}(1) = \tilde{g}(1)$. By Lemma 9.7, there is a path-homotopy $\tilde{F} : \tilde{f} \Rightarrow_p \tilde{g}$. Then $F := p \circ \tilde{F}$ is clearly a path-homotopy from f to g (check the conditions!) so that $[f] = [g]$.

□

10.4 $\pi_1(S_1, b_0) \cong \mathbb{Z}$

Theorem 10.8. *$\pi_1(S_1, b_0)$ is isomorphic to $\mathbb{Z}$.*

Proof. Since $\mathbb{R}$ is simply-connected (Example 9.6), the map $\phi_{e_0} : \pi^1(S^1, b_0) \cong p^{-1}(b_0)$. Let $b_0 := (1, 0)$, then $p^{-1}(b_0) = \mathbb{Z} \subset \mathbb{R}$. If we can show that ϕ_{e_0} is actually a group homomorphism, we are done. Let $e_0 := 0 \in \mathbb{R}$ and $\phi := \phi_{e_0}$. We need to show

$$\phi([f] * [g]) = \phi([f]) + \phi([g]).$$

Let $\tilde{f}, \tilde{g}$ be the lifts of f, g at e_0. Let $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$ so that $\phi([f]) = n$ and $\phi([g]) = m$. Define a path $\tilde{g}' : \mathsf{I} \to \mathbb{R}$ by

$$\tilde{g}'(s) := n + \tilde{g}.$$

Then $\tilde{g}'$ is the lift of g at n so that $\tilde{f} * \tilde{g}'$ is well-defined. Since $p \circ (\tilde{f} * \tilde{g}') = (p \circ \tilde{f}) * p \circ \tilde{g}' = f * g$ by Lemma 8.9, we see that $\tilde{f} * \tilde{g}'$ is the lift of $f * g$ and $(\tilde{f} * \tilde{g}')(1) = n + \tilde{g}(1) = n + m$. Thus $\phi([f] * [g]) = n + m$. □

10.5 Retraction and fixed points theorem

Definition 10.9. Let A be a subspace of X. A continuous map $r : X \to A$ is a ***retraction of** X **to** A* if $r(a) = a$.

Lemma 10.10. *Let $i : A \to X$ be an inclusion of a subspace A of X. If there is a retraction $r : X \to A$, then $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective. Furthermore $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.*

Proof. Notice that $r \circ i = \mathrm{id}_{(A,a_0)}$. By Theorem 9.10 (2), $r_* \circ i_* = \mathrm{id}_{\pi_1(A,a_0)}$. This implies that i_* is injective and r_* is surjective.

Let $f : X \to Y$ and $g : Y \to X$ be maps of sets. If $g \circ f = \mathrm{id}_X$, then f is injective and g is surjective. Why? This is because, if f is not injective, then $g \circ f = \mathrm{id}_X$ can not be injective and if g is not surjective, then $g \circ f = \mathrm{id}_X$ can not be surjective.

□

Theorem 10.11 (§55.2, [Mun])**.** *There is no retraction of B^2 to S^1 where B^2 is the 2-dimensional disk.*

Proof. If there is a retraction, by Lemma , the inclusion $j : S^1 \to B^2$ induces an injective map $j_* : \pi_1(S_1, x_0) \to \pi_1(B^2, x_0)$. Since B^2 is simply-connected (Lemma 9.6) and $\pi_1(S_1, x_0)$, we have a contraction. □

Theorem 10.12 (§55.6, [Mun])**.** *If $f : B^2 \to B^2$ is continuous, then there is $x \in B^2$ such that $f(x) = x$.*

Proof. Suppose that there is no such fixed point, i.e. $f(x) \neq x$ for all $x \in B^2$. Then for each x, consider the half line from $f(x)$ to x. This line intersects with S^1. Let this point be denoted by $r(x)$ So define a map

$$r : B^2 \to S^1, x \mapsto r(x).$$

This map is well-defined because there is no fixed point. This map is continuous.

It is intuitively obvious. Any rigorous proof is welcome!

This map r is a retraction from B^2 to S^1 and it contradict with Theorem 10.11. □

10.6 Deformation retract §58

Lemma 10.13 (58.1)**.** *Let $h, k : X \to Y$ be continuous maps and suppose $h(x_0) = y_0$ and $k(x_0) = y_0$. If there is a homotopy $F : X \times [0, 1] \to Y$ from h to k such that $F(x_0, t) = y_0$ for all $t \in [0, 1]$, then $h_* = k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.*

Definition 10.14 (p.361)**.** Let $A \subset X$ be a subspace and let $j : A \hookrightarrow X$ be the inclusion map. A homotopy $H : X \times [0, 1] \to X$ is a **deformation retraction of** X **onto** A if

$$H(x, 0) = x, \quad H(x, 1) \in A, \quad \forall x \in X, \quad \text{and} \quad H(a, t) = a, \forall a \in A.$$

If wefine $r : X \to A$ by $r(x) := H(x, 1) \in A$. Then r is a retraction. Furthermore, H is a homotopy from id_X to $j \circ r$. In this case, A is called a ***deformation retract of*** X.

Theorem 10.15. *If A is a deformation retract of X, then $j_* : \pi_1(A, a) \to \pi_1(X, a)$ is an isomorphism.*

Proof.

1. By Lemma 10.5, j_* is injective.
2. By Lemma 10.13 and definition of deformation retraction, $(j \circ r)_* : \pi_1(X, a) \to \pi_1(X, a)$ is the identity map. By $(j \circ r)_* = j_* \circ r_*$, j^* must be surjective.

□

Example 10.16. S^n is a deformation retract of $\mathbb{R}^{n+1} - \vec{0}$. Thus the inclusion $j : S^n \to \mathbb{R}^{n+1} - \vec{0}$ induces an isomorphism of the fundamental groups.

Let $X := \mathbb{R}^{n+1} - \vec{0}$. Consider $H : X \times [0, 1] \to X$ defined by

$$H(x, t) := (1 - t)x + tx/\|x\|.$$

It is a continuous map. Thus it is a homotopy. $H(x, 0) = x$ and $H(x, 1) = x/\|x\|$. So if we define $r : X \to S^n$ by $r(x) := H(x, 1)$, then H is a homotopy from id_X to $j \circ r$. Since for all $a \in S^n$, $H(a, t) = (1 - t)a + ta//\|a\| = (1 - t)a + ta = a$, H is a deformation retraction.

11 Application and more computations of π_1

11.1 Homotopy invariance of fundamental groups

Theorem 11.1 (Lemma 58.4)**.** *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (X, x_0) \to (Y, y_1)$ be based continuous maps. If h and k is homotopic, then there is a path α from y_0 to y_1 such that $h_* \circ \hat{\alpha} = k_*$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(X,x) & \xrightarrow[h_*]{} & \pi_1(Y,y_0) \\ & \searrow^{k_*} & \downarrow{\scriptstyle \hat{\alpha}} \\ & & \pi_1(Y,y_1). \end{array}$$

Proof. We will prove $k_*([f]) = \hat{\alpha} \circ h_*([f])$, i.e. $[k \circ f] = [\bar{\alpha} * (h \circ f) * \alpha]$, which is equivalent to $[\alpha * (k \circ f)] = [(h \circ f) * \alpha]$. Consider

$$I \times [0,1] \xrightarrow[G]{} [0,1] \times [0,1] \xrightarrow[F]{} X \times [0,1] \xrightarrow[H]{} Y.$$

where

- G is a path-homotopy between $(s,0) * (1,s)$ (go right and then up $\beta_0 * \gamma_1$) and $(0,s) * (s,1)$ (go up and then right $\gamma_0 * \beta_1$).
- $F(s,t) := (f(s),t)$. In particular, $F(0,t) = F(1,t) = (x_0,t) =: c(t)$. Then $c = F \circ \gamma_1 = F \circ \gamma_0$.
- H is a homotopy from h to k.

Since $H : X \times [0,1] \to Y$ is a homotopy from h to k, $H(x_0,0) = h(x_0) = y_0$ and $H(x_0,1) = k(x_0) = y_1$. Thus $\alpha := H|_{\{x_0\}\times[0,1]}$ is a path from y_0 to y_1, i.e. $\alpha(t) := H(x_0,t) = H \circ c(t)$.

Then we show $H \circ F \circ G$ is a path-homotopy between $(h \circ f) * \alpha$ and $\alpha * (k \circ f)$.

