

Texas Hold'em

Poker is one of the most popular card games, especially among betting games. While poker is played in a multitude of variations, Texas Hold'em is the version played most often at casinos and is the most popular among the “community cards” variants of poker. It is also the variant played at the World Series of Poker and on the World Poker Tour.

Rules

Each hand is played with a whole deck. One player is the dealer (this role rotates clockwise after each hand), and bets are placed in a clockwise order starting with the player on the dealer's left. Each hand has four stages, and after each stage there is a round of betting. The four stages are:

1. (Pre-flop) Everyone gets two cards (dealt face down).
2. (Flop) Three cards are dealt face up in the middle of the table.
3. (Turn) A fourth card is dealt face up in the middle of the table.
4. (River) A fifth card is dealt face up in the middle of the table.

The object of the game is to form the best five-card hand possible using the player's two cards and the five “community cards” dealt in the middle of the table. A hand is won by having the best hand among the players who did not fold (i.e. refuse to match an opponent's bet, as described below), or by having everyone else fold.

We are going to use a \$1/\$2 betting structure. Before the pre-flop, the two players to the left of the dealer must bet \$1 (these mandatory bets are called blinds, since the player must make them before she sees her cards). Then, following the betting order, each player may raise the bet, up to four times per player per betting round. (The blinds act as a bet, so in the pre-flop betting round, the first player to act will be the person three seats to the left of the dealer). Whenever a player raises the bet, the other players must call (that is, accept the raise), fold (that is, give up and lose the money already bet) or raise the bet even more. On the pre-flop and flop, the players bet \$1 at a time, while on the turn and river they bet \$2 at a time.

The hand ends when all but one player has folded or when all the cards have been dealt and the last betting round is over. In this last case, the players must show their cards and the player with the highest hand wins.

Ranking of Poker Hands

From highest to lowest, the possible five card hands in poker are ranked as follows:

- **Straight Flush:** Five cards in sequence all of the same suit, for example 7 - 8 - 9 - 10 - J all in hearts. Aces can be treated as either above kings or below twos. If multiple players both have straight flushes, the players compare the high cards of their straight flushes (higher card wins). A straight flush with A - K - Q - J - 10 is called a royal flush and is the highest hand in the game.

- **Four of a Kind:** Four cards of the same rank (and one card of some other rank). When comparing four of a kind hands the four of a kind of higher rank wins.
- **Full House:** Three matching cards of one rank and two matching cards of a different rank. If multiple players both have full houses the player whose set of three cards has higher rank wins.
- **Flush:** All five cards of the same suit. If multiple players both have flushes, the player with the single card of highest rank wins. In situations where players both have the same rank high card they then compare the rank of their second highest cards, then third highest cards, etc. Two flushes with cards of the same ranks tie.
- **Straight:** All five cards in sequence. As with straight flushes, aces can be high or low and two straights are compared by looking at high cards.
- **Three of a Kind:** Three cards of the same rank and two unmatched cards. Three of a kind hands are compared by looking at the rank of the set of three cards.
- **Two Pair:** Two cards of the same rank, two cards of another rank (matching each other but not the first two cards), and one card with a third rank. Two of a kind hands are compared by looking at each player's highest ranking pair (higher ranks wins). If players have the same high pair then they compare the rank of their low pairs. If this still does not decide a winner then the rank of the unmatched card (also called the "kicker") is compared.
- **One Pair:** Two cards of the same rank and three cards all of different ranks (both from the pair and from each other). One pair hands are compared by looking first at the rank of each player's pair, and if needed then considering the rank of each player's unmatched cards in the same way as for a flush.
- **High Card:** A set of five cards that does not fit into any of the above categories. High cards hands are compared by considering the rank of each player's cards as described above in the case of a flush.

There is no ranking of suits in poker, so two players who have identical hands but in different suits tie the hand and split the pot. The two cards that a player does not use in making his five card hand are ignored; they are not used to break ties between five card hands.

Some Calculations

Pot odds are the odds you get when you analyze the current size of the pot against the cost of your next call. The general idea is to compare your chance of winning to your pot odds. You have good pot odds if your chance of winning is significantly bigger than the ratio of the bet to the pot size. For example, say you are on the turn, you have two hearts in your hand, and you have one opponent still in the hand. The community cards have two hearts, so any of the nine remaining hearts finishes a flush for you. We say that you have 9 "outs" (outs are the cards still unseen that will improve your hand) out of a total of 46 unseen cards. The ratio $9/46$ is approximately $1/5$. Suppose your opponent raises \$2 and the pot you get if you call and win is \$20. The ratio $2/20$ is 1 in 10, which is smaller than your 1 in 5 chance of hitting the flush, so pot odds say that calling is the right move.

