Elementary components of Hilbert schemes of points

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Abstract

Consider the Hilbert scheme of points on a higher-dimensional affine space. Its component is elementary if it parameterizes irreducible subschemes. We characterize reduced elementary components in terms of tangent spaces and provide a computationally efficient way of finding such components. As an example, we find an infinite family of elementary and generically smooth components on the affine four-space. We analyse singularities and formulate a conjecture which would imply the non-reducedness of the Hilbert scheme. Our main tool is a generalization of the Białynicki-Birula decomposition for this singular scheme.

MSC classes: 14C05, 14L30, 13D10

1 Introduction

While the Hilbert scheme of points on a smooth connected surface is smooth and irreducible [Fog68], little is known about the irreducible components of Hilbert schemes of points on higher-dimensional varieties [Ame10], despite much recent interest in their geometry. Following [Iar73], a component is elementary if it parameterizes subschemes supported at a single point. All components are generically étale-locally products of elementary ones.

Up to now, the only known method of finding elementary components is to construct a locus \( \mathcal{L} \) inside the Hilbert scheme, and verify that the tangent space to the Hilbert scheme at a point of \( \mathcal{L} \) has dimension \( \dim \mathcal{L} \), to conclude that \( \mathcal{L} \) contains an open neighbourhood of this point. Roughly speaking, the construction step is conceptual and the verification step is algorithmic. Iarrobino [Iar73, Iar84, IK99] obtained loci \( \mathcal{L} \) by choosing general forms of prescribed degrees and taking their apolar algebra. The obtained algebras are called compressed. Recently, Huibregtse [Hui17] extended Iarrobino’s method in certain cases, taking into account the automorphisms of the ambient variety. All known elementary components come from the constructions of these two authors. Verification of tangent space dimension was done in [IE78] for an explicit degree 8 scheme, then extended [Sha90] for algebras of multiplicity two, and afterwards conducted for several other explicit cases [IK99, EV10, Hui17], see Remark 6.10. It is conjectured that the tangent space has correct dimension in greater number of compressed cases, see [IE78, §2.3]. The main limitation of this approach is that one needs to construct the locus \( \mathcal{L} \). As a side effect, all loci \( \mathcal{L} \) obtained so far are isomorphic to open subsets of products of Grassmannians.

The aim of the present paper is to avoid the construction step entirely. We answer the following question:

Question 1.1. How to check that a given point \([R] \in \text{Hilb}_{pt}(\mathbb{A}^n)\) lies on an elementary component?

The \(n\)-dimensional additive group acts on \(\mathbb{A}^n\) and \(\text{Hilb}_{pt}(\mathbb{A}^n)\) by translations. Let \(R \subset \mathbb{A}^n\) be a finite subscheme supported at the origin. The tangent map of the \(\mathbb{A}^n\)-orbit of \([R]\) is

\[
\text{span}(\partial_1, \ldots, \partial_n) \to \text{Hom}(I_R, \mathcal{O}_R)_{<0}.
\]

We say that \(R\) has trivial negative tangents if this tangent map is surjective or, equivalently, \(T^1(R)_{<0} = 0\). If the characteristic is zero or \(I_R\) is homogeneous, then \(R\) has trivial negative tangents if and only if \(\dim \text{Hom}(I_R, \mathcal{O}_R)_{<0} = n\). Our main result is that having trivial negative tangents is intimately connected to lying on an elementary component.

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We say that $T$ be smooth even if $T$ has trivial negative tangents and that it is a rich source of smooth points on elementary components. Of course, the point table; see Example 6.9. This makes Corollary 1.3 effective: computer algebra experiments show $\gamma$ same Gröbner fan with respect to the standard torus action, and a general point of $R$ is a point.

Corollary 1.3

Let $R \subset \mathbb{A}^n$ be a finite subscheme supported at the origin. If $R$ has trivial negative tangents and $T^2(R)_{\geq 0} = 0$ then $[R]$ is a smooth point of $\text{Hilb}_R(\mathbb{A}^n)$ lying on a unique elementary component.

In a number of cases, the vanishing of $\text{Ext}^1(I_R, \mathcal{O}_R)_{\geq 0}$ is forced by the degrees in the Betti table; see Example 6.9. This makes Corollary 1.3 effective: computer algebra experiments show that it is a rich source of smooth points on elementary components. Of course, the point $[R]$ may be smooth even if $T^2(R)_{\geq 0} \neq 0$. A subtler relative smoothness criterion is given in Corollary 4.13. Applying it, we obtain the following result.

Corollary 1.3 (Corollary 4.6) Let $R \subset \mathbb{A}^n$ be a finite subscheme supported at the origin. If $R$ has trivial negative tangents and $T^2(R)_{\geq 0} = 0$ then $[R]$ is a smooth point of $\text{Hilb}_R(\mathbb{A}^n)$ lying on a unique elementary component.

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$$s = \sum_{i,j} c_{ij} x_1^i x_2^j y_1^{e-1-i} y_2^{e-1-j}, \text{ where } c_{ij} \in \mathbb{k}.$$ 

We say that $s$ is general if the $e \times e$ matrix $[c_{ij}]$ is invertible. Fix a general form $s$ and let $R(e) \subset M(e)$ be the subscheme cut out by this form. The degree of $R(e)$ is $d := \binom{e+1}{2}^2 - 1$. The subscheme $R(e)$ deforms freely in $M(e)$ and so all points $[R(e)]$ corresponding to different choices of $s$ lie in the same irreducible component of $\text{Hilb}_d(\mathbb{A}^4)$.

Theorem 1.4. For all $e \geq 2$ the point $[R(e)] \in \text{Hilb}_d(\mathbb{A}^4)$ is a smooth point on an elementary component $Z(e)$ of dimension $4e \deg R(e) - (e - 1)(e + 5) = e^4 + 2e^3 - 4e + 1$.

The component $Z(3)$ gives a negative answer to the following open question, stated at [Ame10]: Is the Gröbner fan a discrete invariant that distinguishes the components of $\text{Hilb}_d(\mathbb{A}^n)$? Namely, we construct a curve $C \subset \text{Hilb}_{35}(\mathbb{A}^4)$ such that all subschemes corresponding to points of $C$ share the same Gröbner fan with respect to the standard torus action, and a general point of $C$ is a smooth point on $Z(3)$, while the special point on $C$ lies in the intersection of $Z(3)$ and another component, see Example 6.8.
A central idea of the proof of Theorem 1.2 is to consider a scheme $\text{Hilb}^+_{pt}(A^n)$ that is a generalization of the Białynicki-Birula decomposition for the scheme $\text{Hilb}_{pt}(A^n)$ with the $G_m$-action coming from the standard $G_m$-action on $A^n$. Intuitively, the scheme $\text{Hilb}^+_{pt}(A^n)$ parameterizes families that have a limit at infinity, see Section 3 for details. In particular:

- the $k$-points of $\text{Hilb}^+_{pt}(A^n)$ correspond to subschemes $R \subset A^n$ supported at the origin,
- the tangent space to $\text{Hilb}^+_{pt}(A^n)$ at a $k$-point $[R]$ is $\text{Hom}(I_R, O_R)_{\geq 0}$,
- deformations of $[R] \in \text{Hilb}^+_{pt}(A^n)$ have an obstruction space $T^2(R)_{\geq 0}$.

The scheme $\text{Hilb}^+_{pt}(A^n)$ comes with a forget-about-the-limit-and-translate map

$$\theta: \text{Hilb}^+_{pt}(A^n) \times A^n \to \text{Hilb}_{pt}(A^n), \quad (1.1)$$

A finite subscheme $R$ has trivial negative tangents if and only if the tangent map $d\theta$ is surjective at $([R], 0)$ and in this case $\theta$ is an open immersion onto a neighbourhood of $R$; see Theorem 1.3. The proof of Theorem 1.2 crucially depends on representability of $\text{Hilb}^+_{pt}(A^n)$ by a finite type scheme.

Apart from finding new elementary components of $\text{Hilb}_{pt}(A^n)$ we also investigate its singularities. It is widely expected that $\text{Hilb}_{pt}(A^n)$ has arbitrary bad singularities, as defined in [Vak06, p. 570]. However, absolutely no hard evidence for this exists. The following questions remain open.

**Question 1.5** (Fog68, CEVV09, Ame10). Is $\text{Hilb}_{pt}(A^n)$ always reduced?

In this paper, we do not answer Question 1.5 directly, however we reduce it to an algorithmic tangent-space-level Conjecture 1.6. We recall some background results. Vakil [Vak06] proved the Murphy’s Law for all Hilbert schemes other than the Hilbert schemes of points. His method depends on the information in the projective embedding and does not apply to the zero-dimensional case. Using [Vak06], Erman [Erm12] proved that the Murphy’s Law holds for $\text{Hilb}^{G_m}_{pt}(A^n)$. His idea is that homogeneous deformations of a high enough truncation of a cone correspond to deformations of this cone and the associated projective variety. Erman’s result does not imply anything about $\text{Hilb}_{pt}(A^n)$, as he explicitly states in [Erm12] p. 1278. The problem is that $\text{Hilb}^{G_m}_{pt}(A^n)$ is a closed, nowhere dense, subset of $\text{Hilb}_{pt}(A^n)$.

We extend Erman’s idea to $\text{Hilb}^+_{pt}(A^n)$ and use the map $\text{(1.1)}$ to compare of $\text{Hilb}^+_{pt}(A^n) \times A^n$ and $\text{Hilb}_{pt}(A^n)$. Our argument can be made on the infinitesimal level, however for clarity we will keep using the geometric language. The first step is to prove that $\text{Hilb}^+_{pt}(A^n)$ satisfies Murphy’s Law. This is done in Theorem 5.1 by finding a smooth locus of the pass-to-the-limit retraction $\text{Hilb}^+_{pt}(A^n) \to \text{Hilb}^{G_m}_{pt}(A^n)$. The second step is to compare $\text{Hilb}^+_{pt}(A^n) \times A^n$ with $\text{Hilb}_{pt}(A^n)$ using the map $\theta$ from $\text{(1.1)}$. By Theorem 1.2 if the subscheme $[R]$ has trivial negative tangents, then $\theta$ is an open embedding near $[R]$. It is not clear why a chosen point $[R] \in \text{Hilb}^+_{pt}(A^n)$ with pathological deformation space should have trivial negative tangents. Therefore, we put forward a conjectural method of modifying any point of $\text{Hilb}_{pt}(A^n)$ to a point having trivial negative tangents.

**Conjecture 1.6.** Let $S$ be a polynomial ring over $k$ and $I \subset S$ be an ideal of regularity $r_0$. For all $r \geq r_0 + 2$, there exists an integer $t$, a polynomial ring $T = S[x_1, \ldots, x_t]$, and a linear subspace $L \subset T_r$ such that the finite scheme $R$ given by the ideal $I \cdot T + L + T_{d+r+1}$ has trivial negative tangents.

For $I = 0$, the obtained ideals are compressed and in this case the conjecture was formulated already in [IE78] with much more quantitative precision. Shafarevich [Sha90] proved this more precise version for $I = 0$ and $L$ spanned by quadrics. Not much more is known. Since $r \geq r_0 + 2$, the $G_m$-invariant deformations of $I$ and $I_R$ are smoothly equivalent. By semicontinuity, if the conjecture holds for one particular $L \subset T_r$, then also for a general $L' \subset T_r$ of the same codimension;
the particular choice of $L$ brings little information apart from ensuring that all deformations of $R$ are supported on a single point, by Theorem 1.2(1). Conjecture 1.6 implies that $\text{Hilb}_{pt}(\mathbb{A}^n)$ is highly singular, in particular non-reduced, answering Question 1.5, see Section 5.

Let us mention another open question. It is known [Iar72] that $\text{Hilb}_d(\mathbb{A}^3)$ is reducible for $d \geq 78$. However none of its components other than the smoothable one is known. Also the minimal $d$ such that $\text{Hilb}_d(\mathbb{A}^3)$ is reducible is not known, except for the bound $12 \leq d \leq 78$, see [Ame10] [DJNT17]. By Theorem 1.2 to prove that $d \leq d_0$ it is enough to find a degree $d_0$ subscheme having trivial negative tangents.

**Question 1.7.** What is the smallest $d$ such that $\text{Hilb}_d(\mathbb{A}^3)$ is reducible?

Similarly, new examples of finite Gorenstein subschemes of $\mathbb{A}^4$ and $\mathbb{A}^5$ would likely improve the bounds given [BB14] §8.1 and perhaps allow to extend the smoothability results on finite Gorenstein algebras presented in [CN11] [CN14]. It would also be very interesting to know how the Białynicki-Birula decompositions could be used to improve explicit computations on the Hilbert schemes [BCR17] [LR11].

Our focus on $\mathbb{A}^n$ is without loss of generality as Hilbert schemes of points on other smooth varieties have the same components, see [Art76] p.4 or [BJ17]. The idea of proving smoothness using the Białynicki-Birula decomposition is very general; we plan to investigate its applications to other moduli spaces, such as Quot schemes. We believe that the characteristic zero assumption in Theorem 1.2(2) can be removed. Upon completion of this work we learned that the Białynicki-Birula decomposition for algebraic spaces was constructed earlier by Drinfeld [Dri13], by an entirely different method. His method was extended [JS18] to actions of groups other than $\mathbb{G}_m$. While this paper was in review, a preprint [Jel18] appeared, which in particular claims to answer Question 1.5.