$G : \beta_0 * \gamma_1 \Rightarrow_p \gamma_0 * \beta_1$ implies $F \circ G : F \circ (\beta_0 * \gamma_1) \Rightarrow_p F \circ (\gamma_0 * \beta_1)$ since F is a continuous map. Thus

$$F \circ G : (F \circ \beta_0) * (F \circ \gamma_1) \Rightarrow_p (F \circ \gamma_0) * (F \circ \beta_1).$$

$$H \circ F \circ G : \underbrace{(H \circ F \circ \beta_0)}_{h \circ f} * \underbrace{(H \circ F \circ \gamma_1)}_{\alpha} \Rightarrow_p \underbrace{(H \circ F \circ \gamma_0)}_{\alpha} * \underbrace{(H \circ F \circ \beta_1)}_{k \circ f}$$

□

Definition 11.2. X and Y have the same ***homotopy type*** if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \cong \mathrm{id}_Y$ and $g \circ f \cong \mathrm{id}_X$. In this case, $f : X \to Y$ is called a ***homotopy equivalence*** and g is called a ***homotopy inverse*** of f

Example 11.3. If A is a deformation retract of X, then the retraction map $r : X \to A$ and the inclusion map $j : A \hookrightarrow X$ are homotopy equivalences and A and X have the same homotopy type. To see this, consider $r : X \to A$ and $j : A \hookrightarrow X$. Since $r \circ j = \mathrm{id}_A$, obviously $r \circ j$ is homotopic to id_A. The deformation retraction is a homotopy from id_X to $j \circ r$.

Example 11.4.

1. The figure-eight is a deformation retract of $\mathbb{R}^2 - p - q$.
2. The theta figure $S^1 \cup (0 \times [-1,1])$ is a deformation retract of $\mathbb{R}^2 - p - q$.

Theorem 11.5 (58.7)**.** *If $f : X \to Y$ is a homotopy equivalence and $f(x) = y$, then $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ is an isomorphism.*

Proof. Let $g : Y \to X$ be a homotopy inverse to f, i.e. there are homotopies $f \circ g \cong \mathrm{id}_Y$ and $g \circ f \cong \mathrm{id}_X$. Pick some $x_0 \in X$ and consider the maps

$$(X, x_0) \xrightarrow[f]{} (Y, y_0) \xrightarrow[g]{} (X, x_1) \xrightarrow[f]{} (Y, y_1),$$

where y_0, x_1, y_1 are consecutively picked as $y_0 := f(x_0)$, $x_1 := g(y_0)$ and $y_1 := f(x_1)$. They induces

$$\pi_1(X, x_0) \xrightarrow[f_*]{} \pi_1(Y, y_0) \xrightarrow[g_*]{} \pi_1(X, x_1) \xrightarrow[f_*]{} \pi_1(Y, y_1).$$

By Theorem 11.1, $f \circ g \cong \mathrm{id}_Y$ and $g \circ f \cong \mathrm{id}_X$ implies

$$f_* \circ g_* = \hat{\alpha} \circ (\mathrm{id}_Y)_* = \hat{\alpha}, \quad g_* \circ f_* = \hat{\beta} \circ (\mathrm{id}_X)_* = \hat{\beta}$$

for some paths α and β. Since $\hat{\alpha}$ is an isomorphism, g_* is injective and f_* is surjective. Since $\hat{\beta}$ is an isomorphism, g_* is surjective and f_* is injective. Therefore, f_* and g_* are isomorphisms. □

Corollary 11.6. *If X and Y are path-connected and have the same homotopy type, then their fundamental groups are isomorphic.*

11.2 Fundamental group of S^n

Theorem 11.7 (59.1)**.** *Let $X = U \cup V$ where U, V are open sets and let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be inclusions.* Suppose that $U \cap V$ is path-connected. *Let $x_0 \in U \cap V$. Then images of*

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \quad \textit{and} \quad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$, i.e. any element is a product of elements in $\mathrm{Im}\, i_* \cup \mathrm{Im}\, j_*$.

Theorem 11.8 (59.3)**.** *If $n \geq 2$, S^n is simply-connected.*

Proof. Let $S^n := \{\vec{x} \in \mathbb{R}^{n+1}, |\vec{x}| = 1\}$ and $\vec{p} := (0, \cdots, 0, 1), \vec{q} := (0, \cdots, 0, -1) \in S^n$. Let $U := S^n - \{\vec{p}\}$ and $V := S^n - \{\vec{q}\}$.

1. The stereographic projection is the map $f_p : U \to \mathbb{R}^n$ defined by

   $$(x_1, \cdots, x_n, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}}(x_1, \cdots, x_n).$$

   It is a homeomorphism because $g : \mathbb{R}^n \to U$ defined by

   $$\vec{y} = (y_1, \cdots, y_n) \mapsto \left(\frac{2y_1}{1 + |\vec{y}|^2}, \cdots, \frac{2y_n}{1 + |\vec{y}|^2}, 1 - \frac{2}{1 + |\vec{y}|^2} \right)$$

   is the inverse of f. Thus U is path-connected and $\pi_1(U, \vec{x}_0)$ is trivial. Similar for V.

2. The intersection $U \cap V = S^n - \{\vec{p}, \vec{q}\}$ is path-connected, since $f|_{U \cap V} : S^n - \{\vec{p}, \vec{q}\} \to \mathbb{R}^n - \vec{0}$ is a homeomorphism and $\mathbb{R}^n - \vec{0}$ is path-connected from Example .

3. Applying Theorem 11.7, $\pi_1(S^n, \vec{x}_0)$ is generated by the images of the fundamental groups of U and V. But both of them are trivial, so $\pi_1(S^n, \vec{x}_0)$ is trivial. Since S^n is path-connected by Example , S^n is simply-connected.

□

11.3 Fundamental theorem of algebra

Lemma 11.9. *Let $f : S^1 \to S^1, z \mapsto z^n$ where $S^1 := U(1) \subset \mathbb{C}$. Then $f_* : \pi_1(S^1, 1) \mapsto \pi_1(S^1, 1)$ is given by $[f] \mapsto [f]^n = [f] * \cdots * [f]$.*

Proof. Under the isomorphism in Theorem 10.8, we must prove $f(1) = n$: Since $\mathbb{Z}$ is generated by 1, it is enough to show $f(1) = n$, i.e. $f(m) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = mf(1) = mn$. $1 \in \mathbb{Z}$ is given by $g : \mathsf{I} \to S^1$, $g(s) = \cos 2\pi s + i \sin 2\pi s = e^{2\pi i s}$ since the lift $\tilde{g}$ at $0 \in \mathbb{R}$ is then given by $\tilde{g} : I \to \mathbb{R}, s \mapsto s$. Now $f \circ g(s) = e^{2\pi n s}$ and the lift of $f \circ g$ is $\widetilde{f \circ g}(s) = ns$. Therefore $f_*([g]) = [f \circ g] = \widetilde{f \circ g}(1) = n$. □

Lemma 11.10 (55.3)**.** *For a continuous map $h : S^1 \to X$, the following conditions are equivalent:*

1. *h is nullhomotopic, i.e. homotopic to a constant map.*
2. *h extends to a continuous map $k : B^2 \to X$, i.e. if $j : S^1 \to B^2$ is the natural inclusion, then $h = k \circ j$.*
3. *h_* is the trivial homomorphism, i.e. $h_*([f]) = 1$ for all $[f] \in \pi_1(S_1, b)$.*

Proof. (1 ⇒ 2) Let $H : S^1 \times [0, 1] \to X$ be a homotopy from h to a constant map. Define a continuous map $\pi : S^1 \times [0, 1] \to B^2$ by $\pi(b, t) := (1 - t)b$. Since π is constant on $S^1 \times \{1\}$ and injective elsewhere, H factors through π:

$$\begin{array}{ccc} S^1 \times [0,1] & \xrightarrow{H} & X \\ & \searrow^{\pi} & \uparrow k \\ & & B^2 \end{array}$$

Since π is a quotient map (*1), k must be a continuous map (*2). Since $\pi|_{S^1 \times \{0\}}$ is the natural inclusion of S^1 into B^2, k is an extension of h.

*1 Let $\pi' : S^1 \times [0, 1] \to S^1/\sim$ be the quotient map collapsing $S^1 \times \{1\}$ to a point. Then π factors through π', inducing a bijection $j : S^1/\sim \longrightarrow B^2$, which is continuous from *2. Since $S^1/\sim$ is compact and B^2 is Hausdorff, j is a homeomorphism. Thus π must be a quotient map too.