Implied odds take into account the fact that betting will continue throughout the rest of the hand, so you have the potential to gain more money from your opponents in future rounds of betting (and also you may have to pay more money to stay in the hand in later rounds of betting). In the example above, if you feel that your opponent will call a bet after the river, then if hit your flush you will be able to earn an additional \$2. If you do not hit your flush you can fold the hand and not lose any additional money. So in this case your implied odds are 2/22, or 1 in 11, even better than your pot odds.

Problems

For all of the following problems we assume that all of the cards not in a player's hand or in the collection of community cards are drawn with equal probability. This is a valid assumption if we have no knowledge of the other players' cards (see the Blackjack lesson, problem 3 for further details). In actuality it may be possible to infer some information about an opponent's hand based on her betting patterns or behavior.

1. If you are dealt two hearts and the flop contains exactly two hearts, what is the probability that you get a flush on the turn or the river? If the flop contains only one heart, what is the probability that you get hearts on both the the turn and the river to make your flush?
2. You are dealt a pair of eights and the flop comes up 1 - 7 - 2. What the is probability that you will have four a kind after the river? A full house?
3. Your are dealt a 6 - 4 and the cards on the table are 7 - K - 3 - 10. There are two opponents still in the game. The pot is currently \$20 and your have been raised \$2. Assuming that you win if you hit your straight and lose if you do not, what do pot odds tell you to do? Assuming further that both of your opponents will call a \$2 bet after the river, what do implied odds tell you to do?
4. It is sometimes useful to know the frequency of each of the different poker hands. In Texas Hold'em, each player is making a hand out of seven available cards. To determine the probability of each hand occurring we can count the number of distinct ways of obtaining each hand and divide by the total number of possible hands. This requires thinking about all of the different ways of obtaining a given hand and coming up with an orderly process for counting these different ways. The easiest way to do this involves heavy use of combinations $C_{n,m}$ (the number of ways of choosing m objects (in any order) from a collection of n objects. These are also called binomial coefficients. See the probability review for more details). For example, the total number of possible seven cards hands is equal to the number of ways of choosing seven distinct cards out of a collection of 52, giving a total of $C_{52,7} = 133,784,560$ hands. Note that even though two identical hands in different suits have the same value in poker they are being counted as distinct hands.

The difficulty of calculating these frequencies varies significantly by hand. The high ranking hands such as four of a kind and straight flush can only be obtained in a limited number of different ways and therefore it is therefore easier to calculate their frequencies. Try calculating these frequencies first. Once you get the hang these types of calculations, if you are up for a challenge you can attempt some of the more involved calculations.

Solutions

1. Since you have two hearts in your hand and there are two hearts in community cards there are nine remaining hearts among the 47 unknown cards. The probability of the turn card being a heart is then $9/47 \approx 0.19$. The probability of the turn card not being a heart but getting a heart on the river is $\frac{38}{47} \cdot \frac{9}{46} \approx 0.16$. Therefore the total probability of getting a flush is approximately $0.19 + 0.16 = 0.35$. We can also do this problem by calculating the probability of not getting a flush. The chance of not drawing a flush on either the turn or the river is $\frac{38}{47} \cdot \frac{37}{46} \approx 0.65$, so the probability of getting a flush is $1 - 0.65 = 0.35$, the same value calculated above.

In the second situation there are 10 remaining hearts among the 47 unknown cards. You need to draw a heart on both the turn and the river, so the probability of getting a flush is $\frac{10}{47} \cdot \frac{9}{46} \approx 0.04$.

2. The only way that you can end up with four of a kind is by drawing eights on the turn and river. There are only two eights left in the deck, so this occurs with probability $\frac{2}{47} \cdot \frac{1}{46} \approx 0.0009$.

There are many different types of full houses that you could draw in this situation. We calculate the probability of each type and sum the results to find the total chance of getting a full house.

- Three 8's, Two 1's. This requires getting an eight and a one as the last two cards. The chance of drawing an eight and then a one is $\frac{2}{47} \cdot \frac{3}{46} \approx 0.0028$, and the probability of drawing a one and then an eight is $\frac{3}{47} \cdot \frac{2}{46} \approx 0.0028$. So this full house occurs with probability 0.0056.
- Three 8's, Two 7's or Three 8's, Two 2's. By a calculation identical to the one above, these both occur with probability 0.0056.
- Three 1's, Two 8's. This requires getting a one on both the turn and the river, which occurs with probability $\frac{3}{47} \cdot \frac{2}{46} \approx 0.0028$.
- Three 7's, Two 8's or Three 2's, Two 8's. These are both analogous to the three 1's, two 8's situation and occur with probability 0.0028.