The organization of the paper is linear. In Section 2 we discuss all algebraic preliminaries. In Section 3 we construct $\text{Hilb}_{pt}(\mathbb{A}^n)$ and, in Section 4 obstruction theories. Finally, in Section 5 we discuss singularities and Conjecture 1.6. We conclude with Section 6 which contains some explicit examples.

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**2 Preliminaries**

Throughout the article, $k$ is an arbitrary field, $S$ is a finitely generated $k$-algebra, $\mathbb{A} := \text{Spec } S$ and $\text{Hilb}_{pt}(\mathbb{A}) = \bigsqcup_d \text{Hilb}_d(\mathbb{A})$ is the Hilbert scheme of points on $\mathbb{A}$. For closed subschemes $R \subset M \subset \mathbb{A}$ by $I_M \subset I_R \subset S$ we denote the respective ideals and by $O_R = S/I_R$, $O_M = S/I_M$ the corresponding
algebras. We define $J = I_R/I_M$ so that there are short exact sequences

$$0 \to I_M \to I_R \to J \to 0,$$
$$0 \to J \to \mathcal{O}_M \to \mathcal{O}_R \to 0.$$

The associated long exact sequences form the following commutative diagram (2.1) of $S$-modules with exact rows and columns. Here and elsewhere $\text{Ext} := \text{Ext}_S$ and $\text{Hom} := \text{Hom}_S$.

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}(J, J) \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}(J, \mathcal{O}_M) \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}(J, \mathcal{O}_R) \\
\downarrow & & \downarrow \\
0 & \to & \text{Ext}^1(J, J) \\
\downarrow & & \downarrow \\
\text{Ext}^1(J, J) & \to & \text{Ext}^1(J, \mathcal{O}_M) \\
\downarrow & & \downarrow \\
\text{Ext}^1(J, \mathcal{O}_R) & \to & \text{Ext}^2(J, J) \\
\end{array}
$$

Now we fix assumptions and conventions regarding the torus action. In this paper, we eventually restrict to $S = k[x_1, \ldots, x_n]$ graded by $\deg x_i > 0$, so it might be helpful to have this example in mind. For a homogeneous ideals the definitions below are well-known, but non-homogeneous ideals and their $\text{Hom}$’s are used substantially in the proof of Theorem [1,2].

We fix an $\mathbb{N}$-grading on $S$ and assume that $S_0 = k$. The origin of $S$ is the distinguished $k$-point of $\text{Spec} S$ given by the ideal $S_+ = \bigoplus_{i>0} S_i$. Consider the algebraic torus $\mathbb{G}_m := \text{Spec} k[t^{\pm 1}]$ and the affine line $\overline{\mathbb{G}}_m := \text{Spec} k[t^{-1}]$ with the natural $\mathbb{G}_m$-action. The grading on $S$ induces a $\mathbb{G}_m$-action on $A = \text{Spec} S$. The origin is the unique $\mathbb{G}_m$-fixed point. For a homogeneous ideal $I_R \subset S$, we denote

$$\text{Hom}(I_R, \mathcal{O}_R)_i = \{ \varphi \in \text{Hom}(I_R, \mathcal{O}_R) \mid \varphi((I_R)_j) \subset (\mathcal{O}_R)_{i+j} \text{ for all } j \in \mathbb{N} \}.$$ 

Under the above convention, we have $t \cdot \varphi = t^{-i} \varphi$ for all $t \in \mathbb{G}_m(k)$ and all $\varphi \in \text{Hom}(I_R, \mathcal{O}_R)_i$.

Let $I_R \subset S$ be an ideal, not necessarily homogeneous, supported at the origin. We now define $\text{Hom}(I_R, \mathcal{O}_R)_{\geq 0}$ and $\text{Ext}^1(I_R, \mathcal{O}_R)_{\geq 0}$. While $\text{Hom}_{\geq 0}$ is easy, $\text{Ext}^1_{\geq 0}$ requires some care and a change of perspective. For all $k \geq 0$, we define the $k$-subspaces $S_{\geq k} := \bigoplus_{i \geq k} S_i \subset S$, $(I_R)_{\geq k} := I_R \cap S_{\geq k} \subset I_R$ and $(\mathcal{O}_R)_{\geq k} = (S_{\geq k} + I_R)/I_R \subset \mathcal{O}_R$. We set

$$\text{Hom}(I_R, \mathcal{O}_R)_{\geq 0} := \{ \varphi \in \text{Hom}(I_R, \mathcal{O}_R) \mid \varphi((I_R)_{\geq k}) \subset (\mathcal{O}_R)_{\geq k} \text{ for all } k \in \mathbb{N} \}.$$  

If the ideal $I_R$ is homogeneous, then $\text{Hom}(I_R, \mathcal{O}_R)_{\geq 0} = \bigoplus_{k \geq 0} \text{Hom}(I_R, \mathcal{O}_R)_k$ as expected. We also define

$$\text{Hom}(I_R, \mathcal{O}_R)_{< 0} := \text{Hom}(I_R, \mathcal{O}_R)/\text{Hom}(I_R, \mathcal{O}_R)_{\geq 0}.$$ 

and again for homogeneous ideals we have $\text{Hom}(I_R, \mathcal{O}_R)_{< 0} = \bigoplus_{k < 0} \text{Hom}(I_R, \mathcal{O}_R)_k$ as expected.

The group $\text{Ext}^1_{\geq 0}$ could be defined using the formula (2.2) and constructing a filtered free resolution [Bjo87]. However, we have not found a suitable reference for the independence of the resolutions, so we take a different path. Consider the ring $S[t^{\pm 1}]$. Ideals of $S$ correspond bijectively to homogeneous ideals of $S[t^{\pm 1}]$. Explicitly, for an element $i \in I_R$, with $i = \sum i_k$ where $i_k \in S_k$, we define $i^h := \sum t^{-k} i_k$ and $I^h_R := (i^h \mid i \in I_R)S[t^{\pm 1}]$. The ideal $I^h_R \subset S[t^{\pm 1}]$ is the homogeneous
ideal corresponding to \( I_R \). Setting \( O_R^{\text{hom}} := S[t^{\pm 1}]/I_R^{\text{hom}} \), we have a canonical isomorphism

\[
\text{Hom}(I_R, O_R) \cong \left( \text{Hom}_{S[t^{\pm 1}]}(I_R^{\text{hom}}, O_R^{\text{hom}}) \right)_0. \tag{2.3}
\]

Let \( I_R^h := I_R^{\text{hom}} \cap S[t^{-1}] \) and \( O_R^h := S[t^{-1}]/I_R^h \subset O_R^{\text{hom}} \). On the right side of (2.3), the condition (2.2) translates into \( \varphi(I_R^h) \subset O_R^h \) and we obtain canonically

\[
\text{Hom}(I_R, O_R)_{\geq 0} = \left( \text{Hom}_{S[t^{-1}]}(I_R^h, O_R^h) \right)_0. \tag{2.4}
\]

Since \( R \) is supported at the origin, for large enough \( k \) we have \( S_{\geq k} \subset I_R \). Therefore, \( S_{\geq k}[t^{-1}] \subset I_R^h \), so the algebra \( O_R^h \) is finite over \( k[t^{-1}] \). The algebra \( O_R^h \) is flat over \( k[t^{-1}] \); see [Eis95, p. 346].

Geometrically, the family \( \text{Spec} O_R^h \subset \mathbb{A} \times \mathbb{G}_m \to \mathbb{G}_m \) corresponds to a morphism \( \mathbb{G}_m \to \text{Hilb}_{\text{pt}}(\mathbb{A}) \) that is the closure of the \( \mathbb{G}_m \)-orbit of the point \([R] \). The tangent space to \([R] \in \text{Hilb}_{\text{pt}}(\mathbb{A}) \) is isomorphic to the space of \( \mathbb{G}_m \)-invariant vector fields on the orbit \( \mathbb{G}_m[R] \). The space \( \text{Hom}(I_R, O_R)_{\geq 0} \) is the space of \( \mathbb{G}_m \)-invariant vectors fields on \( \mathbb{G}_m[R] \). It consists of the vector fields in \( \text{Hom}(I_R, O_R) \) that extend from \( \mathbb{G}_m[R] \) to \( \mathbb{G}_m[R] \). The advantage of (2.4) over (2.2) is that it easily extends from \( \text{Hom} \) to \( \text{Ext} \). We define

\[
\text{Ext}^1(I_R, O_R)_{\geq 0} := \left( \text{Ext}^1_{S[t^{-1}]}(I_R^h, O_R^h) \right)_0.
\]

We will also use Schlessinger’s \( T^2 \) functor, so we recall its construction [Har10 §1.3]. For a finite subscheme \( R \subset \mathbb{A} \) we fix a surjection \( j : F \to I_R \) from a free \( S \)-module \( F \). Let \( G = \ker j \) and \( K \subset F \) be the submodule generated by \( j(a)b - aj(b) \) for all \( a, b \in F \). We have \( K \subset G \). We define \( T^2(R) \) to be the \( S \)-module \( \text{Hom}(G/K, O_R)/\text{Hom}(F, O_R) \). As \( \text{Ext}^1(I_R, O_R) = \text{Hom}(G, O_R)/\text{Hom}(F, O_R) \), we have an inclusion, which is usually strict

\[
T^2(R) \subset \text{Ext}^1(I_R, O_R).
\]

We also need a homogenized version. Let \( I_R^{\text{hom}} \subset S[t^{\pm 1}] \) and \( I_R^h \subset S[t^{-1}] \) be defined as in (2.3). By localization we obtain \( T^2(O_R^h) \to T^2(S[t^{\pm 1}]/I_R^{\text{hom}}) \cong T^2(O_R[t^{\pm 1}]) \). We define \( T^2(R)_{\geq 0} \) to be \( T^2(O_R^h)_{\geq 0} \). As a result we have a canonical inclusion

\[
T^2(R)_{\geq 0} = T^2(O_R^h)_{\geq 0} \to T^2(O_R[t^{\pm 1}])_{\geq 0} \cong T^2(R).
\]

Following [BBKT13 Definition 5.15], we say that a finite subscheme \( R \subset \mathbb{A}^n \) is cleavable if it is a limit of geometrically reducible subschemes of \( \mathbb{A}^n \). Here and elsewhere, geometrically means “after base change from \( k \) to its algebraic closure \( \overline{k} \). If \( k \) is algebraically closed, this is vacuous, while for non-algebraically closed fields geometric reducibility is better behaved than reducibility [Liu02 §3.2.2]. Put differently, the subscheme \( R \) is cleavable if the point \([R] \) lies on a non-elementary component of \( \text{Hilb}_{\text{pt}}(\mathbb{A}^n) \).

Finally, we recall the notion of a \( \mathbb{G}_m \)-limit. Let \( \infty := \mathbb{G}_m(k) \setminus \mathbb{G}_m(k) \). For a separated scheme \( X \) with a \( \mathbb{G}_m \)-action and a k-point \( x \in X \), we say that the orbit of \( x \) has a limit at infinity, if the orbit map \( \mu : \mathbb{G}_m \ni t \mapsto t \cdot x \in X \) extends to \( \overline{\mu} : \overline{\mathbb{G}_m} \to X \). This extension is unique and we denote the point \( \overline{\mu}(\infty) \in X \) by \( \lim_{t \to \infty} t \cdot x \). When \( X \) is proper, the limit always exists.

## 3 The Białynicki-Birula decomposition

The Białynicki-Birula decomposition in its classical version [BB73 Theorem 4.3] applies to a smooth and proper variety \( X \) with a \( \mathbb{G}_m \)-action. In this setup, the locus \( X^{\mathbb{G}_m} \) is also smooth and, for each
its component $Y_i \subset X^{G_m}$, the decomposition associated to $Y_i$ is a smooth locally closed subscheme of $X$ defined by

$$Y_i^+ := \{ x \in X \mid \lim_{t \to \infty} t \cdot x \text{ exists and lies in } Y_i \}.$$ 

The Białynicki-Birula decomposition is $Y^+ = \bigcup_i Y_i^+$. In this section, we generalize the decomposition to the case of Hilbert scheme of points $\Hilb_{\text{pt}}(\mathbb{A})$, which is singular and non-proper. In contrast with [RB73], we are interested also in the local theory. We define a functor $\Hilb_{\text{pt}}^+(\mathbb{A}) : \text{Sch}^\text{op} \to \text{Set}$ by

$$\Hilb_{\text{pt}}^+(\mathbb{A})(B) := \{ \varphi : \mathbb{G}_m \times B \to \Hilb_{\text{pt}}(\mathbb{A}) \mid \varphi \text{ is } G_m\text{-equivariant} \}.$$ (3.1)

Consider $\mathbb{A}' := \mathbb{A} \times \mathbb{G}_m = \text{Spec}(S[t^{-1}])$ with its induced grading and $G_m$-action. We stress that the variable $t^{-1}$ has negative degree. The multigraded Hilbert functor $\mathcal{H}S : \text{Sch}^\text{op} \to \text{Set}$ given by

$$\mathcal{H}S(B) = \{ Z \subset \mathbb{A}' \times B \to B \mid Z \text{ is } G_m\text{-invariant, } \forall i (\pi_i \mathcal{O}_Z)_i \text{ locally free of finite rank} \}
$$

is represented by a scheme with quasi-projective connected components [HS04 Theorem 1.1]. We denote this scheme by $\mathcal{H}S$. We have a natural transformation $\iota : \Hilb_{\text{pt}}^+(\mathbb{A}) \to \mathcal{H}S$. Indeed, by definition

$$\Hilb_{\text{pt}}^+(\mathbb{A})(B) \simeq \{ Z \subset \mathbb{A}' \times B \mid Z \text{ is } G_m\text{-invariant, } Z \to \mathbb{G}_m \times B \text{ flat, finite, } G_m\text{-equivariant} \}.$$ 

The transformation $\iota$ assigns to an embedded family $Z \to \mathbb{G}_m \times B$ the family $Z \to B$. Since $Z \to \mathbb{G}_m \times B$ is finite, flat and $G_m$-equivariant, the pushforward of $\mathcal{O}_Z$ is a locally free $\mathcal{O}_B$-module with finite rank graded pieces.