*2 In general, if we have the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{g} & \uparrow h \\ & & Z \end{array}$$

where f is continuous and g is a quotient map. Then h is a continuous map. We need to show that if $U \subset Y$ is open, then $h^{-1}(U)$ is open. Since $f^{-1}(U) = (h \circ g)^{-1}(U) = g^{-1}(h^{-1}(U))$ is open and g is a quotient map $h^{-1}(U)$ must be open.

(2 ⇒ 3) $h = k \circ j$ implies that h_* factors through

$$h_* : \pi_1(S^1, b) \xrightarrow[j_*]{} \pi_1(B^2, b) \xrightarrow[k_*]{} \pi_1(X, h(b)).$$

Since B^2 is convex, the middle term is trivial, so h_* must be a trivial homomorphism.

(3 ⇒ 1). Let $p : \mathbb{R} \to S^1$ be the standard covering map used in Theorem 10.8. Then $p|_{\mathsf{I}} : \mathsf{I} \to S^1$ is a loop and represent $1 \in \mathbb{Z} \cong \pi_1(S^1, b_0)$. Since h_* is trivial, $h \circ p|_{\mathsf{I}}$ is path-homotopic to a constant loop at $x_0 := h(b_0)$. Let $F : \mathsf{I} \times [0, 1] \to X$ be the path-homotopy. F factors through $p|_{\mathsf{I}} \times \mathrm{id}_{[0,1]}$:

$$\begin{array}{ccc} \mathsf{I} \times [0,1] & \xrightarrow{F} & X \\ & \searrow^{p|_{\mathsf{I}} \times \mathrm{id}} & \vdots\, H \\ & & S^1 \times \mathsf{I} \end{array}$$

Since $p|_I$ is a quotient map, the induced map H is continuous (*2). H is a homotopy from h to a constant map:

$$F(s, 0) = h \circ p|_I(s) = H|_{S^1\times\{0\}} \circ p|_I(s) \quad \Rightarrow \quad H|_{S^1\times\{0\}}(b, 1) = h(b).$$

$$x_0 = F(s, 1) = H|_{S^1\times\{1\}} \circ p|_I(s) \quad \Rightarrow \quad H|_{S^1\times\{1\}}(b, 0) = x_0.$$

□

Theorem 11.11. *A polynomial of degree n with coefficients in $\mathbb{C}$ has n roots (counted with multiplicities).*

Proof. Let $f(z) \in \mathbb{C}[z]$ be a polynomial. Suppose the theorem below, say a is a root. Divide $f(z)$ by $(z - a)$:

$$f(z)/(z - a) = g(z) + R/(z - a)$$

where $g(z)$ is a polynomial of degree $n - 1$ and $R \in \mathbb{C}$ is the remainder. Multiply $(z - a)$:

$$f(z) = g(z)(z - a) + R.$$

Since $f(a) = 0$, $R = 0$. Thus $f(z) = g(z)(z - a)$. Keep this process n-times, we get $f(z) = (z - a_1)\cdots(z - a_n)$. □

Theorem 11.12. *A polynomial of degree n with coefficients in $\mathbb{C}$ has at least one root.*

Proof.

1. Since S^1 is a deformation retract of $\mathbb{C} - 0$, by applying Theorem 10.15 to j, we have $j_* : \pi_1(S^1, 1) \cong \pi_1(\mathbb{C} - \vec{0}, 1)$. By Lemma 11.9, $f : S^1 \to S^1, z \mapsto z^n$ induces $f_* : \pi_1(S^1, 1) \mapsto \pi_1(S^1, 1), [g] \mapsto [g]^n$ which is injective. Therefore $k := j \circ f : S^1 \to \mathbb{C} - 0, z \mapsto z^n$ must induces an injective map $k_* : \pi_1(S^1, 1) \to \pi_1(\mathbb{C} - 0, 1)$ must be injective. Since $\pi_1(S^1, 1) \cong \mathbb{Z}$, k_* must be a non-trivial map. By Lemma 11.10, k is not nullhomotopic.

2. We prove the claim in a special case: let $g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$ where $|a_{n-1}| + \cdots + |a_0| < 1$. Suppose that there is no root. Then regarding B^2 as a disk in $\mathbb{C}$, $G : B^2 \to \mathbb{C} - 0,\ r \mapsto g(z)$ is well-defined. Since $G|_{S^1} : S^1 \to B^2$ is a map extendable to B^2, by Lemma 11.10, $G|_{S^1}$ is nullhomotopic.

3. Define a homotopy $F : S^1 \times [0, 1] \to \mathbb{C} - 0$ by $F(z, t) := z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$. It is well-defined since $F(z, t) \neq 0$:
$$|F(z, t)| \geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_0)| \geq 1 - t|a_{n-1}z^{n-1} + \cdots + a_0|$$
$$\geq 1 - t(|a_{n-1}z^{n-1}| + \cdots + |a_0|) \geq 1 - t(|a_{n-1}| + \cdots + |a_0|) > 0$$
The first inequality uses Remark 11.13 and the last strict inequality uses the condition $|a_{n-1}| + \cdots + |a_0| < 1$. This F is a homotopy from k to $G|_{S^1}$. Since $G|_{S^1}$ is homotopic to a constant map (nullhomotopic), k must be homotopic to a constant map too. But this contradict to (1).

4. Consider the general equation $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$. Let $w = cz$ where $c \neq 0$. The equation becomes
$$(cw)^n + a_{n-1}(cw)^{n-1} + \cdots + a_1(cz) + a_0 = 0 \Leftrightarrow w^n + \frac{a_{n-1}}{c}w^{n-1} + \cdots + \frac{a_1}{c^{n-1}}w + \frac{a_0}{c^n} = 0$$
$z = z_0$ is a root iff $w = w_0$ is a root. By choosing a large c, the w-equation has a root by (3), therefore we have a root for z-equation too.

□

Remark 11.13. For any complex numbers a, b, we have $|a+b| \geq |a|-|b|$: apply triangle inequality to $(a+b)+(-b)$.

$$|a| = |(a + b) + (-b)| \leq |a + b| + | - b| = |a + b| + |b| \quad \Rightarrow \quad |a| - |b| \leq |a + b|.$$

11.4 Fundamental group of torus

Theorem 11.14 (60.1)**.** *$\pi_1(X \times Y, (x, y))$ is isomorphism to $\pi_1(X, x) \times \pi_1(Y, y)$.*

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be projection maps. Define a map

$$\Phi : \pi_1(X \times Y, (x, y)) \to \pi_1(X, x) \times \pi_1(Y, y), \quad [f] \mapsto ([p \circ f], [q \circ f]).$$

Φ is a homomorphism:

$$\Phi([f] * [g]) = \Phi([f * g]) = ([p \circ (f * g)], [q \circ (f * g)]) = ([p \circ f] * [p \circ g], [q \circ f] * [q \circ g])$$

$$= ([p \circ f], [q \circ f]) \cdot ([p \circ g], [q \circ g]) = \Phi([f]) \cdot \Phi([g]).$$

Note that for given groups G and H, the natural group multiplication in $\mathsf{G} \times \mathsf{H}$ is $(g, h) \cdot (g', h') := (gg', hh')$.

- (Φ Surjective) Let $([g], [h]) \in \pi_1(X, x) \times \pi_1(Y, y)$. Defne $[f] \in \pi_1(X \times Y, (x, y))$ by $f(s) := (g(s), h(s))$. Then $\Phi([f]) = ([g], [h])$.

