Centralizers and dynamics in Thompson's group $V$

Francesco Matucci
(with C. Bleak, H. Bowman, A. Gordon, G. Graham, J. Hughes and J. Sapir)

GAGTA 5 - Manresa

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Thompson’s group $F$
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Thompson’s group $T$
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Similar to $F$, but defined on the unit circle: it preserves the cyclic order of the intervals
Thompson’s group \( T \)

Similar to \( F \), but defined on the unit circle: it preserves the cyclic order of the intervals

![Diagram of Thompson's group T](image-url)
Thompson’s group $T$

Similar to $F$, but defined on the unit circle: it preserves the cyclic order of the intervals.
Thompson’s group $V$

Similar to $F$, but not continuous: it permutes the order of the intervals and can be seen as a group of homeomorphisms of the Cantor set $\mathcal{C}$ to itself:
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Motivation: simultaneous conjugacy problem
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Given two $k$-tuples $(y_1, \ldots, y_k), (z_1, \ldots, z_k)$ of elements in a group $G$, can we decide if there is a $g \in G$ such that $gy_i g^{-1} = z_i$?
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Rephrasing: given \((x_1, \ldots, x_{k-1}, y), (x_1, \ldots, x_{k-1}, z)\), decide if there is a \(g \in G\) with \(gyg^{-1} = z\) and \(g \in C_G(\langle x_1, \ldots, x_{k-1} \rangle)\).
Motivation: simultaneous conjugacy problem

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**Rephrasing**: given $(x_1, \ldots, x_{k-1}, y), (x_1, \ldots, x_{k-1}, z)$, decide if there is a $g \in G$ with $gyg^{-1} = z$ and $g \in C_G(\langle x_1, \ldots, x_{k-1} \rangle)$.

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**Theorem (Kassabov-M)**

*SCP is solvable in $F$.*

**Theorem (Barker)**

*SCP is solvable in $V$.*
Motivation: Farrell-Jones conjecture
The Farrell-Jones conjecture relates homology of groups to $K$-theory.

This conjecture is proved under some hypotheses, one of which is that centralizers of elements of finite order have good homological finiteness properties.

(asked us by R. Geoghegan and M. Varisco)
Centralizers in Thompson’s group $F$
Centralizers in Thompson’s group $F$

**Theorem (Guba-Sapir, 1997)**

Let $\alpha \in F$. Then $C_F(\alpha) \cong F^m \times \mathbb{Z}^n$. 

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Centralizers in Thompson’s group $F$

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Theorem (Guba-Sapir, 1997)

Let $\alpha \in F$. Then $C_F(\alpha) \cong F^m \times \mathbb{Z}^n$.

$F$-terms appear where $\alpha$ is trivial.

$\mathbb{Z}$-terms appear where $\alpha$ is non-trivial.
Centralizers in Thompson’s group $T$

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Centralizers and dynamics in Thompson’s group $V$
Fact: If $\alpha \in T$, there is $q \in \mathbb{N}$ such that $\alpha^q$ has fixed points.
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Theorem (Bleak-Kassabov-M: torsion case)

Let $\alpha \in T$ be such that $\alpha^q = id_{S^1}$. Then $C_T(\alpha)$ is a central extension

$$1 \rightarrow C_q \rightarrow C_T(\alpha) \rightarrow T \rightarrow 1$$

where $C_q$ is the cyclic of order $q$. This extension does not split.
**Fact:** If \( \alpha \in T \), there is \( q \in \mathbb{N} \) such that \( \alpha^q \) has fixed points.

**Theorem (Bleak-Kassabov-M: torsion case)**

Let \( \alpha \in T \) be such that \( \alpha^q = \text{id}_{S^1} \). Then \( C_T(\alpha) \) is a central extension

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1 \rightarrow C_q \rightarrow C_T(\alpha) \rightarrow T \rightarrow 1
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where \( C_q \) is the cyclic of order \( q \). This extension does not split.