**Proposition 3.1.** The functor $\Hilb_{\text{pt}}^+(\mathbb{A})$ is represented by an open subscheme of $\mathcal{H}S$ and $\iota$ is the corresponding open immersion. Abusing notation, we use $\Hilb_{\text{pt}}^+(\mathbb{A})$ to denote both the functor and the scheme representing it.

**Proof.** Choose a connected scheme $B$ and an element of $\mathcal{H}S(B)$ corresponding to a $G_m$-invariant family $Z \subset \mathbb{A}' \times B$. For each point $b \in B$ the family $Z|_b \to b$ corresponds to an element of $\mathcal{H}S(b)$. Suppose a point $b \in B$ is such that the corresponding element lies in $\iota(\Hilb_{\text{pt}}^+(\mathbb{A}))$. This means that $Z|_b \subset \mathbb{A}' \times b$ is a finite $G_m$-equivariant flat family over $\mathbb{G}_m \times b$ so there exists a number $k$ such that the ideal of $Z|_b \subset \mathbb{A}' \times b$ contains $S_{\geq k} \otimes_k \kappa(b)$.

Denote by $p$ the natural $G_m$-equivariant affine map $Z \to \mathbb{G}_m \times B$. We prove that $p_* \mathcal{O}_Z$ is a finitely generated $\mathcal{O}_{\mathbb{G}_m \times B}$-module. The algebra $\kappa[t^{-1}]$ is negatively-graded and $S_{\geq k} \otimes_k \kappa(b)$ is contained in the ideal of $Z|_b$ so $((p_* \mathcal{O}_Z)|_b)_i = 0$ for all $i \geq k$. The $\mathcal{O}_B$-modules $(p_* \mathcal{O}_Z)_i$ are locally free for all $i$ and $B$ is connected, so $(p_* \mathcal{O}_Z)_i = 0$ for all $i \geq k$. The ideal of $Z \subset \mathbb{A}' \times B$ contains $S_{\geq k} \otimes_k \mathcal{O}_B$, so the algebra $\mathcal{O}_Z$ is a quotient of $(S/S_{\geq k}) \otimes_k \kappa[t^{-1}] \otimes_k \mathcal{O}_B$ which is a finitely generated $\mathcal{O}_B[t^{-1}]$-module. Thus, the $\mathcal{O}_{\mathbb{G}_m \times B}$-module $p_* \mathcal{O}_Z$ is finitely generated. Fix a finite set of homogeneous generators of this module and let $\Delta \subset Z$ be the finite set of their degrees.

Pick a finite rank graded free $\mathcal{O}_{\mathbb{G}_m \times B}$-module $F$ and a graded homomorphism $r : F \to p_* \mathcal{O}_Z$ such that $r|_b$ is an isomorphism. Denote by $r_i$ the $i$-th graded piece of $r$. The set $\Delta$ is finite and for all $i \in \Delta$ the map $(r_i)|_b$ is an isomorphism, so by Nakayama’s lemma there exists an open set $U \subset B$ such that for all $i \in \Delta$ the map $(r_i)|_U$ is an isomorphism. Therefore, the map $r|_U$ is surjective. For all $i$ the map $(r_i)|_U$ is as surjection of locally free $\mathcal{O}_B$-modules of the same finite rank, hence is an isomorphism. As a result, the $\mathcal{O}_{\mathbb{G}_m \times B}$-module $p_* \mathcal{O}_Z$ is locally free so the map $p|_U$ is finite flat so the family $Z|_U \to U$ comes from an element of $\iota(\Hilb_{\text{pt}}^+(\mathbb{A}))$. \hfill \Box

**Remark 3.2.** Representability of the functor $\Hilb_{\text{pt}}^+(\mathbb{A})$ follows also from [JST8, Proposition 5.3]
as the Hilbert scheme has a covering by $G_m$-stable affine open subschemes [MS05, Chapter 18]. However, the embedding $i$ is crucial for constructing obstruction theories for $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ in Section 4.

We have a natural transformation $\theta_0: \text{Hilb}_{\text{pt}}^+(\mathbb{A}) \to \text{Hilb}_{\text{pt}}(\mathbb{A})$, given by forgetting about the limit point. More precisely, for a $G_m$-equivariant family $\varphi: \mathbb{A} \times B \to \text{Hilb}_{\text{pt}}(\mathbb{A})$ corresponding to a $B$-point of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$, we take $\theta_0(\varphi): B \to \text{Hilb}_{\text{pt}}(\mathbb{A})$ to be the restriction of $\varphi$ to $\{1_{G_m}\} \times B$. The map $\theta_0$ is a monomorphism because having $\theta_0(\varphi) = \varphi|_{1 \times B}$ we uniquely recover $\varphi|_{G_m \times B}$ and then $\varphi$. In particular, the map $\theta_0$ is injective on $k$-points. By a slight abuse of notation, we identify the $k$-points of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ with their images in $\text{Hilb}_{\text{pt}}(\mathbb{A})$ and we denote by $[R]$ both the point of $\text{Hilb}_{\text{pt}}(\mathbb{A})$ and the corresponding point of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ if the latter exists.

In the following Proposition 3.3 we classify the $k$-points of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ lying in the image of $\theta_0$. The answer is very intuitive: since the grading on $S$ is non-negative, the action of $G_m$ on $\mathbb{A}$ is divergent, with all points except the origin going to infinity. Thus the only points $[R] \in \text{Hilb}_{\text{pt}}(\mathbb{A})$ for which the orbit has a limit at infinity are those corresponding to $R$ supported at the origin.

**Proposition 3.3.** The morphism $\theta_0$ sends $k$-points of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ bijectively to $k$-points of $\text{Hilb}_{\text{pt}}(\mathbb{A})$ corresponding to subschemes supported at the origin.

**Proof.** Since $\theta_0$ is a monomorphism, it is injective on $k$-points. It remains to describe the $k$-points in its image. Consider a $k$-point $[R]$ corresponding to a finite subscheme $R \subset \mathbb{A}$ of degree $d$. The Hilbert-Chow morphism $\rho: \text{Hilb}_d(\mathbb{A}) \to \text{Sym}^d(\mathbb{A})$ is equivariant. The map $\rho$ is a base change of the projective morphism $\bar{\rho}: \text{Hilb}_d(\mathbb{P}) \to \text{Sym}^d(\mathbb{P})$, where $\mathbb{P}$ is a compactification of $\mathbb{A}$, so the fibers of $\rho$ are proper. If $R \subset \mathbb{A}$ corresponds to a scheme supported at the origin, then $[R]$ lies in $\rho^{-1}(0)$, which is proper and $G_m$-invariant, hence the $G_m$-orbit of $[R]$ extends to $\mathbb{A} \times G_m \to \text{Hilb}_{\text{pt}}^+(\mathbb{A})$ and the point $[R]$ is in the image of $\theta_0$.

Conversely, if $[R]$ lies in the image of $\theta_0$, then for every point $v \in \text{Supp} \, R$, the map $\varphi$ induces an $G_m$-equivariant map $\varphi_v: \mathbb{A} \to \mathbb{A}$ with $\varphi_v(1) = v$. Choose an equivariant embedding $\mathbb{A} \subset \mathbb{A}^N = k[x_1, \ldots, x_N]$, where $\deg x_i > 0$ for all $i$. For $v = (v_1, \ldots, v_N) \in \mathbb{A}^N$ we have

$$t \cdot v = (t^{\deg x_1} v_1, \ldots, t^{\deg x_N} v_N),$$

so that $\varphi_v$ exists only if $v = (0, \ldots, 0)$. \hfill $\square$

Our strategy is to gain knowledge about $\text{Hilb}_{\text{pt}}(\mathbb{A})$ by an analysis of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ and the map $\theta_0: \text{Hilb}_{\text{pt}}^+(\mathbb{A}) \to \text{Hilb}_{\text{pt}}(\mathbb{A})$. This is done using obstruction theories in the next section.

We conclude this section by providing a purely algebraic description of $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$. For a $k$-algebra $A$ and a quotient $O_R = S \otimes A/I_R$, we have a natural filtration on $O_R$, where $(O_R)_{\geq i}$ is the image of $S_{\geq i} \otimes A$. Define $\text{gr}(O_R) = \bigoplus (O_R)_{\geq i}/(O_R)_{\geq i+1}$. The scheme $\text{Hilb}_{\text{pt}}^+(\mathbb{A})$ represents the functor

$$(\text{Hilb}_{\text{pt}}^+(\mathbb{A}))(\text{Spec}(A)) = \{O_R = S \otimes A/I_R \mid O_R \text{ and } \text{gr}(O_R) \text{ are finite, } A\text{-flat, and of degree } d\}.$$

Flatness of $\text{gr}(O_R)$ is equivalent to preservation of local Hilbert function. The infinitesimal version of this construction appeared in [Kle98, p. 614]. Since we will not use this description further, we leave the details to the reader.

## 4 Obstruction theories

In this section we construct obstruction theories for Hilbert schemes of points and their Bialynicki-Birula decompositions. We choose a very explicit approach and follow the notation of [FGI+05, Chapter 6] and, when speaking about Schlessinger’s $T^i$ functors, of [Har10].
The slogan is that the obstruction space for the Białynicki-Birula decomposition of $\text{Hilb}_{pt}(\mathbb{A}^n)$ is the non-negative part of the obstruction space for $\text{Hilb}_{pt}(\mathbb{A}^n)$. An important feature is that the non-negative part frequently vanishes; in particular this happens in the setting of Theorem 1.4.

Let $\text{Art}_k$ be the category of finite local $k$-algebras with residue field $k$. For an algebra $A \in \text{Art}_k$ a small extension is an algebra $B \in \text{Art}_k$ together with a surjective homomorphism of algebras $f: B \to A$ such that the ideal $K = \ker f$ is annihilated by the maximal ideal of $B$. A small extension gives rise to an exact sequence of $B$-modules $0 \to K \to B \to A \to 0$.

Let $X$ be a $k$-scheme and $x \in X$ be a $k$-point. The deformation functor associated to the pair $(X, x)$ is given on an algebra $B \in \text{Art}_k$ by

$$D_X(B) = \{ \varphi: \text{Spec} B \to X \mid \varphi(\text{Spec} k) = x \},$$

see [FGI+05] Section 6.1. An obstruction theory for $(X, x)$ is a pair of spaces $(T, \text{Ob})$ and a collection of maps $\text{ob}_X$ such that for every small extension $0 \to K \to B \to A \to 0$ we have

$$T \otimes_k K \longrightarrow D_X(B) \longrightarrow D_X(A) \longrightarrow \text{Ob}_X \otimes_k K,$$

see [FGI+05] Definition 6.1.21]. We call $T$ and $\text{Ob}$ the tangent and obstruction space, respectively. For a morphism $\varphi: (X, x) \to (Y, y)$ a map of obstruction theories from $(T_X, \text{Ob}_X)$ to $(T_Y, \text{Ob}_Y)$ consists of maps $T_\varphi: T_X \to T_Y$ and $\text{Ob}_\varphi: \text{Ob}_X \to \text{Ob}_Y$ such that for every small extension we have a commutative diagram

$$\begin{array}{ccc}
T_X \otimes_k K & \longrightarrow & D_X(B) \\
\downarrow T_\varphi \otimes \text{id}_K & & \downarrow \text{id} \\
T_Y \otimes_k K & \longrightarrow & D_Y(B)
\end{array} \quad \begin{array}{ccc}
D_X(A) & \longrightarrow & \text{Ob}_X \otimes_k K \\
\downarrow \text{id} & & \downarrow \text{id} \\
D_Y(A) & \longrightarrow & \text{Ob}_Y \otimes_k K
\end{array}$$

We begin with recalling a natural obstruction theory of the Hilbert scheme of points. It employs Schlessinger’s $T^2$ functor, which was recalled in Section 2. This theory appears in [FGI+05] Theorem 6.4.5] but with a larger obstruction space.

