Solutions for homework 1

§1.1, #5

Consider the matrix $\begin{bmatrix} 1 & -4 & 5 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$ as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

Solution: First, create a zero in row 2, column 3 by **adding 3 * row 3 to row 2**. Second, create a zero in row 1, column 3 by **subtracting 5 * row 3 from row 1**.

§1.1, #11

Solve the system.

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

Solution: The augmented matrix is $\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix}$.

Swapping rows 1 and 2 yields $\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix}$.

Subtracting 3 * row 1 from row 3 yields $\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix}$.

Adding 2 * row 2 to row 3 yields $\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

The bottom row corresponds to the equation 0=2, which has no solutions. Thus **the system is inconsistent; it has no solution.**

§1.1, #20

Determine the value(s) of h such that the matrix $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$ is the augmented matrix of a consistent linear system.

Solution: Row reduction yields the echelon matrix $\begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$. The system is consistent so long as this matrix has no pivot in the last column. But the 3 in the (first row, third column) cannot be a pivot because nonzero entries occur to its left, and the 0 in the (second row, third column) can never be a pivot. Thus we don't need any conditions on h to ensure that the system is consistent; **the system is consistent for all values of h**.

§1.1, #24

True or false.

(a) Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

This is **true**; it follows immediately from the definition and discussion of elementary row operations on pages 7 and 8.

(b) Two matrices are row equivalent if they have the same number of rows.

This is **false**; consider for example the matrices $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \end{bmatrix}$. They have the same number of rows, but if they were row equivalent then the corresponding systems $x = 1$ and $x = 0$ would have the same solution set (see the box at the top of page 8); they clearly do not.

(c) An inconsistent system has more than one solution.

This is **false**. An inconsistent system, by the definition on page 4, has no solutions.

(d) Two linear systems are equivalent if they have the same solution set.

This is **true**. It's exactly the definition of equivalence given on page 3.

§1.1, #27

Suppose the system $\left\{ \begin{array}{r} x_1 + 3x_2 = f \\ cx_1 + dx_2 = g \end{array} \right\}$ is consistent for all possible values of f and g. What can you say about the coefficients c and d?

Solution: Row-reducing the augmented matrix yields $\begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-fc \end{bmatrix}$. The system is consistent for all f and g, so the $g - fc$ in the (second row, third column) is never a pivot. Thus, for all f and g, either $d - 3c/neq0$ or $g - fc = 0$. But, regardless of c and d, we can always find an f and g so that $g - fc \neq 0$, so it must be the case that $\mathbf{d - 3c \neq 0}$.

§1.2, #2c

Determine whether the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is in reduced echelon form or only in echelon form.

Solution: It is **neither**. The leading entry of row 2 is in column 1, the same column as the leading entry of row 1. This violates property 2 in the definition on page 14.

§1.2,#4

Row reduce the matrix $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$ to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

Solution: Subtracting 3 * row 1 from row 2 gives $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{bmatrix}$.

Subtracting 5 * row 1 from row 3 gives $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}$.

Dividing row 2 by -4 yields $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix}$.

Adding 8 * row 2 to row 3 yields $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$.

Dividing row 3 by -10 leaves $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Subtracting 3 * row 3 from row 2 leaves $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Subtracting 7 * row 3 from row 1 leaves $\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Finally, subtracting 3 * row 2 from row 1 gives us $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, which is in reduced echelon form. The pivot columns are **columns 1, 2, and 4**; the pivot positions are (row 1, column 1), (row 2, column 2), and (row 3, column 4).

§1.2, #15a

Suppose the matrix $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$ represents the augmented matrix for a system of linear equations. Determine if the system is consistent. If the system is consistent, determine if the solution is unique.

Solution: There is no pivot in the last column, so the system **is consistent**. There are pivots in every other column, so there are no free variables, so the solution **is unique**.

§1.2, #19

Choose h and k such that the system $\left\{ \begin{matrix} x_1 + hx_2 = 2 \\ 4x_1 + 8x_2 = k \end{matrix} \right\}$ has (a) no solution, (b) a unique solution, and (c) many solutions.

Solution: Row-reducing the augmented matrix yields $\begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$.

(a) There is no solution when there is a pivot in the third column, i.e., when $8 - 4h = 0$ and $k - 8 \neq 0$, i.e., when $\mathbf{h = 2}$ **and** $\mathbf{k \neq 8}$.

(b) There is a unique solution when there are pivots in columns one and two, but not in column three, i.e., when $8 - 4h \neq 0$, i.e., when $\mathbf{h \neq 2}$.

(c) There are many solutions when neither column two nor column three contain a pivot, i.e. when $8 - 4h = k - 8 = 0$, or when $\mathbf{h = 2}$ **and** $\mathbf{k = 8}$.

§1.2, #29

A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system.* Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

Solution: Let A represent the augmented matrix for the system. We have:

$$\begin{aligned} \text{The number of pivots of } A &\leq \text{ The number of rows of } A \\ &= \text{The number of equations in the system} \\ &\lneqq \text{The number of unknowns in the system} \\ &= (\text{The number of columns of } A) - 1 \end{aligned}$$

Thus A has at least two columns without pivots; at least one of these must correspond to a free variable in our system. Since the system is consistent and has a free variable, it must have infinitely many solutions.