- (Φ Injective) For a group homomorphism, if its kernel is trivial, then it is injective (Lemma 7.16). Let $[f] \in \ker \Phi$, i.e. there are path-homotopies $G : p \circ f \Rightarrow_p e_x$ and $H : q \circ g \Rightarrow_p e_y$. We need to show $F : f \cong_p e_{(x,y)}$. Define $F : \mathsf{I} \times [0, 1] \to X \times Y$ by $F(s, t) := (G(s, t), H(s, t))$. Then F is a path homotopy from f to the constant loop $e_{(x,y)}$:

$$F(s, 0) = (G(s, 0), H(s, 0)) = (g(s), h(s)), \quad F(s, 1) = (G(s, 1), H(s, 1)) = (e_x(s), e_y(s)) = e_{(x,y)}(s).$$

□

Corollary 11.15 (60.2)**.** *$\pi_1(T, b) \cong \mathbb{Z} \times \mathbb{Z}$ where T is the torus $S^1 \times S^1$.*

12 Fundamental groups of surfaces

12.1 Fundamental groups of a double torus

Theorem 12.1. *The fundamental group of the eight figure is a free group generated by two elements.*

Proof. The simply connected cover of the eight figure is given by "rose".
`http://en.wikipedia.org/wiki/Rose_(topology)`.
It is an infinite *tree* graph, i.e. no loop in the graph. We want to show the rose is simple-connected. Since the image of a path is compact so that it is contained in a finite graph, it is suffice to show that every finite tree graph is contractible. Take a finite tree graph. First contract the edges that has a vertex with no other edges. Then keep this process until there are no edges. By Theorem 10.7, there is a bijection between the fundamental group and the fiber of the covering. Let f and g are loops for each circle. Then if you lift any two distinct words, the ending points of them must be different. Thus there can not be a relation among the words in f and g. □

In general, the fundamental group of a wedge of circles are known to be a free group

Theorem 12.2 (71.1)**.** *Let X be a union of circles $S_1, \cdots, S_n$ where p is the only common point of circles. Then $\pi_1(X, p)$ is a free group generated by $f_1, \cdots, f_n$ where f_i is a generator of $\pi_1(S_i, p)$.*

Definition 12.3. Let X_1 and X_2 be a topological surface. A ***connected sum*** $X_1 \sharp X_2$ is given by taking an open disc from each X_i and pasting the remaining pieces along their edges. Note that here we have essentially two choices of how we glue.

Corollary 12.4. *Let $T\sharp T$ be a double torus. Then $\pi_1(T\sharp T, b)$ contains a free group generated by two elements as a subgroup. In particualr, it is non-abelian group.*

Proof. There is a retraction from $T\sharp T$ to the eight figure (Figure 60.2 [Mun]). Thus by Lemma 10.5, the induced map from the inclusion

$$j_* : \pi_1(\infty, b) \to \pi_1(T\sharp T, b),$$

is an injective map. □

12.2 Constructing various surfaces by identifying edges of polytopes

Let Δ be an $2m$-gon polytope with edges $e_1, \cdots, e_{2m}$ (numbered counter clockwisely). Create m-pairs among $\{e_i\}$, each labeled by $a_1, \cdots, a_m$. Orient the boundary of Δ counter clockwise. Assign $\epsilon_i := \pm 1$ to each e_i. Orient e_i compatibly with the orientation on $\partial\Delta$ if $\epsilon = +1$. Orient oppositely if $\epsilon = -1$. All these information is written on the right hand side of

$$(e_1, \cdots, e_{2m}) = (a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m}}^{\epsilon_{2m}}).$$

Now identify paired edges consistently with the orientation and obtain $X := \Delta/\sim$. We have

Theorem 12.5 (74.1)**.** *$X = \Delta/\sim$ is a compact topological surface.*

12.2.1 Fundamental group of surfaces constructed from polytopes

Theorem 12.6 (72.1)**.** *Let X be a Hausdorff space and A a closed path-connected subspace of X (with inclusion $i : A \hookrightarrow X$). If there is a continuous map $h : B^2 \to X$ such that $h|_{\mathrm{Int}\, B^2}$ is a bijection onto $X - A$ and $h|_{\partial B^2} : S^1 \to A$ is a map into A. Then*

$$i_* : \pi_1(A, a) \to \pi_1(X, a)$$

is surjective and the kernel is the least normal subgroup containing $(h|_{\partial B^2})_*(\gamma)$ where γ is a generator of $\pi_1(S^1, b)$ with $h(b) = a$.

Remark 12.7 (**Outline of how to compute the fundamental group of surface** $\Delta/\sim$)**.** Apply the theorem to the construction in the previous section. Let $X = \Delta/\sim$ and since $B^2 \cong \Delta$, regard Δ as B^2 in the theorem. Thus $h : \Delta \to X$ is the quotient map. Now $\partial\Delta \cong S^1$ and let $A := h(\partial\Delta)$. Then A is the wedge of some circles with a common point b. Let $\alpha_1, \cdots, \alpha_n$ be generators of each circle in A. Thus by Theorem 12.2, $\pi_1(A, b)$ is the free group generated by $\alpha_1, \cdots, \alpha_n$. Let $[\gamma]$ be the generator of $\partial\Delta$ presented by a counter clockwise loop. If we consider the edges are counter-clockwise paths on the boundary of Δ, then $\gamma = e_1 * \cdots * e_{2m}$. We can regard $a_{i_k}^{\epsilon_k}$ to be $h \circ e_k$. Thus $h \circ \gamma = a_{i_1}^{\epsilon_1}, \cdots, a_{i_2m}^{\epsilon_{2m}}$. Then $\alpha_1, \cdots, \alpha_n$ are presented by compositions of $h \circ e_k$, so $[h \circ \gamma] = 1$ gives a relation among $\alpha_1, \cdots, \alpha_n$ in the free group.

12.2.2 Torus, n-fold torus, $\mathbb{RP}^2$, Klein Bottle

Theorem 12.8 (73.1)**.** *A torus T is given by the identification date $(\alpha, \beta, \alpha^{-1}, \beta^{-1})$ for a 4-gon. We have*

$$\pi_1(T, a) \cong \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1} = 1 \rangle$$

Theorem 12.9 (74.3)**.** *An n-fold torus $X = T\sharp \cdots \sharp T$ is given by $(\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \cdots, \alpha_n, \beta_n, \alpha_n^{-1}, \beta_n^{-1})$ for $4n$-gon. We have*

$$\pi_1(T, a) \cong \langle \alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \mid [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n] = 1 \rangle$$

where $[\alpha_i, \beta_i] = \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$.

Theorem 12.10 (74.4)**.** *A n-fold projective space $X = \mathbb{RP}^2\sharp \cdots \sharp\mathbb{RP}^2$ is given by $2n$-gon with*

$$(\alpha_1, \alpha_1, \cdots, \alpha_n, \alpha_n).$$

We have

$$\pi_1(X, a) = \langle \alpha_1, \cdots, \alpha_n \mid \alpha_1^2 \cdots \alpha_n^2 = 1 \rangle.$$

Exercise 12.11 (EX3, p.454)**.** The klein bottle K is given by 4-gon with $(\alpha, \beta, \alpha^{-1}, \beta)$. We have

$$\pi_1(K, a) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta = 1 \rangle$$

12.3 Properly discontinuous actions, covering spaces, and fundamental groups

Definition 12.12 (p.490 [Mun])**.** Let G be a discrete group continuously acting on X. The action is ***properly discontinuous*** if

$$\text{(PdC) } \forall x \in X, \exists U_x \text{ an open nbhd of } x \text{ such that } g(U) \cap U = \emptyset, \forall g \in \mathsf{G} \text{ with } g \neq 1_{\mathsf{G}}.$$

Theorem 12.13 (81.5 [Mun])**.** *Let X be path-connected, locally path-connected. Let G be a discrete group continuously acting on X. Then the G-action is properly discontinuous if and only if the quotient map $\pi : X \to X/\mathsf{G}$ is a covering map.*

Proof.

- *Suppose that the action is p.d.* For $x \in X$, let U_x be the nbhd of x in (PdC). Then

  $$\pi^{-1}(\pi(U_x)) = \{y \in X, \ \ y = gx', x' \in U_x, g \in \mathsf{G}\} = \cup_{g\in\mathsf{G}} g(U_x) = \sqcup_{g\in\mathsf{G}} g(U_x).$$

  This implies (1) $\pi(U_x)$ is an open nbhd of $\pi(x)$ because π is a quotient map and the right hand side is an open set, (2) $\pi^{-1}(\pi(U_x))$ is a disjoint union of open sets. The restriction $\pi|_{g(U_x)} : g(U_x) \to \pi(U_x)$ is a homeomorphism: it is continuous. It is bijective since otherwise it contradicts to the disjointness. The inverse is continuous because π is an open map: if $U \subset X$ is open, then $\pi^{-1}(\pi(U)) = \cup_{g\in\mathsf{G}} g(U_x)$ (not necessarily disjoint), is a union open sets, so open.

- *Suppose that q is a covering map.* ....exercise.