Proposition 4.1. The scheme $(\text{Hilb}_{pt}(\mathbb{A}^n), [R])$ has an obstruction theory $(\text{Hom}(I_R, \mathcal{O}_R), T^2(R))$.

Proof. For the tangent space and other details of the construction we refer to [FGI+05] Theorem 6.4.5], which we follow closely. We abbreviate $S \otimes_k (-)$ to $S_{(-)}$. We begin by constructing the obstructions. Fix a small extension $0 \to K \to B \to A \to 0$ and a deformation $\mathcal{R}$ of $R$ over $A$. The ring $\mathcal{O}_R$ is a quotient of $S_A$ by an ideal $\mathcal{I}_R$. We form a commutative Diagram 1.1 with exact rows and columns. Its top row comes from applying $(-) \otimes_k K$ to $0 \to I_R \to S \to \mathcal{O}_R \to 0$, its bottom row comes from the deformation $\mathcal{R}$, and its column comes from applying $S_{(-)}$ to the small extension above. The subquotient $\ker \beta/\text{im} \alpha$ is an $S_A$-module. The obstruction class $\text{ob} \in \text{Ext}^1_S(I_R, \mathcal{O}_R)$ is

$$\begin{array}{ccc}
0 & \longrightarrow & I_R \otimes_k K \\
\alpha \downarrow & & \downarrow \beta \\
S_B & \longrightarrow & \mathcal{O}_R \otimes_k K \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{I}_R \longrightarrow S_A \longrightarrow \mathcal{O}_R \longrightarrow 0
\end{array}$$

Diagram 4.1: Constructing obstruction
defined as the extension

\[ 0 \to O_R \otimes_{\mathbb{k}} K \to \frac{\ker \beta}{\im \alpha} \otimes_A \mathbb{k} \to I_R \otimes_A \mathbb{k} \to 0. \quad (4.2) \]

We show that this element lies in \( T^2(R) \). Let \( i_1, \ldots, i_r \) be the generators of \( I_R \). Fix a rank \( r \) free \( S \)-module \( F \) with basis \( e_1, \ldots, e_r \) and a surjection \( j: F \to I \) given by \( j(e_a) = i_a \). Let \( G = \ker j \).

After a choice of lifting \( \gamma: F \to \frac{\ker \beta}{\im \alpha} \otimes_A \mathbb{k} \) we obtain the commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \to & G & \to & F & \to & I_R & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & O_R \otimes_{\mathbb{k}} K & \to & \frac{\ker \beta}{\im \alpha} \otimes_A \mathbb{k} & \to & I_R \otimes_A \mathbb{k} & \to & 0
\end{array}
\]

The obstruction \( ob \) lies in \( T^2(R) \) if and only if \( \gamma(i_a e_b - i_b e_a) = 0 \) for all \( 1 \leq a \leq r \) and all \( 1 \leq b \leq r \). Consider the following commutative diagram of surjections of \( S_B \)-modules

\[
\begin{array}{ccc}
\ker \beta & \to & I_R \\
\downarrow & & \downarrow \pi & & \downarrow (-) \otimes_A \mathbb{k} \\
\frac{\ker \beta}{\im \alpha} \otimes_A \mathbb{k} & \to & I_R
\end{array}
\]

Choose lifts \( j_1, \ldots, j_r \in \ker \beta \) of \( i_1, \ldots, i_r \in I_R \) respectively. For every element \( f \) of \( (\ker \beta/\im \alpha) \otimes_A \mathbb{k} \) we have \( \pi(j_a)f = i_a f \). Choose \( \gamma \) so that \( \gamma(e_a) = \pi(j_a) \) for each \( a = 1, \ldots, r \). It follows that \( \gamma(i_a e_b - i_b e_a) = i_a \gamma(e_b) - i_b \gamma(e_a) = \pi(j_a) \pi(j_b) - \pi(j_b) \pi(j_a) = 0 \). Hence, we see that \( \gamma(K) = 0 \), so \( ob \in T^2(R) \), which concludes the proof. \( \Box \)

Consider the finite subscheme \( R \subset \mathbb{A} \) supported at the origin. We show that the scheme \((\text{Hilb}_{\text{pt}}^+(\mathbb{A}), [R])\) has an obstruction theory with obstruction space \( T^2(R)_{\geq 0} \). As explained in the introduction, the restriction from \( T^2(R) \) to \( T^2(R)_{\geq 0} \) is crucial for proving smoothness.

**Theorem 4.2.** The scheme \((\text{Hilb}_{\text{pt}}^+(\mathbb{A}), [R])\) has an obstruction theory with tangent space \( \text{Hom}(I_R, \mathcal{O}_R)_{\geq 0} \) and obstruction space \( T^2(R)_{\geq 0} \subset \text{Ext}^1(I_R, \mathcal{O}_R)_{\geq 0} \).

**Proof.** The open embedding \( \iota \) from Proposition 3.1 sends \([R]\) to a family \( \mathcal{Z} \subset \mathbb{A} \times \mathbb{G}_m \) given by the ideal \( I_R^h \); see Section 2. Applying the argument of Proposition 4.1 to \((\mathcal{H}_S, [\mathcal{Z}])\) and taking into account \( \mathbb{G}_m \)-invariance, we obtain an obstruction theory

\[
\left( \text{Hom}_{S[t^{-1}]}(I_R^h, \mathcal{O}_R^h) \right)_0, \quad (T^2_{\mathcal{Z}})_0.
\]

These spaces are isomorphic to \( \text{Hom}_{S}(I_R, \mathcal{O}_R)_{\geq 0} \) and \( T^2(R)_{\geq 0} \) respectively, see Section 2. \( \Box \)

Below we consider \( \mathcal{A} \) equal to \( \mathbb{A}^n \) with positive grading. In algebraic terms, we consider \( \mathcal{A} = \text{Spec} S \) for a graded polynomial ring \( S = \mathbb{k}[x_1, \ldots, x_n] \) with \( \text{deg}(x_i) > 0 \) for all \( i \). Recall from Section 3 the natural forget-about-the-limit map \( \theta_0: \text{Hilb}_{\text{pt}}^+(\mathbb{A}^n) \to \text{Hilb}_{\text{pt}}(\mathbb{A}^n) \). We form the map

\[
\theta: \text{Hilb}_{\text{pt}}^+(\mathbb{A}^n) \times \mathbb{A}^n \to \text{Hilb}_{\text{pt}}(\mathbb{A}^n),
\]

that sends \(([R], v)\) to the subscheme \([R]\) translated by the vector \( v \). The map \( \theta \) is the forget-about-the-limit-and-translate map defined in the introduction.

**Lemma 4.3.** The morphism \( \theta \) is injective on \( K \) points for all fields \( K \supset \mathbb{k} \).
Proof. After base change to $K$, we may assume $K = k$. Let $([R], v)$ be a $k$-point of $\text{Hilb}^+_\text{pt}(\mathbb{A}^n) \times \mathbb{A}^n$. Then $v$ is the support of $\theta([R], v)$. From $\theta([R], v)$ we recover $\theta_0([R])$ as the scheme $\theta([R], v)$ translated by $-v$. But $\theta_0$ is a monomorphism, hence is injective on $k$-points. This concludes the proof. \qed

Remark 4.4. The map $\theta$ is a monomorphism of schemes when $k$ has characteristic zero. It is not a monomorphism for $\text{char } k = p > 0$, because the action of $\mathbb{A}^n$ on $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ is not free; for example the stabilizer of $(x_1^2, x_2, \ldots, x_n)$ is not reduced.

The tangent bundle to $\mathbb{A}^n$ is trivial, spanned by global sections corresponding to partial derivatives $\partial_i$. The tangent map $d\theta_{[R]}$ sends $\partial_i$ to a homomorphism $\overline{\partial}_i : I_R \to \mathcal{O}_R$ defined by the formula $(\overline{\partial}_i)(s) := \partial_i(s) + I_R$. We obtain the linear subspace

$$\langle \overline{\partial}_1, \overline{\partial}_2, \ldots, \overline{\partial}_n \rangle \subset \text{Hom}(I_R, \mathcal{O}_R).$$

The image of $d\theta_{[R]}$ in $\text{Hom}(I_R, \mathcal{O}_R)$ is equal to $\text{Hom}(I_R, \mathcal{O}_R)_{\geq 0} + \langle \overline{\partial}_i | 1 \leq i \leq n \rangle$.

Now we prove the key theorems comparing $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ and $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$. Following [BBKT15 Definition 1.15], we say that a finite subscheme $R \subset \mathbb{A}^n$ is cleavable, if it is a limit of geometrically reducible subschemes. Put differently, the subscheme $R$ is cleavable if and only if the point $[R]$ lies on a non-elementary component of $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$.

Theorem 4.5. Let $R \subset \mathbb{A}^n$ be a subscheme supported at the origin and with trivial negative tangents. Then the map

$$\theta : \text{Hilb}^+_\text{pt}(\mathbb{A}^n) \times \mathbb{A}^n \to \text{Hilb}^+_\text{pt}(\mathbb{A}^n)$$

is an open embedding of a local neighbourhood of $([R], 0)$ into $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$. Hence, every component of $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ containing $[R]$ is elementary. In particular, if $R \not\subset \text{Spec } k$, then the scheme $R$ is not smoothable or cleavable.

Proof. For brevity, we denote $H := \text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ and $H^+ := \text{Hilb}^+_\text{pt}(\mathbb{A}^n)$. Since $R$ has trivial negative tangents, the tangent map $d\theta_{[R]}$ is surjective. Fix a small extension $0 \to K \to B \to A \to 0$. By Theorem 4.2 the map $\theta$ induces a map of obstruction theories

$$\begin{array}{cccccc}
T_{H \times \mathbb{A}^n, [R]} \otimes_k K & \longrightarrow & (H^+ \times \mathbb{A}^n)(B) & \longrightarrow & (H^+ \times \mathbb{A}^n)(A) & \longrightarrow & \text{Ext}(I_R, \mathcal{O}_R)_{\geq 0} \otimes_k K \\
\downarrow d\theta_{[R]} & & \downarrow & & \downarrow & & \left\langle \theta \right\rangle \\
T_H \otimes_k K & \longrightarrow & H(B) & \longrightarrow & H(A) & \longrightarrow & \text{Ext}(I_R, \mathcal{O}_R) \otimes_k K.
\end{array}$$

By a diagram chase, the map $\theta^\# : \hat{\mathcal{O}}_{H, [R]} \to \hat{\mathcal{O}}_{H^+, [R]}$ satisfies infinitesimal lifting. Consequently, the map $\theta$ is smooth at $[R]$. By Lemma 4.3 this map is universally injective [sta17, Tag 01S2], hence it is étale at $[R]$ so it is étale at some neighbourhood $U$ of $[R]$. But then $\theta_{[U]} : U \to \text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ is an open immersion [sta17, Tag 02LC] and so by Proposition 4.3 every component of $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ containing $[R]$ is elementary. \qed

Corollary 4.6. Let $R \subset \mathbb{A}^n$ be a finite subscheme supported at the origin and with trivial negative tangents. Suppose further that $T^2([R])_{\geq 0} = 0$ or $\text{Ext}^1(I_R, \mathcal{O}_R)_{\geq 0} = 0$. Then $[R]$ is a smooth point of $\text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ lying on a unique elementary component of dimension $n + \dim [R] \text{Hilb}^+_\text{pt}(\mathbb{A}^n) = \dim_k \text{Hom}(I_R, \mathcal{O}_R)$.

Proof. By Theorem 4.2 the point $[R] \in \text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ is smooth. By Theorem 4.5 the map $\theta$ is an open immersion near $[R]$ so $[R] \in \text{Hilb}^+_\text{pt}(\mathbb{A}^n)$ is smooth as well. Other assertions follow immediately. \qed

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In the following corollary we show that having trivial negative tangents can be, in most cases, deduced from the Hilbert function of the tangent space.

**Corollary 4.7.** Let $S$ be standard graded and let $R \subset \mathbb{k}^n$ be a finite subscheme supported at the origin. Suppose that the characteristic of $\mathbb{k}$ is zero or the ideal $I_R$ is homogeneous. Then the subscheme $R$ has trivial negative tangents if and only if $\dim \ker \text{Hom}(I_R, \mathcal{O}_R)_{<0} = n$.

**Proof.** If $R$ has trivial negative tangents, then by Theorem 4.5 the map $d\theta_{|R}$ is bijective, so $\dim \ker \text{Hom}(I_R, \mathcal{O}_R)_{<0} = \dim \text{span}(\partial_1, \ldots, \partial_n) = n$. Suppose $\dim \ker \text{Hom}(I_R, \mathcal{O}_R)_{<0} = n$. We are to prove that $\overline{\partial}_1, \overline{\partial}_2, \ldots, \overline{\partial}_n$ are linearly independent elements of $\text{Hom}(I_R, \mathcal{O}_R)/\text{Hom}(I_R, \mathcal{O}_R)_{\geq 0}$. Suppose it is not so. After a coordinate change we see that $\overline{\partial}_1 \in \text{Hom}(I_R, \mathcal{O}_R)_{\geq 0}$. By definition of $\text{Hom}_{\geq 0}$, the operator $\partial_1$ satisfies for all $k$ the containment

$$\partial_1((I_R)_{\geq k}) \subset I_R + S_{\geq k}. \quad (4.3)$$

Suppose that the characteristic is zero and take a smallest $k$ such that $(I_R)_{\geq k} = S_{\geq k}$. Then $\partial_1((I_R)_{\geq k}) = S_{\geq k-1}$, so $S_{\geq k-1} \subset I_R$, a contradiction with the choice of $k$.