**Theorem (Bleak-Kassabov-M: non-torsion case)**

Let \( \alpha \in T \) be such that \( \alpha^q \neq \text{id}_{S^1} \). Then \( C_T(\alpha) \) is an extension

\[
1 \rightarrow F^r \times \mathbb{Z}^s \rightarrow C_T(\alpha) \rightarrow C_m \rightarrow 1
\]

where \( C_m \) is a suitable cyclic group of order \( m \).
Centralizers in Thompson’s group $V$
Centralizers in Thompson’s group $V$

Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)
Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)

Let $\alpha \in V$. Then, there exist non-negative integers $s, t, n_1, \ldots, n_s, q_1, \ldots, q_t$ so that

$$C_V(\alpha) \cong \left( \prod_{i=1}^{s} \text{Maps}(\mathcal{C}, C_{n_i}) \rtimes V \right) \times \left( \prod_{j=1}^{t} (A_j \times \mathbb{Z}) \wr \text{Sym}(q_j) \right)$$
**Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)**

Let $\alpha \in V$. Then, there exist non-negative integers $s, t, n_1, \ldots, n_s, q_1, \ldots, q_t$ so that

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- $Maps(\mathcal{C}, C_{n_i})$ is a group of continuous maps from $\mathcal{C}$ to the cyclic group $C_{n_i}$.
- $A_j$ is the group of torsion elements in the centralizer of a “component of a flow graph” of $\alpha$.
- $Sym(q_j)$ full symmetric group on the “isomorphic components of a flow graph”.

Francesco Matucci (with C.Bleak, H.Bowman, A.Gordon, G.C)  Centralizers and dynamics in Thompson’s group $V$.
Elements of $V$ can be represented by pairs of binary trees and a permutation. Here is an example:
Thompson’s group $V$

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The tree pair indicates how to map $\mathcal{C}$ to itself.
Thompson’s group $V$

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![Diagram of binary trees representing elements of $V$.]
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Elements of $V$ can be represented by pairs of binary trees and a permutation. Here is an example:
Multiplication of tree pairs
Multiplication of tree pairs
Multiplication of tree pairs

1 2 5 3 1 5
3 4

4 →

0

4 →

1 2
2 1
Multiplication of tree pairs
Multiplication of tree pairs

![Diagram of tree pairs multiplication]

- Tree 1: 1, 2, 5, 3, 34
- Tree 2: 4, 2
- Resulting tree: 1, 5, 1, 234

The diagrams show the process of multiplying tree pairs.
Multiplication of tree pairs

\[
\begin{array}{ccccccccc}
& & & & & 1 & 5 & & & \\
& & & & & 4 & 2 & & & \\
1 & 2 & 5 & 3 & & & & & \\
3 & 4 & & & & & & & \\
\end{array}
\]
Multiplication of tree pairs
Multiplication of tree pairs

1 2 5 3 1 5 5 6 3 4 2 1 2 3 4 5 6
Multiplication of tree pairs

Centralizers and dynamics in Thompson's group V
Multiplication of tree pairs
Multiplication of tree pairs

Centralizers and dynamics in Thompson's group V
Multiplication of tree pairs

1 2 → 6
3 4 5 2
3 4

1 6 → 1 6 5 2
4 3 1 6

Centralizers and dynamics in Thompson's group V
Multiplication of tree pairs

\[
\begin{array}{c}
1 & 2 & 6 \\
5 & 3 & 4 \\
1 & 6 \\
5 & 2 \\
4 & 31 & 6 \\
\end{array}
\]
Multiplication of tree pairs

\[
\begin{array}{cccc}
1 & 2 & 6 \\
5 & 3 & 4 \\
\end{array}
\rightarrow
\begin{array}{ccccc}
5 & 2 \\
4 & 3 & 1 & 6 \\
\end{array}
\]
An element of $T$
An element of $F$

[Diagram with labeled nodes and arrows, illustrating the dynamics and centralizers in Thompson's group $V$.]