□

Example 12.14 (60.3-4)**.** The antipodal quotient map $S^2 \to \mathbb{RP}^2$ is a covering map and $\pi_1(\mathbb{RP}^2, b) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$ where $\mathbb{Z}_2 := \{1, -1\}$ acts on S^2 by $-1 : x \mapsto -x$. This action is continuous because $\mathbb{R}^3 \to \mathbb{R}^3, x \mapsto -x$ is a homeomorphism. It is also a properly discontinuous action since the distance between x and $-x$ is 2 (consider the metric topology induced from $\mathbb{R}^3$ and use the ϵ-ball to separate x and $-x$). Thus the quotient map $\S^2 \to \mathbb{RP}^2$ is a covering map. Since S^2 is simply-connected and the cardinality of a fiber is 2, by Theorem 10.7, $\pi_1(\mathbb{RP}^2, b_0)$ has cardinality 2. The group of cardinality 2 must be $\mathbb{Z}/2\mathbb{Z}$. □

Example 12.15. Let $S^3 := \{(z, w) \in \mathbb{C}^2, |z|^2 + |w|^2 = 1\}$. Let $\mathbb{Z}_3 = \{e^{2\pi i \frac{k}{3}}, k = 0, 1, 2\} \subset U(1)$. Define $\mathbb{Z}_3$-action on S^3 by $\omega : (z, w) \mapsto (\omega z, \omega^2 w)$. This action is properly discontinuous. Similarly to the argument in Theorem 12.14, $\pi_1(S^3/\mathbb{Z}_3) \cong \mathbb{Z}_3$.

Example 12.16. Let $S^5 := \{(z, w, v) \in \mathbb{C}^2, |z|^2 + |w|^2 + |v|^3 = 1\}$. Let $\mathbb{Z}_4 = \{e^{2\pi i \frac{k}{4}}, k = 0, 1, 2, 3\} \subset U(1)$. Define $\mathbb{Z}_4$-action on S^5 by $\omega : (z, w, v) \mapsto (\omega z, \omega^2 w, \omega^3 v)$. This action is properly discontinuous. As a set $\pi_1(S^3/\mathbb{Z}_3) \cong \mathbb{Z}_4$. But there are exactly two groups of cardinality 4. $\mathbb{Z}_2 \times \mathbb{Z}_2$ has also cardinality 4. The following theorem can be proved by studying classifying spaces and the ***covering transformation***.

Theorem 12.17 (7.3, p.151, [Bredon])**.** *If X is simple-connected and locally path-connected and a discrete group* G *acts on X properly discontinuously, then* $\pi_1(X/\mathsf{G}, [x_0]) \cong \mathsf{G}$.

Proof. (outline)

1. Define the group of transformation (Deck transformations) of a covering $p : E \to B$: the group of homeomorphisms f of E which satisfy $p \circ f = p$.

2. We show that the group of transformation is isomorphic to the fundamental group of B if E is simply-connected (Cor 81.4), [Mun].

3. We show that the group of transformation of $X \to X/\mathsf{G}$ is isomorphic to G.

4. We show that the quotient space of E by the action of the group of transformations is homeomorphic to B.

□

13 Cauchy Integral Formula, Jordan Curve Theorem and Winding Number Theorem

13.1 Cauchy Integral Formula §66 [Mun]

Definition 13.1. A ***complex analytic function*** is a complex valued function $f(z)$ defined on an open set D in $\mathbb{C}$ such that f is infinitely differentiable and the taylor series at $z_0 \in D$ converges to $f(z)$ for z in the nbhd of z_0.

A ***holomorphic function*** is a complex valued function $f(z)$ defined on D which is differentiable by z everywhere in D. It is exactly a function $f(x+iy) = u(x,y) + v(x,y)$ such that partial derivatives of u and v are continuous and satisfies the Cauchy-Riemann equations $u_x - v_y = 0, v_x + u_y = 0$. A big theorem in complex analysis is that complex analytic functions are exactly holomorphic functions.

There is some analogy between Green's theorem and the following theorems (Cauchy-Riemann equations seems saying the the divergence and the circulation density of the vector field $\langle v, u\rangle$ is zero.)

Theorem 13.2 (Theorem 5, p.92, § 1.2 [Ahlfors])**.** *Let D be a simply connected open set in $\mathbb{C}$.*

(1) (Cauchy's theorem) If $h(z)$ is analytic in D then for any closed curve γ in D,

$$\int_\gamma h(z)dz = 0.$$

(2) If $h(z)$ is analytic in $D - \{z_0\}$ and $\lim_{z\to z_0}(z-z_0)h(z) = 0$, then for any closed curve γ in $D - \{z_0\}$,

$$\int_\gamma h(z)dz = 0.$$

Now we define so-called winding numbers. It is a mathematically rigorous definition of how many times a loop γ in $\mathbb{C}$ goes around a point $a \in \mathbb{C}$. This number is crucial when we compute a line integral of a complex analytic function.

Definition 13.3. Let $\gamma : \mathsf{I} \to \mathbb{R}^2$ be a loop in $\mathbb{R}^2$ and let $a \in \mathbb{R}^2$ such that $a \notin \gamma(\mathsf{I})$. Define a loop $g : \mathsf{I} \to S^1 \subset \mathbb{R}^2$ in S^1 by

$$g(s) := \frac{\gamma(s) - a}{|\gamma(s) - a|}.$$

Consider the standard covering map $p : \mathbb{R} \to S^1, t \to e^{2\pi i t}$. Take any lift $\tilde{g}$, then $\tilde{g}(1) - \tilde{g}(0)$ is always an integer. Moreover it doesn't depend on the choice of lifting, because if $\tilde{g}$ is a lifting, then the uniqueness of liftings implies that other liftings are given by $\tilde{g}(s) + m$ for some $m \in \mathbb{Z}$. Define the ***winding number of γ around*** $a \in \mathbb{R}^2$ to be

$$n(\gamma, a) := \tilde{g}(1) - \tilde{g}(0).$$

The following lemma is a consequence of an easy computation, although it is quite essential computation we can apply to more general integrals.

Theorem 13.4 (Lemma 66.3)**.** *Let $\gamma : \mathsf{I} \to \mathbb{C}$ be a piecewise differentiable loop in $\mathbb{C}$. Let $a \in \mathbb{C}$ such that $a \notin \gamma(\mathsf{I})$. Then*

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}.$$

Proof. Let $\tilde{g}$ be a lifting of g, so that, for $t \in \mathsf{I}$,

$$e^{2\pi i \tilde{g}(t)} = g(t) = \frac{\gamma(t) - a}{|\gamma(t) - a|}.$$

Let $r(t) := |\gamma(t) - a|$, then

$$\gamma(t) = r(t)\cdot e^{2\pi i \tilde{g}(t)} + a, \quad \gamma'(t) = r'(t)\cdot e^{2\pi i \tilde{g}(t)} + r(t)\cdot 2\pi i e^{2\pi i \tilde{g}(t)} \cdot \tilde{g}'(s).$$

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-a}dt = \frac{1}{2\pi i}\int_0^1 \left(\frac{r'(t)}{r(t)} + 2\pi i\tilde{g}'(t)\right)dt$$
$$= \frac{1}{2\pi i}\Big[\log r(t) + 2\pi i\tilde{g}(t)\Big]_0^1 = \frac{1}{2\pi i}\Big[2\pi i\tilde{g}(t)\Big]_0^1 = \tilde{g}(1) - \tilde{g}(0).$$

□

As a corollary of above two theorems, we obtain the Cauchy Integral Formula which seems a bit insufficient because of the appearance of the winding number.

Theorem 13.5 (Theorem 6, § 2.2, p.95 [Ahlfors]). *Let $f(z)$ be a analytic function over an open disk D in $\mathbb{C}$. Let $\gamma : \mathsf{I} \to D \subset \mathbb{C}$ be a piecewise-differentiable loop in D. For $a \in D$ such that $a \notin \gamma(\mathsf{I})$, we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = n(\gamma, a)\cdot f(a).$$

Proof. The proof is basically the application of the previous theorem. Let

$$h(z) := \frac{f(z) - f(a)}{z-a}.$$

Then $h(z)$ is analytic on $D - \{a\}$. However

$$\lim_{z\to a}(z-a)h(z) = \lim_{z\to a} f(z) - f(a) = 0.$$

Thus for any closed curve γ in $D - \{a\}$,

$$\int_\gamma \frac{f(z) - f(a)}{z-a}dz = 0,$$

which, together with Theorem 13.4, implies the formula. □

When $n(\gamma, \alpha) = 1$, we have the classicall Cauchy Integral Formula,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a).$$

The really topological problem is "when $n(\gamma, a) = 1$?" To formulate the Cauchy Integral Formula without the winding number, we need the following theorems in topology that seems trivial but is not easy to prove at all.