Suppose that $I_R$ is homogeneous. In this case Equation (4.3) implies that $\partial_1(I_R) \subset I_R$ so that $\text{char} \mathbb{k} = p$ is positive and $I_R$ is generated by elements of $\mathbb{k}[x^p_1, x_2, \ldots, x_n] \cap I_R$. Recall that for all $1 \leq i \leq n$ the derivation $\partial_i$ is a first element of a sequence of differential operators $\partial_i^{[j]}$ on $S$, called Hasse derivatives. They are defined first as operators on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\partial_i^{[j]} := \frac{\partial^j}{j! \partial x_i^j}$$

and then descended to $S$ by $\mathbb{k}$-linearity. For all $s \in \mathbb{N}$ and $f, g \in S$ they satisfy the identity $\partial_i^{[s]}(fg) = \sum_{j=0}^s \partial_i^{[j]}(f)\partial_i^{[s-j]}(g)$.

Let $Q = (q_1, \ldots, q_n)$ be the lex-largest sequence of powers of $p$ such that after some coordinate change the ideal $I_R$ is generated by $I_R \cap \mathbb{k}[x_1^{q_1}, \ldots, x_n^{q_n}]$. Let $S' = \mathbb{k}[x_1^{q_1}, \ldots, x_n^{q_n}]$. For all $i$ and $j < q_i$ we have $\partial_i^{[j]}(I_R) \subset I_R$ so by the identity above the operator $\partial_i^{[q_i]}$ descends to a $S$-module homomorphism $\partial_i^{[q_i]}: I_R \to \mathcal{O}_R$ of degree $-q_i$. The operator $\partial_i^{[q_i]}$ acts on $S'$ as a partial derivative with respect to $i$-th coordinate. We claim that the homomorphisms $\{\partial_i^{[q_i]} | i = 1, \ldots, n\}$ are linearly independent. Suppose not. By homogeneity, the dependence occurs between operators $\{\partial_i^{[q_i]} | i = 1, \ldots, n, q_i = q\}$ for some fixed $q$. After a coordinate change we have $\partial_i^{[q_i]} = 0$. If $\partial_i^{[q_i]}(I_R) \subset I_R$, then the ideal $I_R \cap S'$ is preserved by the $i$-th partial derivative so the sequence $Q$ is not maximal, a contradiction. We conclude that all homomorphisms $\partial_i^{[q_i]}$ are linearly independent. We claim that also the elements

$$\left\{x_i^{q_i}\partial_i^{[q_i]} \mid i = 1, 2, \ldots, n, j = 0, 1, \ldots, q_i - 1 \right\} \quad (4.4)$$

are linearly independent in $\text{Hom}(I_R, S/I_R)_{<0}$. Suppose it is not so. Since $I_R$ is generated by $I_R \cap S'$, the quotient $S/I_R$ is naturally graded by $G = \mathbb{Z}/q_1 \times \mathbb{Z}/q_2 \times \ldots \times \mathbb{Z}/q_n$. The homomorphism $x_i^{q_i}\partial_i^{[q_i]}$ is non-zero and has $G$-degree $(0, \ldots, 0, j, 0, \ldots, 0)$, so the linear dependence may occur only between elements of $\{\partial_i^{[q_i]} | i = 1, \ldots, n\}$, but that was excluded above. Therefore, there are no linear dependencies and the set (4.4) is a basis of a $(q_1 + \ldots + q_n)$-dimensional subspace of $\text{Hom}(I_R, \mathcal{O}_R)_{<0}$. But $q_1 + \ldots + q_n \geq p + (n - 1) > n$, a contradiction. \qed

**Remark 4.8.** We do not know whether the equivalence of Lemma 1.7 holds also for non-homogeneous ideals in positive characteristic, though we would guess so.
Theorem 4.9. Let $Z \subset \text{Hilb}_{pt}(\mathbb{A}^n)$ be an irreducible component. Suppose that $Z$ is generically reduced and that $k$ has characteristic zero. Then $Z$ is elementary if and only if a general point of $Z$ has trivial negative tangents.

Proof. If any point of $Z$ has trivial negative tangents, then $Z$ is elementary by Theorem 1.3. Conversely, suppose that $Z$ is elementary, so it lies in the image of $\theta$. Take an irreducible component $W$ of $\text{Hilb}^+_{pt}(\mathbb{A}^n) \times \mathbb{A}^n$ which dominates $Z$. By assumptions, a general point of $Z$ is smooth. Choose a point $[R] \in \theta(W)$ which is a smooth point of $\text{Hilb}_{pt}(\mathbb{A}^n)$. The characteristic is zero, so the tangent map

$$d\theta_{[R]} : T_{\text{Hilb}_{pt}(\mathbb{A}^n) \times \mathbb{A}^n,[R]} \to T_{\text{Hilb}_{pt}(\mathbb{A}^n),[R]}$$

is injective at $[R]$, see proof of Corollary 4.7. Since $[R]$ is smooth, we have $\dim T_{\text{Hilb}_{pt}(\mathbb{A}^n),[R]} = \dim_{[R]} \text{Hilb}_{pt}(\mathbb{A}^n) = \dim Z$. Since $\theta$ is dominating near $[R]$, we have $\dim T_{\text{Hilb}^+_{pt}(\mathbb{A}^n) \times \mathbb{A}^n,[R]} \geq \dim_{[R]} (\text{Hilb}^+_{pt}(\mathbb{A}^n) \times \mathbb{A}^n) \geq \dim Z$. The source of the injective map $d\theta_{[R]}$ has dimension at least equal to the dimension of the target, so this map is surjective as well. \hfill \square

Example 4.10. An earlier version of this paper incorrectly stated that in the proof of Theorem 4.9 we may take $[R] \in Z$ to be any smooth point of $\text{Hilb}_{pt}(\mathbb{A}^n)$. This is false. For example take $I = (x_1,x_2)^2 + (x_3,x_4)^2 + (x_1x_3+x_2x_4) \subset S = \mathbb{Q}[x_1,x_2,x_3,x_4]$, which is a smooth point on an elementary component by [CEVV09] or Theorem 1.3. Introduce a grading on $S$ by $\deg(x_1) = \deg(x_3) = 3$ and $\deg(x_2) = \deg(x_4) = 1$. Then a direct computation shows that $\text{Spec}(S/I)$ does not have trivial negative tangents. In this example, the grading on $S$ is non-standard. We do not know whether a similar example exists for the standard grading. Let us pass to the flag case. We return to the slightly more general setup: we consider subschemes of $A$ instead of just $\mathbb{A}^n$. Consider the flag Hilbert scheme $\text{HilbFlag}$, parametrizing pairs $R \subset M$ of subschemes of $A$, see [Ser06] Section 4.5. As in Proposition 3.1 we see that its Bialynicki-Birula decomposition $\text{HilbFlag}^+$ exists and is embedded by a map $\tau_{\text{flag}}$ as an open subset of a multigraded flag Hilbert scheme $\mathcal{HSFlag}$ defined functorially by

$$\mathcal{HSFlag}(B) = \{ Y \subset Z \subset A' \times B \xrightarrow{\pi} B \mid Y, Z \text{ are } G_m\text{-invariant,}$$

$$\forall_i (\pi_* O_Y)_i \text{ and } (\pi_* O_Z)_i \text{ are locally free of finite rank} \}.$$  \hfill (4.5)

The scheme $\text{HilbFlag}$ comes with forgetful maps $\pi_M([R \subset M]) = [M]$ and $\pi_R([R \subset M]) = [R].$ We obtain a diagram of schemes

$$
\begin{array}{ccc}
\text{Hilb}^+_M & \xleftarrow{\pi^+_M} & \text{HilbFlag}^+ & \xrightarrow{\pi^+_R} & \text{Hilb}^+_R \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hilb}_M & \xleftarrow{\pi_M} & \text{HilbFlag} & \xrightarrow{\pi_R} & \text{Hilb}_R,
\end{array}
$$

\hfill (4.6)

where $\text{Hilb}_R = \text{Hilb}_M = \text{Hilb}_{pt}(A)$ and the subscript indicates the $k$-point of interest.

Fix subschemes $R \subset M \subset A$ supported at the origin and denote by $p = [R \subset M]$ the obtained $k$-point of $\text{HilbFlag}^+$. We construct an obstruction theory for the scheme $(\text{HilbFlag}^+, p)$ and maps between the obstruction theories of the schemes of the upper row of Diagram (4.6).

Theorem 4.11. Suppose that the map $\phi_{\geq 0}$ in Diagram (2.1) is surjective. The pointed scheme $(\text{HilbFlag}^+, [R \subset M])$ has an obstruction theory $(T_{\text{flag}}, \text{Ob}_{\text{flag}})$ given by the following pullback dia-
The projections $\pi_M$ and $\pi_R$ are maps of obstruction theories.

Proof. Using the embedding $\iota_{\text{flag}}$, see \([4.5]\), it is enough to produce an obstruction theory for $\mathcal{HSFlag}$. Let $T = S[t^{-1}] = H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$. The tangent space is computed in \([Ser06, \text{Proposition 4.5.3}]\). Our construction of the obstruction space below is a straightforward generalization of Proposition \([4.1]\). We abbreviate $T \otimes_k (-)$ to $T(-)$. Consider a small extension $0 \to K \to B \to A \to 0$ and an element $d \in D_{\mathcal{HSFlag}}(A)$ corresponding to a $\mathbb{G}_m$-invariant deformation $\mathcal{R} \subset \mathcal{M} \subset \text{Spec}(T_A)$. Diagrams \([4.1]\) for deformations $\mathcal{R}$ and $\mathcal{M}$ jointly form the commutative Diagram \([4.7]\) with exact rows and columns. By Theorem \([4.2]\) the obstructions to deforming $R$ and $M$ are elements $e_M \in T^2(\mathcal{M})_0 \subset \text{Ext}^1(I_M, \mathcal{O}_M)$ and $e_R \in T^2(\mathcal{R})_0 \subset \text{Ext}^1(I_R, \mathcal{O}_R)$ respectively. By tracing their construction in \([4.2]\) and comparing with Diagram \([4.7]\) we see that the images of both $e_R$ and $e_M$ in $\text{Ext}^1(I_M, \mathcal{O}_R)$ are canonically isomorphic to

$$0 \to \mathcal{O}_R^h \otimes_k K \to 0$$

thus we obtain an obstruction class $e_{\text{flag}} = (e_M, e_R) \in \text{Ob}_{\text{flag}}$. It remains to prove that the vanishing of $e_{\text{flag}}$ is necessary and sufficient for $d \in D_{\mathcal{HSFlag}}(A)$ to lie in the image of $D_{\mathcal{HSFlag}}(B)$. Necessity follows from Theorem \([4.2]\). Suppose that $e_{\text{flag}} = 0$. It this case, $\pi_M(d) \in D_{\text{Hilb}_m^+(A)}$ and $\pi_R(d) \in D_{\text{Hilb}_m^+(A)}$ come from elements $d_1' \in D_{\text{Hilb}_m^+(B)}$ and $d_2' \in D_{\text{Hilb}_m^+(B)}$ respectively. These elements correspond to extensions $\mathcal{J}_M, \mathcal{J}_R \subset T_B$, which give a commutative Diagram \([4.8]\) with exact rows and columns.

To obtain an element of $D_{\mathcal{HSFlag}}(B)$, we need to ensure that $\mathcal{J}_M \subset \mathcal{J}_R$. In other words, we need the induced $T_B$-module homomorphism $f: \mathcal{J}_M \to \mathcal{O}_R$ to be zero, see Diagram \([4.8]\). Since $0 = f \circ i_M = s_R \circ f$, the map $f$ is induced from a $T_B$-module homomorphism $\mathcal{I}_M \to \mathcal{O}_R^h \otimes_k K$. 