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Revealing tree pairs describe dynamics

Francesco Matucci (with C.Bleak, H.Bowman, A.Gordon, G.C)

Centralizers and dynamics in Thompson’s group V
Brin introduced a special type of tree pair for elements called revealing pair.
Revealing tree pairs describe dynamics

Brin introduced a special type of tree pair for elements called revealing pair.

From this pair it is easy to find

- repellors and attractors,
- their neighborhoods: sources and sinks,
- periodic orbits.
A non-revealing pair
A better representative

Centralizers and dynamics in Thompson's group V
Corollary (Burillo-Cleary-Stein-Taback)

Let \( \alpha \in F, T \) or \( V \). Then \( \alpha \) has finite order if and only if it has a representative tree pair \((D, R, \sigma)\) with \( D = R \).
Revealing pairs: cyclic orbits

[Diagram with numbered vertices and arrows connecting them.]

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Centralizers and dynamics in Thompson's group V
Revealing pairs: attracting leaves
Revealing pairs: repelling leaves and orbits
Revealing pairs: sources flow to sinks
Revealing pairs: the flow graph

Left part: non-torsion part. Right part: torsion part
Revealing pairs: a partition of $\mathcal{C}$
Given \( \alpha \sim (D, R, \sigma) \), we can decompose \( \mathcal{C} \) as \( T_\alpha \cup Z_\alpha \):
Revealing pairs: a partition of $\mathcal{C}$

Given $\alpha \sim (D, R, \sigma)$, we can decompose $\mathcal{C}$ as $T_\alpha \cup Z_\alpha$:

$T_\alpha$: subset of $\mathcal{C}$ lying under the leaves that are on cyclic orbits
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- $T_\alpha$: subset of $\mathcal{C}$ lying under the leaves that are on cyclic orbits
- $Z_\alpha$: subset of $\mathcal{C}$ lying under the remaining leaves (including attractors and repellers)
Revealing pairs: a partition of $\mathcal{C}$

Given $\alpha \sim (D, R, \sigma)$, we can decompose $\mathcal{C}$ as $T_\alpha \cup Z_\alpha$:

$T_\alpha$: subset of $\mathcal{C}$ lying under the leaves that are on cyclic orbits

$Z_\alpha$: subset of $\mathcal{C}$ lying under the remaining leaves (including attractors and repellers)

Break the centralizers of an element $\alpha$ in two direct components:

$$C_V(\alpha) \cong C_{V_{T_\alpha}(\alpha_T)} \times C_{V_{Z_\alpha}(\alpha_Z)}$$
Results on torsion elements

Francesco Matucci (with C.Bleak, H.Bowman, A.Gordon, G.C) Centralizers and dynamics in Thompson’s group V
Replace a torsion element by a suitable simpler conjugate.
Results on torsion elements

Replace a torsion element by a suitable simpler conjugate.

Lemma

If $\alpha$ is torsion then, up to conjugation, one has $\alpha \sim (D, D, \sigma)$ and
Replace a torsion element by a suitable simpler conjugate.

**Lemma**

If $\alpha$ is torsion then, up to conjugation, one has $\alpha \sim (D, D, \sigma)$ and 1 leaves of $D$ are divided into disjoint cycles of varying lengths.
Replace a torsion element by a suitable simpler conjugate.

Lemma

If $\alpha$ is torsion then, up to conjugation, one has $\alpha \sim (D, D, \sigma)$ and

1. leaves of $D$ are divided into disjoint cycles of varying lengths.
2. for any $k$, there is at most one cycle of length $k$. 
Replace a torsion element by a suitable simpler conjugate.