Theorem 13.6 (Jordan Curve Theorem). *Let $\gamma : \mathsf{I} \to \mathbb{R}^2$ be a loop such that $\gamma(t) = \gamma(t')$ iff $t = t'$ or $t = 0, t' = 1$ (simple loop). Then $\mathbb{R}^2 - \operatorname{Im}\gamma$ has two connected component, one is bounded and the other is bounded. Furthermore, the boundary of each component coincides with* $\operatorname{Im}\gamma$.

Theorem 13.7 (Winding number theorem). *If γ is a simple loop and a is a point in the bounded component of $\mathbb{R}^2 - \operatorname{Im}\gamma$, then $n(\gamma, a) = \pm 1$. We say γ orient the loop counter clockwise with respect to a if $n(\gamma, a) = 1$.*

With these theorems, we have

Theorem 13.8. *Let $f(z)$ be an analytic function on an open region D. Then*

$$\int_\gamma \frac{f(z)}{z-a}dz = 2\pi i$$

for all loops γ in D such that γ is counter clockwise around a and the bounded component of $\mathbb{R}^2 - \operatorname{Im}\gamma$ is contained in D.

13.1.1 Residue Theorem

Definition 13.9 (Wiki, Residues)**.** The residue of an analytic function f at an isolated singularity a, denoted $\mathrm{Res}_{z=a}(f, a)$ is the unique value R such that $f(z) - \frac{R}{z-a}$ has an analytic antiderivative in a small punctured disk $0 < |z-a| < \delta$. Alternatively, residues can be calculated by finding Laurent series expansions, and are sometimes defined in terms of them.

Theorem 13.10 (Residue Theorem)**.** *Let $f(z)$ be an analytic function on an open region $D - \{a\}$. Then*

$$\int_\gamma f(z)dz = 2\pi i \cdot n(\gamma, a) \cdot \mathrm{Res}_{z=a}(f, a)$$

for all loops γ in D and the bounded component of $\mathbb{R}^2 - \mathrm{Im}\,\gamma$ is contained in D.

13.2 Jordan Curve Theorem

Jordan Curve Theorem states something we want to believe without proving it. It's amazing how complicated to prove such a statement. However, on the other hand, there is the following theorem we haven't prove:

Theorem 13.11 (Peano Space Filling Curve, §44)**.** *There exists a continuous map $f : [0, 1] \to [0, 1] \times [0, 1]$ such that* $\mathrm{Im}\, f = [0, 1] \times [0, 1]$.

The point here is that the intuition is important to guide our investigation but it can not be trusted completely unless it is proved.

Definition 13.12. A loop $\gamma : \mathsf{I} \to X$ is a simple loop if $\gamma(t) = \gamma(t')$ iff $t = t'$ or $t = 0, t' = 1$. γ factors though the standard map $p : \mathsf{I} \to S^1, t \mapsto e^{2\pi i t}$ and the induced map $h : S^1 \to X$ must induces a homeomorphism $S^1 \cong \mathrm{Im}\,\gamma$.

Theorem 13.13 (25.5 [Mun])**.** *If X is locally-path connected, then its connected components and its path-connected components are the same.*

Lemma 13.14. *If γ is a simple loop, then $S^2 - \mathrm{Im}\,\gamma$ is locally path-connected. Thus its connected components and its path-connected components are the same.*

Proof. S^2 is locally path connected. Any open set in a locally path connected space is locally path connected. Thus the second clam follows from the theorem above. □

Lemma 13.15.

(1) *(61.1) Let C be a compact subspace of S^2 and $b \in S^2 - C$. Let $h : S^2 - \{b\} \cong \mathbb{R}^2$ be a homeomorphism, which also induces a homeomorphism $h : S^2 - \{b\} - C \cong \mathbb{R}^2 - C$. Consider a connected component $U \subset S^2 - C$. Then $h(U)$ is an unbounded connected component of $\mathbb{R}^2 - h(C)$ iff $b \in U$. In particular, the unbounded connected component of $\mathbb{R}^2 - h(C)$ is unique.*

(2) *(61.2) Let A be a compact space and $g : A \to \mathbb{R}^2 - \{\vec{0}\}$ a continuous map. If $\vec{0}$ lies in the unbounded connected component of $\mathbb{R}^2 - g(A)$, then g is homotopic to a constant map.*

(3) *(Borsuk Lemma 62.2) Let $a, b \in S^2$ and A a compact space. If there is a continuous injective map $f : A \to S^2 - \{a, b\}$ which is homotopic to a constant map, then a, b lie in the same connected component.*

Proof.

(2) Consider a big ball B centered at $\vec{0}$ in $\mathbb{R}^2$ so that $g(A)$ is contained in B (it's possible to take such a ball, since $g(A)$ is compact so the distance from $g(A)$ is bounded). If $\vec{p} \in \mathbb{R}^2 - B$, then $\vec{p}$ must lie in the unbounded component of $\mathbb{R}^2 - g(A)$. Thus $\vec{p}$ and $\vec{0}$ lie in the same unbounded component. Let $\alpha : \mathsf{I} \to \mathbb{R}^2 - g(A)$ be a path from $\vec{0}$ to $\vec{p}$. Define

$$G : A \times [0, 1] \to \mathbb{R}^2 - \{\vec{0}\}, \quad G(x, t) := g(x) - \alpha(t).$$

$G(x,t) \neq \vec{0}$ follows from the fact that α is a path in $\mathbb{R}^2 - g(A)$. G is a homotopy from g to $k : A \to \mathbb{R}^2 - \{\vec{0}\}, k(x) = g(x) - \vec{p}$. Now define

$$H : A \times [0,1] \to \mathbb{R}^2 - \{\vec{0}, \quad H(x,t) := tg(x) - \vec{p}.$$

It is a homotopy from a constant map to k. Thus we have a homotopy from g to a constant map.

□

Theorem 13.16 (Jordan separation theorem)**.** *If $\gamma : \mathsf{I} \to S^2$ is a simple loop, then $S^2 - \operatorname{Im}\gamma$ is not connected.*

Proof.

1. By Lemma 13.14, it suffices to show that $S^2 - \operatorname{Im}\gamma$ is not path-connected. Assume that it is path-connected.
2. $\operatorname{Im}\gamma = A \cup B$ where A and B are image of some paths and $A \cap B = \{a, b\}$. Let $U := S^2 - A$ and $V := S^2 - B$, then
$$U \cap V = S^2 - \operatorname{Im}\gamma.$$
Thus by the assumption, $U \cap V$ is path-connected, so we can apply the first van Kampen Theorem.
3. $U \cup V = S^2 - \{a, b\}$ which is homeomorphic to $\mathbb{R}^2 - \{\vec{0}\}$. So its fundamental group is isomorphic to $\mathbb{Z}$. On the other hand, we will prove that the inclusions $U \hookrightarrow U \cup V$ and $V \hookrightarrow U \cup V$ induce trivial homomorphisms on fundamental groups so that, by the first van Kampen Theorem, $U \cup V$ has the trivial fundamental group. This leads to a contradiction.
4. Let $f : \mathsf{I} \to U = S^2 - A$ be a loop. It factors through the standard quotient map $p : \mathsf{I} \to S^1, t \mapsto e^{2\pi i t}$. Let $h : S^1 \to S^2 - A$ be the map such that $h \circ p = f$. Let $i : S^2 - A \hookrightarrow S^2 - \{a, b\}$ and $j : S^2 - A \hookrightarrow S^2$ be the natural inclusions. Then since $i \circ h(S^1)$ doesn't not intersect with A, we know that a and b are connected by the path A so that they are in the same path-connected component of $S^2 - j \circ h(S^1)$. We can now show that $i \circ h : S^1 \to S^2 - \{a, b\}$ is null-homotopic (*), so that it induces a trivial homomorphism by Lemma 11.10. Therefore
$$i_* \circ h_*([p]) = i_*([h \circ p]) = i_*([f]) = 1.$$
5. (*) is basically Lemma 13.15 (1) and (2).

□

Theorem 13.17 (A non-separation lemma)**.** *If $\gamma : \mathsf{I} \to S^2$ is a simple path, then $S^2 - \operatorname{Im}\gamma$ has exactly one component.*

Proof. Since $\operatorname{Im}\gamma$ is contractible (*), $\mathrm{id}_{\operatorname{Im}\gamma} : \operatorname{Im}\gamma \to \operatorname{Im}\gamma$ is homotopic to a constant map. It implies that if $a, b \in S^2 - \operatorname{Im}\gamma$, then the inclusion $g : \operatorname{Im}\gamma \hookrightarrow S^2 - \{a, b\}$ is homotopic to a constant map. By Lemma 13.15 (3), a and b are in the same component. Thus $S^2 - \operatorname{Im}\gamma$ has no more than one connected component.