Diagram 4.7: Constructing obstructions.
Diagram 4.8: Obstruction equal to zero.

which comes from a $T$-module homomorphism $h: I^h_M \to \mathcal{O}^h_R \otimes_k K$. By construction, $h$ lies in

$$\text{Hom}_T(I^h_M, \mathcal{O}^h_R) \otimes_k K \simeq \text{Hom}_S(I_M, \mathcal{O}_R) \otimes_k K.$$ 

By assumption on $\varphi \geq 0$, such a homomorphism lifts to a $\mathbb{G}_m$-invariant homomorphism $I^h_M \to \mathcal{O}^h_M \otimes_k K$ and hence it gives a $\mathbb{G}_m$-invariant homomorphism $f: J_M \to \mathcal{Q}_M$. By [FGI+05, Theorem 6.4.5] the element $d_1' - f \in D_{\text{Hilb}^+_M}(B)$ is another $\mathbb{G}_m$-invariant extension of $\pi_M(d)$. We replace $J_M$ by the ideal $J'_M$ corresponding to $d_1' - f$. By a diagram chase, we check that the map $J'_M \to \mathcal{Q}_R$ is zero so $J'_M$ is contained in $J_R$ and we obtain an element of $D_{\text{HSFlag}}(B)$. \hfill \square

In the remaining part of this section we concentrate on the coarse obstruction spaces $\text{Ext}^1$ and not $T^2$. The following theorem summarizes our discussion and gives a rich source of smooth components of $\text{Hilb}^{pt}(\mathbb{A})$. The idea is to take a smooth point $[M]$, so that any obstruction $e = e_{\text{flag}}$ from Theorem 4.11 satisfies $\pi_M(e) = 0$ and so $\pi_R(e)$ lies in the kernel of $\text{Ext}^1(I_R, \mathcal{O}_R) \to \text{Ext}^1(I_M, \mathcal{O}_R)$. This kernel vanishes in a number of cases, one of them discussed in Remark 4.15.

Recall from Diagram (2.1) the homomorphisms $\varphi \geq 0: \text{Hom}(I_M, \mathcal{O}_M) \to \text{Hom}(I_M, \mathcal{O}_R) \otimes$ and $\partial \geq 0: \text{Hom}(I_M, \mathcal{O}_R) \to \text{Ext}^1(I_R/ I_M, \mathcal{O}_R)$. \hfill \square

**Theorem 4.12.** Suppose that $[M] \in \text{Hilb}^{pt}(\mathbb{A})$ is a smooth $\mathbb{G}_m$-invariant point and the maps $\varphi \geq 0, \partial \geq 0$ are both surjective. For all $p = [R \subset M] \in \text{HilbFlag}^+$, the points $[R] \in \text{Hilb}^{pt}(\mathbb{A})$ and $p \in \text{HilbFlag}^+$ are smooth and the map $\pi_R^+: \text{HilbFlag}^+ \to \text{Hilb}^{pt}(\mathbb{A})$ is smooth at $p$. Moreover

$$\dim_{[R]} \text{Hilb}^{pt}(\mathbb{A}) = \dim_{[M]} \text{Hilb}^{pt}(\mathbb{A}) - \dim \text{Ext}^1(J, \mathcal{O}_R) \geq 0$$

$$+ \dim \text{Hom}(J, \mathcal{O}_R) \geq 0 - \dim \text{Hom}(I_M, J) \geq 0.$$

Before we prove Theorem 4.12 we put forward its main consequence.

**Corollary 4.13.** In the setting of Theorem 4.12 assume additionally that $\mathbb{A}$ is equal to $\mathbb{A}^n$ with positive grading and that $R$ has trivial negative tangents. The point $[R] \in \text{Hilb}^{pt}(\mathbb{A}^n)$ is smooth lying on an elementary component of dimension

$$n + \dim_{[R]} \text{Hilb}^{pt}(\mathbb{A}^n) = n + \dim \text{Hom}(I_M, \mathcal{O}_M) \geq 0 - \dim \text{Ext}^1(J, \mathcal{O}_R) \geq 0$$

$$+ \dim \text{Hom}(J, \mathcal{O}_R) \geq 0 - \dim \text{Hom}(I_M, J) \geq 0.$$

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Proof of Theorem 4.12. We prove that every obstruction class $e_{\text{flag}}$ obtained in Theorem 4.11 is actually a zero element of the obstruction group. We have an exact sequence

$$\text{Hom}(I_M, O_R)_{\geq 0} \xrightarrow{\partial_{\geq 0}} \text{Ext}^1(J, O_R)_{\geq 0} \xrightarrow{i} \text{Ext}^1(I_R, O_R)_{\geq 0} \xrightarrow{\nu} \text{Ext}^1(I_M, O_R)_{\geq 0}.$$

On the one hand, since $[M] \in \text{Hilb}_{pt}(A)$ is a smooth $G_m$-invariant $k$-point, the completed local ring $A = \hat{O}_{\text{Hilb}_{pt}(A),[M]}$ is regular. Also the ring $\hat{O}_{\text{Hilb}_{pt}^+(A),[M]} = A/(A_{>0})$ is regular and so the scheme $\text{Hilb}_{pt}^+(A)$ is smooth at $[M]$. There are no obstructions for deforming $[M] \in \text{Hilb}_{pt}^+(A)$ so the obstruction class $e_{\text{flag}}$ lies in $\ker \nu$. On the other hand, by surjectivity of $\partial_{\geq 0}$, the map $i$ is zero, hence $\ker \nu = 0$, so $e_{\text{flag}}$ is zero and $\text{HilbFlag}^+$ is smooth at $p$ by the infinitesimal lifting criterion.

Since $\phi_{\geq 0}$ is surjective, the tangent map $d\pi^+_R$ is surjective at $p$ so the morphism $\pi^+_R$ is smooth at $p$ and $[R] \in \text{Hilb}_{R}^+$ is a smooth point, see [Gro67] Theorem 17.11.1d, p. 83. It remains to prove Equality (4.9). The claim follows from the description of tangent spaces in Theorem 4.2 and a chase on Diagram (2.1). Indeed, since $\phi_{\geq 0}$ and $\partial_{\geq 0}$ are surjective, we have

$$\dim \text{Hom}(I_R, O_R)_{\geq 0} = \dim \text{Hom}(J, O_R)_{\geq 0} + \dim \text{Hom}(I_M, O_R)_{\geq 0} - \dim \text{Ext}^1(J, O_R)$$

$$\dim \text{Hom}(I_M, O_M)_{\geq 0} = \dim \text{Hom}(I_M, O_R)_{\geq 0} + \dim \text{Hom}(I_M, J)_{\geq 0}. \quad \square$$

Proof of Corollary 4.13. Directly from Theorem 4.12 and Corollary 4.16. \hfill \square

Remark 4.14. Jan O. Kleppe, during his investigation of $G_m$-invariant deformations of embedded schemes using Laudal deformation theory and flag Hilbert schemes in [Kle06, Proposition 4] and [Kle81, Theorem 1.3.4], constructed an obstruction space analogous to the one from Theorem 4.11 and derived a theorem analogous to Theorem 4.12. Roughly speaking, he considers $(\text{Hilb}_{pt}^{G_m}(A), \phi_0, \partial_0)$ instead of $(\text{Hilb}_{pt}(A), \phi_{\geq 0}, \partial_{\geq 0})$.

Remark 4.15. Let us discuss the conditions of Corollary 4.13 in a special case. Let $R \subset M \subset \mathbb{A}^n$ be $G_m$-invariant finite subschemes. Let $d$ be the maximal number such that $(O_M)_d \neq 0$ and suppose that $J = I_R/I_M$ is concentrated in degree $d$. In this case

$$\text{Ext}^1(J, O_R) \simeq \bigoplus \text{Ext}^1(k[-d], O_R) = \bigoplus \text{Ext}^1(k, O_R)[d]$$

is concentrated in negative degrees, so the map $\partial_{\geq 0}$ is automatically surjective. Similarly,

$$\text{Ext}^1(I_M, J) \simeq \text{Ext}^1(I_M, \mathbb{k})[-d] \simeq \text{Ext}^2(O_M, \mathbb{k})[-d] \simeq \text{Tor}_2(O_M, \mathbb{k})^\vee[-d]$$

and this space is negatively graded exactly when there are no second syzygies of $O_M$ in degrees $\leq d$. Finally, $R$ having trivial negative tangents seems to be the most subtle assumption and we do not see any interesting sufficient conditions for it yet.

5 Singualrities

In this section we prove that equicharacteristic Murphy’s Law holds for $\text{Hilb}_{pt}^+(\mathbb{A}^n)$ and discuss Conjecture 1.6. Here, equicharacteristic is used to underline that we work over $\mathbb{k}$, while Vakil’s work over $\mathbb{Z}$.

Let us recall the main notions. A pointed scheme $(X,x)$ is a scheme $X$ of finite type over $\mathbb{k}$ together with a point $x \in X$. A morphism of pointed schemes $(X,x) \rightarrow (Y,y)$ is a morphism of schemes $f : X \rightarrow Y$ such that $f(x) = y$. A retraction is a pair $i : (Y,y) \rightarrow (X,x)$ and $\pi : (X,x) \rightarrow (Y,y)$ such that $\pi \circ i = \text{id}_X$ and $i$ is closed immersion.
Vakil [Vak06] defines an equivalence relation on pointed schemes by declaring \((X, x) \sim (Y, y)\) to be equivalent if there exists a pointed scheme \((Z, z)\) and smooth morphisms \((X, x) \leftarrow (Z, z) \rightarrow (Y, y)\). An equivalence class of \(\sim\) is called an equicharacteristic singularity. The equicharacteristic Murphy’s Law holds for \(\mathcal{M}\) if every equicharacteristic singularity appears on \(\mathcal{M}\).

The key to investigation of singularities of \(\text{Hilb}^m_{\text{pt}}(A^n)\) is its relation with \(\text{Hilb}^m(A^n)\). Namely, there is a functorial retraction \(\text{Hilb}^+_m(A^n) \to \text{Hilb}^m_{\text{pt}}(A^n)\). To construct it, observe that \(\text{Hilb}^m(A^n)\) represents the functor of \(m\)-morphism. A family \(\phi: B \to \text{Hilb}^m_{\text{pt}}(A^n)\), where the \(G_m\)-action on \(B\) is trivial. Recall from (3.1) that \(\text{Hilb}^m_{\text{pt}}(A^n)\) represents the functor

\[
\text{Hilb}^+_m(A^n)(B) = \{ \varphi: \mathbb{G}_m \times B \to \text{Hilb}^m_{\text{pt}}(A^n) \mid \varphi \text{ is } G_m\text{-equivariant} \}.
\]

We have a functorial map \(i: \text{Hilb}^m_{\text{pt}}(A^n) \to \text{Hilb}^+_m(A^n)\), which sends a family \(\varphi_0: B \to \text{Hilb}^m_{\text{pt}}(A^n)\) to \(\varphi_0 \circ \text{pr}_2: \mathbb{G}_m \times B \to \text{Hilb}^m_{\text{pt}}(A^n)\) and a functorial map \(\pi: \text{Hilb}^+_m(A^n) \to \text{Hilb}^m_{\text{pt}}(A^n)\), which sends a family \(\varphi: \mathbb{G}_m \times B \to \text{Hilb}^m_{\text{pt}}(A^n)\) to \(\varphi|_{\mathbb{G}_m \times B}: B \to \text{Hilb}^m_{\text{pt}}(A^n)\). We have \(\pi \circ i = \text{id}\) by construction. A family \(\varphi: \mathbb{G}_m \times T \to \text{Hilb}^m_{\text{pt}}(A^n)\) is equal to \(i(\varphi_0)\) if and only if \(\varphi|_{\mathbb{G}_m \times T} = \varphi|_{\mathbb{G}_m \times T}\). Since \(\text{Hilb}^m_{\text{pt}}(A^n)\) is separated, \(i\) is a closed immersion. For every \(k\)-point \([R]\) of \(\text{Hilb}^m_{\text{pt}}(A^n)\), the morphism \(\pi: (\text{Hilb}^m_{\text{pt}}(A^n), [R]) \to (\text{Hilb}^m(A^n), [R])\) induces a map of obstruction theories, which is just the projection \(T^2(R)|_{R} \to T^2(R)\).

**Theorem 5.1.** The equicharacteristic Murphy’s Law holds for \(\text{Hilb}^m_{\text{pt}}(A^5)\) and for \(\text{Hilb}^+_m(A^5)\).

**Proof.** The proof for \(\text{Hilb}^m_{\text{pt}}(A^5)\) is build around the ideas of [Erm12], who actually proved that Murphy’s Law holds for \(\bigsqcup_n \text{Hilb}^m_{\text{pt}}(A^n)\). Our contribution in this case, if any, is the reduction to embedding dimension five.