Lemma

If $\alpha$ is torsion then, up to conjugation, one has $\alpha \sim (D, D, \sigma)$ and

1. leaves of $D$ are divided into disjoint cycles of varying lengths.
2. for any $k$, there is at most one cycle of length $k$.
3. if $\beta \in C_V(\alpha)$, then $\beta$ sends each cycle of $\alpha$ to itself.
A torsion element with more cycle lengths

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Centralizers and dynamics in Thompson’s group V
A torsion element with more cycle lengths

Here we have

- domain tree = range tree,
- leaf cycles (1 2 3) and (a b).
A torsion element with more cycle lengths

Here we have

- domain tree = range tree,
- leaf cycles (1 2 3) and (a b).

Restrict to elements with **only one** cycle.
General Strategy: an induced action
Let $\alpha \in V$. Let $H = \langle \alpha \rangle$. Define the “Fundamental Domain” of the action of $\alpha$ to be the space $\mathcal{C}/H$. If an element commutes with $\alpha$, it will induce an action on $\mathcal{C}/H$. 
Let $\alpha \in V$. Let $H = \langle \alpha \rangle$. Define the “Fundamental Domain” of the action of $\alpha$ to be the space $\mathcal{C}/H$. If an element commutes with $\alpha$, it will induce an action on $\mathcal{C}/H$.

We get a short exact sequence:

$$1 \rightarrow K \rightarrow C_V(\alpha) \rightarrow Q \rightarrow 1$$
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$$1 \to \mathcal{K} \to C_V(\alpha) \to Q \to 1$$

$\mathcal{K}$ represents the elements in $C_V(\alpha)$ which act on $\mathcal{C}$ in such a way that their induced action on $\mathcal{C}/H$ is trivial.
Let $\alpha \in V$. Let $H = \langle \alpha \rangle$. Define the “Fundamental Domain” of the action of $\alpha$ to be the space $C/H$. If an element commutes with $\alpha$, it will induce an action on $C/H$.

We get a short exact sequence:

$$
1 \to \mathcal{K} \to C_V(\alpha) \to \mathcal{Q} \to 1
$$

$\mathcal{K}$ represents the elements in $C_V(\alpha)$ which act on $C$ in such a way that their induced action on $C/H$ is trivial.

$\mathcal{Q}$ is the quotient of $C_V(\alpha)$ by the image of the inclusion map.
The fundamental domain
Suppose the tree pair below represents $\alpha$
The fundamental domain

Suppose the tree pair below represents \( \alpha \)

```
      1
     / \
    2   3
```

```
      3
     / \
    1   2
```
The fundamental domain

Suppose the tree pair below represents $\alpha$

```
      1
     / \  \
   2   3
```

The subset underlying any leaf of this element could be used to represent the fundamental domain, that is $\mathcal{C}/\langle \alpha \rangle \cong \mathcal{C}$.
This labelled tree determines a (continuous) map from $\mathcal{C} \to C_3$.

(Warning: The diagram below does not give an element of $V$!)
The bottom group $K$

This labelled tree determines a (continuous) map from $\mathcal{C} \to C_3$. 

(Warning: The diagram below does not give an element of $V$!)
This labelled tree determines a (continuous) map from $C \to C_3$.  
(Warning: The diagram below does not give an element of $V$!)

Given the map above, to build a centralizer do this:
This labelled tree determines a (continuous) map from $\mathcal{C} \to C_3$. 
(Warning: The diagram below does not give an element of $V$!)

Given the map above, to build a centralizer do this:

Attach the diagram above to each of the leaves of $\alpha$ and shift leaves according to the map $\mathcal{C} \to C_3$ above
The bottom group $\mathcal{K}$: an example
The bottom group $\mathcal{K}$: an example
The bottom group $\mathcal{K}$: an example
The bottom group $\mathcal{K}$: an example
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The bottom group $\mathcal{K}$: an example
The top group $Q$
We can pick any element of $V$ and attach it on the leaves of $\alpha$.

Hence, the top group is isomorphic with $V$. 