- γ is simple if it is continuous injective map. Since I is compact and S^2 is Hausdorff, γ is a topological embedding, i.e. I is homeomorphic to $\operatorname{Im}\gamma$.
- We can show there is at least one component, i.e. $S^2 \neq \operatorname{Im}\gamma$. We know that $\operatorname{Im}\gamma$ is not homeomorphic to S^2 by taking one point out, one is connected but the other is not. If $\operatorname{Im}\gamma \hookrightarrow S^2$ is a continuous injective and both spaces are compact and Hausdorff, if surjective, then it must be homeomorphism. Constradiction.

□

Theorem 13.18 (63.1, converse to the first Seifert-van Kampen)**.** *Let $X = U \cup V$ where U, V are open sets in X. Suppose $U \cap V = A \sqcup B$ where A, B are open sets.*

(1) Let $a \in A$ and $b \in B$. If α is a path in U from a to b and β is a path in V from b to a, then the loop $f := \alpha * \beta$ at a generates an infinite cyclic subgroup of $\pi_1(X, a)$.

(2) *Let $a, a' \in A$. If γ is a path in U from a to a' and δ is a path in V from a' to a, then the loop $g := \gamma * \delta$ generates a subgroup of $\pi_1(X, a)$ which intersect the subgroup generated by $[f]$ trivially, i.e. $\langle [g] \rangle \cap \langle [f] \rangle = \{1\}$.*

Theorem 13.19 (Jordan Curve Theorem)**.** *Let $\gamma : \mathsf{I} \to S^2$ be a simple loop. Then $S^2 - \operatorname{Im}\gamma$ has exactly two connected components W_1 and W_2. Furthermore, the boundary of each component coincides with $\operatorname{Im}\gamma$.*

Proof.

1. By Separation theorem, there are at least two (path-)connected components. Decompose $\operatorname{Im}\gamma$ into two simple paths C_1 and C_2 with $C_1 \cap C_2 = \{x, y\}$. Let $U_i := S^2 - C_i$. By non-separation theorem, they are connected. Then $U_1 \cap U_2 = S^2 - \operatorname{Im}\gamma$.

2. We assume that $S^2 - \operatorname{Im}\gamma$ has more than two connected components and derive a contradiction. Say A_1, A_2 are two distinct components and B is the union of other components.

3. Let $a \in A_1$, $a' \in A_2$ and $b \in B$. Let α be a path in U from a to a' and γ in U from a to b. Let β be a path in V from a' to a and δ in V from b to a. Consider the loops in $U \cup V$ at a, $f : \alpha * \beta$ at a and $g = \gamma * \delta$.

4. By Theorem 13.18 (1) applied to $U \cap V = (A_1 \cup A_2) \sqcup B$, g generates an infinite cyclic subgroup of $\pi_1(S^2 - \{x, y\}, a)$ and by Theorem 13.18 (1) applied to $U \cap V = A_1 \sqcup (A_2 \sqcup B)$, f generates an infinite cyclic subgroup of $\pi_1(S^2 - \{x, y\}, a)$.

5. Since $\pi_1(S^2 - \{x, y\}, a) \cong \pi_1(\mathbb{R}^2 - \{0\}, p) \cong \mathbb{Z}$, if s is its generator, then $[g] = s^m$ and $[f] = s^n$. Thus $[g]^n = [f]^m$ which contradict with Theorem 13.18 (2).

□

13.3 Winding Number Theorem

Lemma 13.20 (65.2, Winding number theorem)**.** *] Let C be a simple closed curve (the image of a simple loop) in S^2. If $p, q \in S^2$ lie in different components of $S^2 - C$, then the inclusion map $j : C \hookrightarrow S^2 - \{p, q\}$ induces an isomorphism of fundamental groups. In other words, let C be a simple closed curve in $\mathbb{R}^2$. If p lie in the bounded component of $\mathbb{R}^2 - C$, then the inclusion map $j : S^2 \hookrightarrow \mathbb{R}^2 - p$ induces an isomorphism of fundamental groups.*

Theorem 13.21. *If γ is a simple loop and a is a point in the bounded component of $\mathbb{R}^2 - \operatorname{Im}\gamma$, then $n(\gamma, a) = \pm 1$. We say γ orient the loop counter clockwise with respect to a if $n(\gamma, a) = 1$.*

Proof.

- Without loss of generality, we can assume $a = \vec{0}$ because $n(\gamma, a) = n(\gamma - a, \vec{0})$ and a is in the unbounded component of $\mathbb{R}^2 - \gamma(\mathsf{I})$ if and only if 0 is in the unbounded component of $\mathbb{R}^2 - (\gamma - a)(\mathsf{I})$.

- $\gamma : \mathsf{I} \to \mathbb{R}^2 - \vec{0}$ factors through the standard map $p : \mathsf{I} \to S^1$ and induces a map $h : S^1 \to \mathbb{R}^2 - \vec{0}$ such that $h : S^1 \to \gamma(\mathsf{I})$ is a homeomorphism. Since $[p]$ is a generator of $\pi_1(S^1, 0)$, $h_*[p]$ is a generator of $\pi_1(\mathbb{R}^2 - \vec{0})$ if $\vec{0}$ is in the bounded component of $\mathbb{R}^2 - \operatorname{Im}\gamma$ by Lemma 13.20. If $\vec{0}$ is in the unbounded component of $\mathbb{R}^2 - \operatorname{Im}\gamma$, then h is nullhomotopic by Lemma 13.15 so that $h_*[p]$ is trivial by Lemma 11.10.

- Consider the deformation retraction $r : \mathbb{R}^2 - \vec{0} \to S^1, x \to x/|x|$, then the induced map $r_* : \pi_1(\mathbb{R}^2 - \vec{0}) \to \pi^1(S^1)$ is an isomorphism by Theorem 10.15. Then $r_*[\gamma] = [r \circ \gamma]$ is a generator in the bounded case and is trivial in the unbounded case. Since $r \circ \gamma(s) = \gamma(s)/|\gamma(s)|$, $n(\gamma, \vec{0}) = \pm 1$ in the bounded case and $n(\gamma, \vec{0}) = 0$ in the unbounded case.

□

14 Classification of compact topological surfaces.

14.1 Triangulation

Definition 14.1. Let X be a compact topological surface. A ***curved triangle*** in X is a subset A in X together with a homeomophism $h : T \to A$ where T is a closed triangular region in $\mathbb{R}^2$. A ***triangulation*** of X is a collection $\{A_1, \cdots, A_n\}$ of curved triangles such that

(Tr1) $\cup_i A_i = X$

(Tr2) For $i \neq j$, $A_i \cap A_j = \emptyset$, a vertex or an edge of both.

(Tr3) If $A_i \cap A_j$ is an edge, then $h_j^{-1} \circ h_i$ on the corresponding edge of T_i is *linear*.

Theorem 14.2 (c.f. [DM])**.** *Every compact surface is triangulable.*

Theorem 14.3 (78.1)**.** *If X is a compact triangulable surface, then X is homeomorphic to the quotient space obtained from a collection of disjoint triangular regions in $\mathbb{R}^2$ by identifying their edges in pairs.*

Proof. Let $\{A_1, \cdots, A_n\}$ be a triangulation of X. Then consider the map $\pi : T_1 \cup \cdots T_n \to X$ where each T_i maps to X via h_i and which is automatically a quotient map (a surjective map from a compact space E to a Hausdorff space X is a closed map and so a quotient map: a closed set $A \subset E$ is compact since E is comact, the image of a compact subspace is compact, a compact subspace in a Hausdorff space is closed.) There are following two things to prove.

1. For each edge e of A_i, there is exactly one other A_j such that $A_i \cap A_j = e$ so that $h_j^{-1} \circ h_i$ will identify the corresponding edges of T_i and T_j in pairs.
2. There is no additional vertex identification, i.e. if $A_i \cap A_j = v$ is a vertex, then there is a sequence $A_i = A_{i_1}, \cdots, A_{i_r} = A_j$ of triangles having v as a vertex such that $A_{i_k} \cap A_{j_{k+1}}$ is an edge containing v.