Therefore, the total probability of getting full house is $3 \cdot (0.0056) + 3 \cdot (0.0028) = 0.0252$.

3. Only a 5 will complete your straight, so you have four outs and a $(4/46) \approx 0.087$ probability of winning. You are considering a raise of \$2 to a \$20 pot giving a 1/10 ratio. Therefore pot odds say you should not call. However, if you hit your straight you will earn an additional \$2 from each of your opponents for a total pot of \$24. The ratio of the cost of your call to this potential pot is $2/24 \approx .083$ which is less than your 0.087 probability of winning, so implied odds say that you should make the call.
4. To calculate the frequency of four of a kind, first note that there are 13 different ranks in which you can get four of a kind. For any given rank, the possible hands that give four of a kind in that rank all include the four cards of that rank as well as any three additional cards. There are $C_{48,3} = 17,296$ different ways of choosing these three additional cards, so we have a total of $13 \cdot 17,296 = 224,848$ different four of a kind hands. This gives a frequency of $\frac{224,848}{1,337,845,560} = 0.0017$.

To find the frequency of straight flushes, sort all straight flush hands by the high card of the highest straight flush in the hand. For ace high straight flushes in any of the four suits you need the A - K - Q

- J - 10 of the given suit and then any 2 of the remaining 47 cards. This gives a total of $C_{47,2} = 1,081$ distinct hands. For straight flushes that are not ace high the same argument holds except that one of the remaining 47 cards would give you higher straight flush if it were in your hand (for example, if you have 10 - 9 - 8 - 7 - 6 in hearts, if one of your two other cards was a jack of hearts you would have a jack high straight flush). Therefore, in these cases there are only $C_{46,2} = 1,035$ distinct straight flush hands. So the total number of straight flush hands is $(1,081 \cdot 4) + (1,035 \cdot 4 \cdot 9) = 41,584$ hands (the nine in the second parenthesis comes from the fact that there are nine different possible non-ace high cards for straights - a 2,3, or 4 high straight can not occur). The corresponding frequency is then $\frac{41,584}{133,784,560} = 0.00031$.

To count the number of full house hands, we divide up the types of full houses by looking at the two cards that are not used as part of the final hand. These two cards can either be a pair (but of a different rank than the triple or the pair you are using for the full house, or else you would have four of a kind), one of the two cards could be of the same rank as your pair (giving you two triples and one card of some different rank), or the two cards could be of different ranks from each other, the triple, and the pair.

- We first consider the case of the unused cards being a pair. We can choose the rank for the triple in 13 ways. Once a rank is chosen we can pick the three cards for the triple in $C_{4,3} = 4$ ways. We can then choose the two ranks for the two pairs in $C_{12,2} = 66$ ways. For each pair, once we have chosen the rank we can choose the cards for the pair in $C_{4,2} = 6$ ways. So we have a total of $13 \cdot 4 \cdot 66 \cdot 6^2 = 123,552$ full house hands of this type.
- Now we consider the case of two triples. We can choose the ranks for the triples in $C_{13,2} = 78$ ways, and for each triple we can then choose the cards for the triple in $C_{4,3} = 4$ ways. There are then 44 remaining cards from which to choose the last card of the hand, so we have a total of $78 \cdot 4^2 \cdot 44 = 54,912$ hands of this type.
- Finally we consider the case of two cards of different rank from each other, the triple, and the pair. As above, the cards for the triple can be chosen in $13 \cdot C_{4,3} = 52$ ways and the cards for the pair can then be chosen in $12 \cdot C_{4,2} = 72$ ways. We can choose the two ranks for the remaining two cards in $C_{11,2} = 55$ ways, and for each rank we can choose any of the four cards of that rank. This gives a total of $52 \cdot 72 \cdot 55 \cdot 4^2 = 3,294,720$ hands of this type.

Therefore, we have a total of 3,473,184 full house hands. This gives a frequency of $\frac{3,473,184}{133,784,560} = 0.02696$.

For additional calculations, as well as the frequencies for 5-card poker hands (which tend to be significantly easier to calculate), see for example http://en.wikipedia.org/wiki/Poker_probability.