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The top group $Q$

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Francesco Matucci (with C. Bleak, H. Bowman, A. Gordon, G. C) Centralizers and dynamics in Thompson's group $V$
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Francesco Matucci (with C. Bleak, H. Bowman, A. Gordon, G. C. Centralizers and dynamics in Thompson's group $V$)
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The top group $Q$

We can pick any element of $V$ and attach it on the leaves of $\alpha$.

Hence, the top group is isomorphic with $V$. 

![Diagram showing the structure of the top group with elements labeled 1, 2, 3, 4, 5, 6.]
The top group $Q$

We can pick any element of $V$ and attach it on the leaves of $\alpha$.

Hence, the top group is isomorphic with $V$. 

Francesco Matucci (with C. Bleak, H. Bowman, A. Gordon, G. C.) Centralizers and dynamics in Thompson's group $V$
Structure of $C_V(\alpha)$, for $\alpha$ torsion in $V$ with one cycle
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\[ 1 \rightarrow \mathcal{K} \rightarrow C_V(\alpha) \rightarrow \mathbb{Q} \rightarrow 1 \]
Structure of $C_V(\alpha)$, for $\alpha$ torsion in $V$ with one cycle

$$1 \to \mathcal{K} \to C_V(\alpha) \to \mathbb{Q} \to 1$$

\[ \mathcal{K} \cong \text{Maps}(\mathcal{E}, C_k) \]
Structure of $C_V(\alpha)$, for $\alpha$ torsion in $V$ with one cycle

$1 \rightarrow \mathcal{K} \rightarrow C_V(\alpha) \rightarrow Q \rightarrow 1$

- $\mathcal{K} \cong \text{Maps}(\mathcal{E}, C_k)$
- $Q \cong V$. 
Structure of $C_V(\alpha)$, for $\alpha$ torsion in $V$ with one cycle

$$1 \rightarrow \mathcal{K} \rightarrow C_V(\alpha) \rightarrow Q \rightarrow 1$$

- $\mathcal{K} \cong Maps(\mathcal{C}, C_k)$
- $Q \cong V$.

This extension splits: $C_V(\alpha) \cong Maps(\mathcal{C}, C_k) \rtimes V$. 
Structure of $C_V(\alpha)$, for $\alpha$ torsion in $V$ with one cycle

1 $\rightarrow \mathcal{K} \rightarrow C_V(\alpha) \rightarrow Q \rightarrow 1$

- $\mathcal{K} \cong Maps(\mathcal{C}, C_k)$
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This extension splits: $C_V(\alpha) \cong Maps(\mathcal{C}, C_k) \rtimes V$.

**Remark.** $Maps(\mathcal{C}, C_k)$ is not f.g., but $Maps(\mathcal{C}, C_k) \rtimes V$ is.
The general torsion solution

If $\alpha$ has $s$ cycles with distinct lengths, we get:
The general torsion solution

If $\alpha$ has $s$ cycles with distinct lengths, we get:

$$C_V(\alpha) \cong \prod_{i=1}^{s} (\text{Maps}(\mathcal{C}, C_{n_i}) \rtimes V).$$
Decomposing non-torsion actions by flow graphs

Centralizers and dynamics in Thompson's group V

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Decomposing non-torsion actions by flow graphs

Centralizers and dynamics in Thompson's group V
Results on non-torsion elements

Francesco Matucci (with C. Bleak, H. Bowman, A. Gordon, G. C. Centralizers and dynamics in Thompson's group V
Let $\alpha$ be a purely non-torsion element with **only one flow graph component** and let $R_{\alpha}$ be the set of its repellers.
Let $\alpha$ be a purely non-torsion element with **only one flow graph component** and let $\mathcal{R}_\alpha$ be the set of its repellers.

**Lemma**

- $C_V(\alpha)$ acts on the finite set $\mathcal{R}_\alpha$. 
Let $\alpha$ be a purely non-torsion element with only one flow graph component and let $\mathcal{R}_\alpha$ be the set of its repellers.