See the proofs for these claims at page 472 - 475 [Mun]. □

Theorem 14.4 (78.2)**.** *If X is a compact connected triangulable surface, then X is homeomorphic to a space obtained from a polygonal region in $\mathbb{R}^2$ by identifying the edges in pairs.*

Proof. By the preceding theorem, we have a collection of triangular regions $T_1, \cdots, T_n$ in $\mathbb{R}^2$, together with the oriented labels on the edges. Start with two triangles having the same oriented label on edges. By identifying them, we have $n-1$ regions. Next take two distinct regions having the same label and identify them. Continuing this process $n-1$ times, we have a single polygonal region with oriented labels on edges in pairs. □

14.2 Classification of polygon quotients

Recall how to construct a surface out of a polygonal region in $\mathbb{R}^2$.

Let Δ be an $2m$-gon polytope with edges $e_1, \cdots, e_{2m}$ (numbered counter clockwisely). Create m-pairs among $\{e_i\}$, each labeled by $a_1, \cdots, a_m$. Orient the boundary of Δ counter clockwise. Assign $\epsilon_i := \pm 1$ to each e_i. Orient e_i compatibly with the orientation on $\partial\Delta$ if $\epsilon = +1$. Orient oppositely if $\epsilon = -1$. All these information is written on the right hand side of

$$(e_1, \cdots, e_{2m}) = (a_{i_1}^{\epsilon_1}, \cdots, a_{i_2m}^{\epsilon_{2m}}).$$

Now identify paired edges consistently with the orientation and obtain $X := \Delta / \sim$.

Definition 14.5 (Cut and Paste)**.** A convex polygonal region can be cut along a line connecting two vertices and decomposed into two pieces, provided the there are at least four vertices. Now the quotient space is obtained by glueing two polygonal regions, that is, by pasting the cutting section as they were before cutting. This new presentation do not change the resulting quotient. We should generalize the above construction to a collection of polygonal regions with labeling, i.e. let $P_1, \cdots, P_r$ be polygonal regions with $2m$ edges in total.

$$(\underbrace{a_{i_1}^{\epsilon_1}, \cdots,}_{label\ on\ P_1} | \underbrace{, \cdots,}_{labels\ on\ P_2} | \cdots | \underbrace{, \cdots, a_{i_{2m}}^{\epsilon_{2m}}}_{label\ on\ P_r})$$

Call this a labeling scheme for X.

Definition 14.6. Define the following ***elementary operation*** on labeling schemes:

1. (*Cut*) $(a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m}}^{\epsilon_{2m}}), m > 1 \Rightarrow (a_{i_1}^{\epsilon_1}, \cdots, c^{-1} \mid c, \cdots, a_{i_{2m}}^{\epsilon_{2m}})$
2. (*Paste*) $(a_{i_1}^{\epsilon_1}, \cdots, c^{-1} \mid c, \cdots, a_{i_{2m}}^{\epsilon_{2m}}) \Rightarrow (a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m}}^{\epsilon_{2m}})$
3. (*Relabel*) $(a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m}}^{\epsilon_{2m}}) \Rightarrow$ replace a label α_i by β_i, or by α_i^{-1} (so that $(\alpha_i^{-1})^{-1} = \alpha_i$).
4. (*Permute*) $(a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m}}^{\epsilon_{2m}}) \Rightarrow (a_{i_{2m}}^{\epsilon_{2m}}, a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m-1}}^{\epsilon_{2m-1}})$
5. (*Flip*) $(a_{i_1}^{\epsilon_1}, \cdots, a_{i_{2m}}^{\epsilon_{2m}}) \Rightarrow (a_{i_{2m}}^{-\epsilon_{2m}}, \cdots, a_{i_2}^{-\epsilon_2}, a_{i_1}^{-\epsilon_1})$
6. (*RelabelI*) If ab and, ab or $b^{-1}a^{-1}$ appear, we can relabel ab by c and $b^{-1}a^{-1}$ by c^{-1} after combining two edges to one.

Note that 3,4,5 can be applied to the labeling on each P_i.

Theorem 14.7. *The elementary operations don't affect the resulting quotient.*

Proof. It might be just easy to picture them. □

Theorem 14.8. *If X is obtained from a polygonal region in $\mathbb{R}^2$ by identifying edges in pairs, then X is homeomorphic to one of the folloiwng: S^2, $T\sharp\cdots\sharp T$, $\mathbb{RP}^2\sharp\cdots\sharp\mathbb{RP}^2$. Furthermore, those spaces in the list are all non-homeomorphic to each other (non-homotopic to each other).*

Proof. A labeling scheme can be transformed, via elementary operations, to one of the following:

- (a, a^{-1}, b, b^{-1}) (S^2)
- (a, b, a, b) $(\mathbb{RP}^2)$
- $(a_1, a_1, \cdots, a_m, a_m), m > 1$ $(\mathbb{RP}^2\sharp\cdots\sharp\mathbb{RP}^2)$
- $(a_1, a_1^{-1}, b_1, b_1^{-1}, \cdots, a_n, a_n^{-1}, b_n, b_n^{-1})$ $(T\sharp\cdots\sharp T)$

The fundamental groups are given by the corresponding relations and they are all different (the number of generators are different or the order of the generators are different). □

14.3 Euler characteristic

Definition 14.9. The ***Euler characteristic*** of a triangulated surface X is defined by

$$\chi(X) := V - E + F$$

where V is the number of vertices, E is the number of edges and F is the number of triangles ("faces").

Theorem 14.10. *The Euler characteristic is independent of the choice of the triangulation.*

Theorem 14.11. *The Euler characteristics of the spaces in the list of the classification are*

- $\chi(S^2) = 3 - 3 + 2 = 2.$
- $\chi(\mathbb{RP}^2) = 2 - 3 + 2 = 1.$
- $\chi(\underbrace{\mathbb{RP}^2 \sharp \cdots \sharp \mathbb{RP}^2}_{m}) = 2 - m.$
- $\chi(\underbrace{T \sharp \cdots \sharp T}_{n}) = 2 - 2n.$

Proof. In general, for two surfaces S_1 and S_2, we have

$$\chi(S_1 \sharp S_2) = \chi(S_1) + \chi(S_2) \quad \underbrace{-2}_{-\text{one triangle glue } 3 \text{ edges and } 3 \text{ vertices}} \quad \underbrace{+3-3}$$

Thus, once we compute $\chi(S^2) = 2$ and $\chi(\mathbb{RP}^2) = 1$, we have

$$\chi(\underbrace{\mathbb{RP}^2 \sharp \cdots \sharp \mathbb{RP}^2}_{m}) = m - 2(m-1) = 2 - m$$

$$\chi(\underbrace{T \sharp \cdots \sharp T}_{n}) = \chi(S^2 \sharp \underbrace{T \sharp \cdots \sharp T}_{n}) = 2 - 2n.$$

□

14.4 What happen to the Klein bottle?

The Klein bottle is given by the labeling scheme (a, b, a^{-1}, b).

$$aba^{-1}b \xrightarrow[cut]{} abc|c^{-1}a^{-1}b \xrightarrow[perm.]{} cab|c^{-1}a^{-1}b \xrightarrow[flip]{} cab|b^{-1}ac \xrightarrow[paste]{} caac \xrightarrow[perm]{} aacc$$

Thus the Klein bottle is $\mathbb{RP}^2 \sharp \mathbb{RP}^2$.

Exercise 14.12. What in the classification list corresponds to $\mathbb{RP}^2 \sharp T$? Note that $\mathbb{RP}^2 \sharp T$ is given by the labeling scheme

$$(a, b, a, b, c, d, c^{-1}, d^{-1}).$$

References

[Mun] Muncres, *Topology*, 2nd edition

[Set] Basic Set Theory, http://www.math.cornell.edu/~matsumura/math4530/basic set theory.pdf

[DK] Lie Groups. Book.

[P] Paulsen, AN INTRODUCTION TO THE THEORY OF TOPOLOGICAL GROUPS AND THEIR REPRESENTATIONS, www.math.uh.edu/ vern/grouprepn.pdf

[GGK] Ginzburg, Guillemin, Karshon, *Moment Maps, Cobordisms and Group Actions*

[Bredon] Bredon, Topology and Geometry, book

[Ahlfors] Ahlfors, Complex Analysis, book

[DM] Doyle, Moran, A Short Proof that Compact 2-Manifolds Can Be Triangulated, Inventiones math., 5, 160-162 (1968), 3 pages.