Fix an equicharacteristic singularity \(\mathcal{S}\). First, by [Vak06, M3] there is a surface \(V \subset \mathbb{P}^4\) such that singularity class of the corresponding Hilbert scheme of surfaces (\(\text{Hilb}(\mathbb{P}^4), [V]\)) is \(\mathcal{S}\). Let \(p = p(t)\) be its Hilbert polynomial and \(S\) be the homogeneous coordinate ring of \(\mathbb{P}^4\). By Gotzmann Regularity Theorem [HS04, Proposition 4.2] there exists a \(d\) such that \(\text{Hilb}^m(\mathbb{P}^4)\) is isomorphic to the multigraded Hilbert scheme parameterizing deformations of pairs \((I_d, I_{d+1})\) such that \(I_d \subset S_d, I_{d+1} \subset S_{d+1}\) and \(I_d S_d \subset I_{d+1}\), see [HS04, Theorem 3.6 and its proof]. The isomorphism sends \(V\) to \(((I_V)d, (I_V)_{d+1})\). This multigraded Hilbert scheme, in turn, is isomorphic to the locus in \(\text{Hilb}^m_{\text{pt}}(A^5)\) that parameterizes homogeneous ideals with Hilbert function \(h\) satisfying \(h(d-1) = \dim S_{d-1}, h(d) = p(d), h(d+1) = p(d+1), \) and \(h(d+2) = 0\). The isomorphism is given by sending \((I_d, I_{d+1})\) to the ideal

\[
J = I_d \oplus I_{d+1} \oplus S_{d+2}.
\]

Let \(R = \text{Spec } S/J\). We conclude that the singularity type of \((\text{Hilb}^m_{\text{pt}}(A^5), [R])\) is equal to \(\mathcal{S}\). It remains to prove that \(\pi: (\text{Hilb}^m_{\text{pt}}(A^5), i([R])) \to (\text{Hilb}^m_{\text{pt}}(A^5), [R])\) is smooth. This is formal. The homogeneous generators of \(J = I_R\) all have degree at least \(d\) and \((O_R)_{d+2} = 0\). We deduce that \(\text{Ext}^1(I_R, O_R)_{d+2} = 0\). Therefore, the morphism \(T^2(R)_{d+2} \to T^2(R)_0\) induced by \(\pi\) is injective. The tangent map \(d\pi: \text{Hom}(I_R, O_R)_{d+2} \to \text{Hom}(I_R, O_R)_{d+2}\) is clearly surjective. Consequently, the map \(\pi\) is smooth at \(i([R])\) by infinitesimal lifting property, see the proof of Theorem 4.5 or [EFM98, Section 6].

As discussed in the introduction, Theorem 5.1 does not shed light on the singularities of \(\text{Hilb}^m_{\text{pt}}(A^5)\). As a caution, we present the following example.

**Example 5.2.** Consider \(Y = \text{Spec } k[x_1, x_2, x_3]\) and \(G_m\) acting on \(Y\) by \(t \cdot (x_1, x_2, x_3) = (x_1, tx_2, t^{-1}x_3)\). The scheme \(Z = \text{Spec } k[x_1, x_2, x_3]/(x_1^2 - x_2 x_3)\) is clearly reduced and has an action of \(G_m\), but \(Z^m = \text{Spec } k[x_1]/(x_1^2)\) and \(Z^+ = \text{Spec } k[x_1, x_3]/(x_1^2)\) are both non-reduced.
However, Theorem 5.1 strongly suggests that Hilb_{pt}(\mathbb{A}^5) is non-reduced. In contrast, Conjecture 1.6 implies that Hilb_{pt}(\mathbb{A}^n) is non-reduced for large enough n.

Proposition 5.3. If Conjecture 1.6 is true, then the Hilbert scheme of points on some \mathbb{A}^n is non-reduced.

Sketch of proof. Let \mathcal{G} be a non-reduced singularity. Let (I_d, I_{d+1}) be as in the proof of Theorem 5.1. Let I be the ideal generated by them. Let r = d+2. As we assume that Conjecture 1.6 we conclude that there exists a polynomial ring T \supset S, and a subspace L \subset T, such that I' = I \cap T + L + T_{r+1} has trivial negative tangents. One proves directly that the \mathbb{G}_m-equivariant deformations of I' and I + T_{r+1} are smoothly equivalent.

Let \mathbb{A}^n := SpecT and R' = Spec(T/I') \subset \mathbb{A}^n. We scheme that (Hilb^G_{pt}(\mathbb{A}^n), [R']) is non-reduced and R' has trivial negative tangents. Since \pi: (Hilb^G_{pt}(\mathbb{A}^n), [R']) \rightarrow (Hilb^G_{pt}(\mathbb{A}^n), [R]) is a retraction, also Hilb^G_{pt}(\mathbb{A}^n) is non-reduced at [R']. Since R' has trivial tangents, the map \theta: Hilb^G_{pt}(\mathbb{A}^n) \times \mathbb{A}^n \rightarrow Hilb_{pt}(\mathbb{A}^n) is an isomorphism near [R'], see Theorem 1.2. Hence, also [R'] \in Hilb_{pt}(\mathbb{A}^n) is a non-reduced point.

The arxiv version (arXiv:1710.06124v3) of this paper contains some observations potentially useful for the proof of Conjecture 1.6

6 Examples

In this section we describe several examples and prove Theorem 1.4. This theorem follows from Corollary 1.13 once we verify its assumptions in our case. We keep the notation from introduction: S = k[x_1, x_2, y_1, y_2] and the subscheme R(e) \subset M(e) is defined by a single form s so that J = \mathfrak{k}s. The ideals I_M, I_R are bi-graded with respect to

\text{deg} \left(x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2}\right) = (a_1 + a_2, b_1 + b_2),

so we will speak about forms of given bi-degree. Observe that \mathcal{O}_M has a basis consisting of all monomials of bi-degree (a, b) with a, b < e.

Proposition 6.1. Let R_1 \subset Spec k[x_1, x_2] and R_2 \subset Spec k[y_1, y_2] be finite schemes. Then the subscheme R_1 \times R_2 \subset \mathbb{A}^4 is smoothable and [R_1 \times R_2] \in Hilb_{d_1d_2}(\mathbb{A}^4) is a smooth point. In particular dim Hom(I_{R_1 \times R_2}, \mathcal{O}_{R_1 \times R_2}) = 4 deg R_1 deg R_2.

Proof. Let d_i = deg R_i for i = 1, 2. The Hilbert schemes Hilb_{d_1}(\mathbb{A}^2) and Hilb_{d_2}(\mathbb{A}^2) are irreducible \cite{Fog68} so both R_1 and R_2 are smoothable. Consequently, R_1 \times R_2 is smoothable \cite{BJ17, B17}. Since T_{Hilb_{d_1d_2}(\mathbb{A}^4),[R_1 \times R_2]} = \bigoplus_i T_{Hilb_{d_i}(\mathbb{A}^2),[R_i]}, the point [R_1 \times R_2] is smooth. □

Corollary 6.2. For all e \geq 2, the subscheme M(e) is smoothable and [M(e)] \in Hilb_{pt}(\mathbb{A}^n) is a smooth point.

Proof. Apply Proposition 6.1 to M(e) = Spec(k[x_1, x_2]/(x_1, x_2)^e) \times Spec(k[y_1, y_2]/(y_1, y_2)^e). □

We proceed to show that \psi and \phi from Diagram (2.1) are surjective, in all degrees, for M = M(e) and R = R(e). Recall that J \simeq k, or, taking into account the grading, J \simeq k[-(2e-2)].

Proposition 6.3. In Diagram (2.1) applied to R(e) \subset M(e), the homomorphism \psi is surjective.
Proposition 6.4. Proof. We abbreviate \( M(e) \) and \( R(e) \) to \( M \) and \( R \) respectively. Since \( M \) is monomial, it is straightforward to compute that \( \text{Ext}^1(J,\mathcal{O}_M) \) is concentrated in degree \(-1\) and that

\[
\dim \text{Hom}(I_M,\mathcal{O}_M)_{-1} = \dim \text{Ext}^1(J,\mathcal{O}_M)_{-1} = 2e(e + 1).
\]

It is enough to show that \( \text{Hom}(I_R,\mathcal{O}_M)_{-1} \) is zero. Let \( s = \sum_j f_jy_1^jy_2^{e-1-j} \), where \( f_j = \sum_i c_{ij}x_1^ix_2^{e-1-i} \), be the form defining \( M \), as in the introduction. Pick \( \varphi \in \text{Hom}(I_R,\mathcal{O}_M)_{-1} \). The element \( t := \varphi(s) \) of \( \mathcal{O}_M \) has degree \( 2e - 3 \), so it is uniquely written as \( t_1 + t_2 \) where \( t_1, t_2 \in \mathcal{O}_M \) have bi-degree \((e - 2, e - 1)\) and \((e - 1, e - 2)\) respectively. Write \( t_1 = \sum g_jy_1^jy_2^{e-1-j} \), where \( g_j \in \mathbb{k}[x_1, x_2]_{e-1} \). Consider the equation

\[
x_1t_1 = x_1t = \varphi(x_1s) = \sum_j \varphi(x_1f_j)y_1^jy_2^{e-1-j}.
\]

The element \( x_1f_j \in \mathbb{k}[x_1, x_2] \) is a form of degree \( e \), thus \( x_1f_j \in I_M \). Analysing \( \varphi|_{I_M} \in \text{Hom}(I_M,\mathcal{O}_M)_{-1} \) directly, we see that \( \varphi(x_1f_j) \) lies in the image \( \mathbb{k}[x_1, x_2] \) and so it is a form of degree \((e - 1, 0)\). Therefore \( x_1t_1 \) and \( \sum_j y_1^jy_2^{e-1-j}\varphi(x_1f_j) \) are two forms of bi-degree \((e - 1, e - 1)\) equal modulo \( I_M \). Comparing their coefficients next to \( y_1^jy_2^{e-1-j} \), we see that \( \varphi(x_1f_j) = x_1g_j \in \mathcal{O}_M \). The same argument shows that \( \varphi(x_2f_j) = x_2g_j \).

Restrict \( \varphi \) to a homomorphism \( \varphi' : (x_1, x_2)^e \rightarrow \mathcal{O}_M \) and extend \( \varphi' \) to a degree minus one homomorphism \( \varphi' : (x_1, x_2)^{e-1} \rightarrow \mathcal{O}_M \) by imposing, for every \( \lambda_i \in \mathbb{k} \), the condition

\[
\varphi' \left( \sum \lambda_jf_j \right) = \sum \lambda_jg_j.
\]

The syzygies of \((x_1, x_2)^{e-1}\) are linear, so the map \( \varphi' \) sends them to forms of degree \( e - 1 \). No such form lies in \( I_M \), thus \( \varphi' \) lifts to an element of \( \text{Hom}((x_1, x_2)^{e-1}, \mathcal{S})_{-1} \). But \( \text{Hom}((x_1, x_2)^{e-1}, \mathcal{S})_{-1} = 0 \) and so \( \varphi' = 0 \). Therefore, \( \varphi((x_1, x_2)^e) = \varphi'((x_1, x_2)^e) = 0 \). Repeating the argument with \( y_i \) interchanged with \( x_i \), we obtain \( \varphi((y_1, y_2)^e) = 0 \), so \( \varphi = 0 \).

Proposition 6.4. In Diagram \((2.1)\) applied to \( R(e) \subset M(e) \), the homomorphism \( \phi \) is surjective.

Proof. We abbreviate \( M(e) \) and \( R(e) \) to \( M \) and \( R \) respectively. We begin with a series of reductions. Let \( N = (x_1, x_2)^e \oplus (y_1, y_2)^e \), with the surjection \( N \rightarrow I_M \). We have \( \text{Hom}(N,\mathcal{O}_M) \simeq \text{Hom}(I_M,\mathcal{O}_M) \) and \( \text{Hom}(N,\mathcal{O}_R) \simeq \text{Hom}(I_M,\mathcal{O}_R) \) so it is enough to show that \( \text{Hom}(N,\mathcal{O}_M) \rightarrow \text{Hom}(N,\mathcal{O}_R) \) is surjective. Second, it is enough to show that for \( N_0 = (x_1, x_2)^e \) the map

\[
\Phi : \text{Hom}(N_0,\mathcal{O}_M) \rightarrow \text{Hom}(N_0,\mathcal{O}_R)
\]

is surjective. Third, the map \( \Phi \) preserves bi-degree, so we may restrict to homomorphisms of given bi-degree. Fourth, the generators of syzygies of \( N_0 \) are linear of bi-degree \((1, 0)\), the modules \( \mathcal{O}_M \) and \( \mathcal{O}_R \) differ only in bi-degree \((e - 1, e - 1)\) and \( N_0 \) is generated in bi-degree \((e, 0)\). If we consider homomorphisms of bi-degree \((d_1, d_2) \neq (-2, e - 1)\) then the syzygies of \( N_0 \) are mapped into degree \((e, 0) + (d_1, d_2) + (1, 0) \neq (e - 1, e - 1)\) and the map \( \Phi \) is an isomorphism. Hence, we restrict to homomorphisms of bi-degree \((-2, e - 1)\). Each such homomorphism sends generators of \( N_0 \) to elements of bi-degree \((e - 2, e - 1)\).