**Lemma**

- $C_V(\alpha)$ acts on the finite set $\mathcal{R}_\alpha$.
- The action induces an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow C_V(\alpha) \rightarrow Q \rightarrow 0$$

where $Q \leq \text{Sym}(\mathcal{R}_\alpha)$ is finite.
A second exact sequence
Define the following group homomorphism $S : C_V(\alpha) \to \mathbb{Z}$

$$S(g) = \log_3 \left( \prod_{r \in R_\alpha} rg' \right)$$

where $rg'$ denotes the slope of $g$ at the repeller $r$. 
A second exact sequence

Define the following group homomorphism $S : C_V(\alpha) \to \mathbb{Z}$

$$S(g) = \log_3 \left( \prod_{r \in R_\alpha} rg' \right)$$

where $rg'$ denotes the slope of $g$ at the repeller $r$.

From $S$ we deduce the exact sequence

$$0 \to \ker(S) \hookrightarrow C_V(\alpha) \twoheadrightarrow \text{im}(S) = \mathbb{Z} \to 0.$$
Non-torsion elements with one flow graph component

Francesco Matucci (with C. Bleak, H. Bowman, A. Gordon, G. C)

Centralizers and dynamics in Thompson’s group $V$
By analyzing the two exact sequences one can deduce
By analyzing the two exact sequences one can deduce

**Lemma**

\[ C_V(\alpha) \cong \ker(S) \rtimes \mathbb{Z}, \text{ and} \]
By analyzing the two exact sequences one can deduce

**Lemma**

- \( C_V(\alpha) \cong \ker(S) \rtimes \mathbb{Z}, \) and
- \( \ker(S) = \{ \text{torsion elements of } C_V(\alpha) \} \) is finite.
Non-torsion elements with many flow graph components
We define flow graph components of $\alpha$ to be \textit{isomorphic} if a centralizer sends one to the other.

Centralizers permute isomorphic components, so we get
Non-torsion elements with many flow graph components

We define flow graph components of $\alpha$ to be isomorphic if a centralizer sends one to the other.

Centralizers permute isomorphic components, so we get

$$C_V(\alpha) \cong (\ker(S) \rtimes \mathbb{Z}) \wr \text{Sym}(t)$$

where $t = \#\{\text{isomorphic components}\}$. 

Putting it all together

\[ C_V(\alpha) \cong \left( \prod_{i=1}^{s} \text{Maps}(C, C_{n_i}) \rtimes V \right) \times \left( \prod_{j=1}^{t} (A_j \rtimes \mathbb{Z}) \wr \text{Sym}(q_j) \right) \]
Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)

Let $\alpha \in V$. The group $C_V(\alpha)$ is finitely generated.
**Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)**

Let $\alpha \in V$. The group $C_V(\alpha)$ is finitely generated.

**Question:** Must $C_V(\alpha)$ be finitely presented?
Applications

**Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)**

Let $\alpha \in V$. The group $C_V(\alpha)$ is finitely generated.

**Question:** Must $C_V(\alpha)$ be finitely presented?

**Question:** Is it true that for each index $j$, the group $A_j$ is abelian? In most examples, it is cyclic, while in one it is $C_2 \times C_2$. 
Applications

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**Question:** Can one replace the terms $A_i \rtimes \mathbb{Z}$ with $A_i \times \mathbb{Z}$?
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Question: Can one replace the terms $A_i \times \mathbb{Z}$ with $A_i \times \mathbb{Z}$?

Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)

Let $\alpha \in V$ be of infinite order. The group $\langle \alpha \rangle$ is undistorted in $V$. 
Discrete train tracks
Discrete train tracks

Flow graphs contain all the information we need. However, from revealing pairs we can get a more complete graph, a **discrete train track**, representing the conjugacy class of an element.
Discrete train tracks

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Francesco Matucci (with C. Bleak, H. Bowman, A. Gordon, G. C.) Centralizers and dynamics in Thompson's group V
Discrete train tracks
Diagram of the lamination carried by this train track local to the repelling cycle in the lower left corner:
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