Pick a homomorphism \( \varphi \in \text{Hom}(N_0,\mathcal{O}_R) \) of bi-degree \((-2, e - 1)\). For each \( 0 \leq i \leq e \) the element \( \varphi(x_1^{e-i}x_2^i) \) is a form of bi-degree \((e - 2, e - 1)\) so it can be uniquely lifted to a form \( \varphi_i \in \mathcal{S} \) of bi-degree \((e - 2, e - 1)\). Recall that \( I_R = I_M + \mathbb{k}s \). Since \( \varphi \) is a homomorphism to \( \mathcal{O}_R \), the syzygies
between elements of $N_0$ give the following relations between forms of bi-degree $(e-1,e-1)$:

$$x_1\varphi_{i+1} - x_2\varphi_i = \lambda_is \mod I_M \quad i = 0, 1, \ldots, e-1. \quad (6.1)$$

To prove that $\varphi$ is in the image of $\Phi$ it is enough to prove that $\lambda_i = 0$ for all $i$. Since $s$ is general, for appropriate choice of a basis $f_0, \ldots, f_{e-1}$ of $k[y_1, y_2]_{e-1}$ we have

$$s = x_2^{e-1}f_0 + x_2^{e-2}x_1f_1 + \ldots + x_1^{e-1}f_{e-1}.$$ 

Fix $k \in \{0, \ldots, e\}$. Both sides of each Equation (6.1) have the form $\sum_j r_j f_j$, where $r_j$ belong to the image of $k[x_0, x_1]_e$ in $O_M$. Extracting coefficients of $f_k$ from both sides, we obtain equalities in $k[x_1, x_2]_e$:

$$x_1\tau_{i+1} - x_2\tau_i = \lambda_i x_1^{e-1-k}x_2^k \mod (x_1, x_2)^e \quad \text{for } i = 0, \ldots, e, \quad (6.2)$$

where $\tau_i$ is the coefficient of $f_k$ in $\varphi_i$. Let $m_0 := x_1^k x_2^{e-1-k}$. We have $m_0 \cdot (x_1, x_2)^e \subset (x_1^e, x_2^e)$, hence Equation (6.2) for $i = k$ multiplied by $m_0$ gives

$$x_1^{k+1} x_2^{e-k} \tau_{k+1} - x_1^k x_2^{e-k} \tau_k = \lambda_k x_1^{e-1} x_2^{e-1} \mod (x_1, x_2)^e. \quad (6.3)$$

For all monomials $m \in k[x_1, x_2]_{e-1}$ different from $m_0$ we have $x_1^k x_2^{e-1-k} \cdot m \in (x_1^e, x_2^e)$. Multiplying Equation (6.2) for $i = k - 1$ by the monomial $x_1^{k-1} x_2^{e-k}$, we obtain

$$x_1^k x_2^{e-k} \tau_k - x_1^{k-1} x_2^{e-k} \tau_{k-1} = x_1^{k-1} x_2^{e-k} \cdot \lambda_{k-1} x_1^{e-1-k} x_2^k = \lambda_k x_1^{e-2} x_2^e = 0 \mod (x_1, x_2)^e. \quad (6.4)$$

Similarly, Equations (6.2) for $i = k - 1, k - 2, \ldots$ and $i = k + 1, k + 2, \ldots$, give

$$x_1^k x_2^{e-k} \tau_{k+1} - x_1^{k+1} x_2^{e-k} \tau_{k+2} = \ldots = x_1^e \tau_0 = 0 \mod (x_1, x_2)^e. \quad (6.5)$$

Together, Equations (6.3), (6.4), (6.5) imply that $0 = \lambda_k x_1^{e-1} x_2^{e-1} \mod (x_1^e, x_2^e)$, so $\lambda_k = 0$. Since $k$ is arbitrary, we have $\lambda_i = 0$ for all $i$. As a result, the map $\varphi$ lifts to $\Hom(I_M, O_M)$ and the claim follows.

**Proof of Theorem 4.4** By Remark 4.15 Corollary 6.2 and Proposition 6.4 the assumptions of Theorem 4.12 are satisfied for $M := M(e)$ and $R := R(e)$. A chase on Diagram (2.1), taking into account surjectivity of $\psi$ and $\phi$, shows that

$$\Hom(I_R, O_R)_{<0} = \ker (\Hom(I_M, O_R)_{<0} \to \Ext^1(J, O_R)_{<0})$$

$$= \ker (\Hom(I_M, O_M)_{<0} \to \Ext^1(J, O_R)_{<0}) = \Ext^1(J, J)_{<0} \simeq \Ext^1(k, k)_{<0}. \quad (6.6)$$

As a result, we obtain $\dim_k \Hom(I_R, O_R)_{<0} = n$. The ideal $I_R$ is homogeneous, so by Lemma 4.7 the subscheme $R(e)$ has trivial negative tangents. Corollary 4.13 implies that $Z(e)$ is elementary. Formula (4.10) yields

$$\dim Z(e) = 4 + (4 \deg M - 2e(e+1)) - 0 + (e^2 - 1) - (2e + 2) = 4 \deg R - (e-1)(e+5). \quad \square$$

**Remark 6.5.** In the setting of Theorem 4.4 denote by $Z_{\text{flag}}$ the component of $\text{HilbFlag}^+$ containing $p = [R \subset M]$. By Remark 4.15 we have $\Ext^1(J, O_R)_{>0} = 0$ so $\Hom(I_R, O_R)_{>0} \to \Hom(I_M, O_R)_{>0}$ is surjective, so the tangent map of $\tau_M : Z_{\text{flag}} \to \im \tau_M \subset \text{Hilb}^+_M$ is surjective at the point $p$. Counting dimensions, we see that the general fiber of $\tau_M$ is $(e^2 - 1)$-dimensional, so $Z_{\text{flag}}$ is dominated by a family of $\mathbb{P}^{e^2-1}$ and also $Z(e)$ is dominated by such a family.
Example 6.6. The component \( Z(2) \) was discovered in [IE78, Section 2.2]. It is isomorphic to \( \text{Gr}(3, 10) \times \mathbb{A}^4 \). This component was throughly analysed in [CEVV09].

Example 6.7. In contrast with \( Z(2) \), the component \( Z(3) \) was not known before. It is more complicated than \( Z(2) \) and we do not know if it is rational.

The Hilbert function of \( O_{R(3)} \) is \( h = (1, 4, 10, 12, 8) \) and the Hilbert series of \( \text{Hom}(I_{R(3)}, O_{R(3)}) \) is \( 4T^{-1} + 56 + 64T \) so the component \( Z \) is a rank 68 fiber bundle over \( Z^{G_m} \) and \( \dim Z^{G_m} = 56 \).

Let \( \text{Hilb}^h(\mathbb{A}^4) \) be the multi-graded Hilbert scheme [HS04], parameterizing graded subschemes with Hilbert function \( h \). The scheme \( \text{Hilb}^h(\mathbb{A}^4) \) is naturally identified with

\[
F^{\text{res}} := \{(I_3, I_4) \in \text{Gr}(8, S_3) \times \text{Gr}(27, S_4) \mid S_1 \cdot I_3 \subset I_4\}.
\]

Projection to first coordinate maps \( F^{\text{res}} \) onto a determinantal scheme

\[
F := \{I_3 \in \text{Gr}(8, S_3) \mid \dim_k(I_3 \cdot S_1) \leq 27\} \subset \text{Gr}(8, S_3),
\]

which is given by \( 28 \times 28 \) minors, thus its dimension at every point is at least \( 8 \cdot 12 - (35 - 27) \cdot (4 \cdot 8 - 27) = 56 \). By comparing bounds, we see that equality occurs near \( [R(3)] \), \( 56 = \dim[R(3)] F^{\text{res}} = \dim[R(3)] \text{Hilb}^h(\mathbb{A}^4) \)

and so \( Z^{G_m} \) is equal to the unique component of \( F^{\text{res}} \) passing through \( [R(3)] \).

Since \( F \) is determinantal, it has an embedding into the generic determinantal variety \( D \). The map \( F^{\text{res}} \to F \) is the pullback of the resolution of singularities of \( D \) via the embedding \( F \subset D \), see [ACGH85, Section II.2].

Example 6.8. In this example we construct a one-parameter family of finite subschemes \( R_t \) of \( \mathbb{A}^4 \).

The schemes \( R_t \) have degree 35 and satisfy the following conditions

1. the Gr"obner fans of all schemes \( R_t \) are equal,

2. for \( t \neq 0 \) the point \( [R_t] \in \text{Hilb}_{35}(\mathbb{A}^4) \) is smooth lying on an elementary component \( Z(3) \),

3. the point \( [R_0] \) lies in the intersection of two components of \( \text{Hilb}_{35}(\mathbb{A}^4) \) and it is cleavable. In other words, \( R_0 \) is a limit of reducible subschemes.

Let \( k = \mathbb{F}_3 \). Put \( s_t = x_1^2 x_2^2 - x_2^2 x_3^2 + tx_1 x_2 x_3 x_4 - x_1^2 x_3^2 - x_2^2 x_4^2 \) and \( R_t = (x_1, x_2)^3 + (x_3, x_4)^3 + s_t \).

The assertions on \( [R_t] \) for \( t \neq 0 \) follow immediately from Theorem 1.4 as the form \( s_t \) corresponds to a matrix

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & t & 0 \\
-1 & 0 & -1
\end{bmatrix},
\]

which is invertible for \( t \neq 0 \). Also the equality of Gröbner fans is clear. The point \( [R_0] \) lies on \( Z(3) \).

The family

\[
\text{Spec} \frac{k[x_1, x_2, x_3, x_4][z]}{(x_3, x_4)^3 + (x_1 x_2, x_1 x_2 \cdot z - x_1 x_3 x_4, x_1 x_2^2 + z - x_1^3, x_1 x_2 x_3 x_4 \cdot z - s_0)} \to \text{Spec} k[z],
\]

is finite and flat. Its fiber over \( z = 0 \) is \( R_0 \), and its fiber over \( z = 1 \) is reducible and so the image of \( \text{Spec} k[z] \to \text{Hilb}_{35}(\mathbb{A}^4) \) is contained in an irreducible component other that \( Z(3) \) and \( [R_0] \) lies in the intersection of those two components.
Example 6.9. Suppose we know the Betti table of a finite subscheme which has trivial negative tangents. In this example we show how may sometimes deduce that this subscheme is a smooth point of the Hilbert scheme. Let \( S = \mathbb{k}[x,y,z,t] \). Let \( R \subset \mathbb{A}^4 = \text{Spec} \, S \) be given by \( I_R = (x^3, y^3, z^3, t^3, Q_1, Q_2) \), where \( Q_1, Q_2 \) are quartics. For \( \mathbb{k} = \mathbb{F}_2 \) and \( Q_1 = xy^2z + y^2z^2 + x^2yt \), \( Q_2 = yz^2t + xzt^2 \) we have \( H_{\mathcal{O}_R}(T) = 1 + 4T + 10T^2 + 16T^3 + 17T^4 + 8T^5 \), \( \deg R = 56 \) and the following Hilbert series

\[
H_{\text{Hom}(I_R, \mathcal{O}_R)}(T) = 4T^{-1} + 98 + 84T + 32T^2.
\]

The resolution of \( \mathcal{O}_R \) is

\[
S \leftarrow S(-3)^4 \oplus S(-4)^2 \leftarrow S(-6)^{16} \oplus S(-7)^4 \leftarrow \ldots
\]

Comparing (6.7) with \( H_{\mathcal{O}_R} \), we see that \( \text{Ext}^1(I_R, \mathcal{O}_R)_{\geq 0} = 0 \) by degree reasons. Since \( \text{Ext}^1 \) has no non-negative part and \( R \) has only trivial negative tangents, Corollary 4.6 implies that \( [R] \) is a smooth point of an elementary component \( Z \) of dimension \( \dim \text{Hom}(I_R, \mathcal{O}_R) = 218 \). Since \( \deg R = 56 \) and \( 56 \cdot 4 > 218 \), the component \( Z \) is small and elementary. Macaulay2 [GS] experiments suggest that (6.6) is true for a general choice of \( Q_1, Q_2 \). We stress that we deduced \( \text{Ext}^1_{\geq 0} = 0 \) only from the graded Betti numbers of \( \mathcal{O}_R \). The group \( \text{Ext}^1 \) is non-zero in negative degrees, with Hilbert function

\[
H_{\text{Ext}^1(I_R, \mathcal{O}_R)}(T) = 60T^{-3} + 204T^{-2} + 60T^{-1}.
\]

Remark 6.10. Below we discuss examples of elementary components known before this work. To the author’s best knowledge, these are all the examples found in the literature. Trivially, \( \text{Hilb}_1(\mathbb{A}^n) = \mathbb{A}^n \) is elementary. The first two nontrivial examples, with Hilbert function \((1,4,3)\) and \((1,6,6,1)\) respectively, are given in [IE78]. For any integers \((d,e)\) with

\[
3 \leq e \leq \frac{(d-1)(d-2)}{6} + 2
\]

an elementary component with Hilbert function \((1,d,e)\) is given in [Sha90]. Five examples, with Hilbert function \((1,n,n,1)\), \(8 \leq n \leq 12\), are given in [IK99, Lemma 6.21]. Two other examples are given in [IK99, Corollaries 6.28, 6.29]. Five examples are discovered in [Hui17]. Only six of all these elementary components are small: those with Hilbert function \((1,4,3)\), \((1,5,3)\), \((1,5,4)\), \((1,6,6,1)\) and two examples \((1,5,3,4)\), \((1,5,3,4,5,6)\) from [Hui17]. The elementarity of all these examples except the last two, which crucially employ non-graded schemes, can be proved using Corollary 1.3 and the obtained components are products of \( \mathbb{A}^n \) with Grassmannians. Therefore, they are smooth and rational.

Both Iarrobino-Kanev [IK99, Conjecture 6.30], relying on [IE78], and Huibregtsen [Hui17, Conjecture 1.4] state conjectures which would give other infinite families of elementary components. To prove these conjectures it is, in each case, enough to check that the tangent space to the Hilbert scheme at a given point has expected dimension.

References


