A THEOREM ON REFINING DIVISION ORDERS
BY THE REVERSE LEXICOGRAPHIC ORDER

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Let $k$ be an infinite field of any characteristic, and let $S = k[x_1, \ldots, x_n]$ be a graded polynomial ring, where each $x_i$ has degree one. Let $I \subset S$ be a homogeneous ideal.

Let $S_d$ denote the finite vector space of all homogeneous, degree $d$ polynomials in $S$, so $S = S_0 \oplus S_1 \oplus \cdots \oplus S_d \oplus \cdots$. Writing $I$ in the same manner as $I = I_0 \oplus I_1 \oplus \cdots \oplus I_d \oplus \cdots$, we have $I_d \subset S_d$ for each $d$. An order $>^*$ on the monomials of $S_d$ for each $d$ is compatible with the monoid structure on the monomials of $S$ if whenever $x^A > x^B$ for two monomials $x^A, x^B$, then $x^C x^A > x^C x^B$ for all monomials $x^C$. We shall only consider orders satisfying this compatibility condition.

If an order $>^*$ is a strict order on the monomials of each degree, one can use $>^*$ in applying the division algorithm to constructing a standard (Gröbner) basis for $I$. The standard basis for $I$, and its properties, will vary in a crucial way with the choice of order $>^*$. The subject of computing standard or Gröbner bases has a long history; see [Bay85] for a recent survey.

One can generalize the necessary definitions to nonstrict orders $>^*$, which fail to distinguish between all monomials of a given degree: For each polynomial $f \in S$, define $\text{in}(f)$ to be the sum of those terms $cx^A$ of $f$ which are greatest with respect to the order $>^*$. Define $\text{in}(I)$ to be the ideal generated by $\{\text{in}(f) | f \in I\}$. Define $f_1, \ldots, f_r$ to be a standard basis for $I$ with respect to the order $>^*$ if $\text{in}(f_1), \ldots, \text{in}(f_r)$ generate the ideal $\text{in}(I)$. If $>^*$ is a strict order, $\text{in}(I)$ will be a monomial ideal; if $>^*$ is not strict, $\text{in}(I)$ may fail to be a monomial ideal.

A nonstrict order $>_1$ can be refined to a strict order by breaking any ties with a fixed strict order $>_2$; the resulting order $>_3$ is then a compatible order, so the usual division algorithm can be applied to compute standard bases with respect to $>_3$. Let $\text{in}_1, \text{in}_2, \text{in}_3$ correspond to $>_1,>_2,>_3$. We shall see that $\text{in}_3(I) = \text{in}_2(\text{in}_1(I))$, so a standard basis with respect to $>_3$ is already a standard basis with respect to $>_1$. Call $>_3$ the refinement of $>_1$ by $>_2$. Thus, refinements provide a mechanism for computing with nonstrict orders. This has been observed for example in [MoM683], where in the affine setting, homogenizing bases (in the above sense, standard bases with respect to the total degree order) are computed via standard bases with respect to a strict order.

We recall two frequently used strict orders: The lexicographic order is defined by $x^A > x^B$ if the first nonzero entry in $A-B$ is positive. The reverse lexicographic order is defined by $x^A > x^B$ if the last nonzero entry in $A-B$ is positive. The reverse lexicographic order has been used in constructing standard bases with respect to the total degree order.
graphic order is defined by $x^A > x^B$ if the last nonzero entry in $A - B$ is negative. These orders induce the same order $x_1 > x_2 > \cdots > x_n$ on the variables of $S$, but differ in higher degrees.

If $f_1, \ldots, f_r$ is a standard basis for the ideal $I$ with respect to the lexicographic order, then $\{ f_1, \ldots, f_r \} \cap k[x_1, \ldots, x_n]$ is a standard basis for the ideal $I \cap k[x_1, \ldots, x_n]$, and in particular, generates this ideal (Spear ([Spe77], [Zac78]), Trinks [Tri78]). If $f_1, x_r, f_r, x_{r+1}, \ldots, f_s$ is a standard basis for the ideal $I$ with respect to the reverse lexicographic order, with none of $f_{r+1}, \ldots, f_s$ divisible by $x_n$, then $f_1, \ldots, f_s$ is a standard basis for the ideal quotient $(I : x_n)$ [Bay82]. If one wants to compute the syzygies of the ideal $I$, then any strict order may be used (Spear ([Spe77], [Zac78]), Schreyer [Sch80]).

Recall the definition of the regularity of an ideal $I$: The regularity of $I$, $\text{reg}(I)$, is the least $m$ so the $i$th syzygies of $I$ are all of degree $< m + i$, for $i \geq 1$, and so $I$ is generated in degrees $< m$. We have $\text{reg}(\text{in}(I)) \geq \text{reg}(I)$ [Bay82]. Regularity is a fair measure of the complexity of an ideal, and is inextricably involved with the problem of computing standard bases: In [BaSt87], it is shown that in characteristic zero, for a generic choice of coordinates $x_1, \ldots, x_n$, and for any strict order, the highest degree of a standard basis element of $I$ is given by $\text{reg}(\text{in}(I))$. Furthermore, for a generic choice of coordinates in any characteristic, $\text{reg}(\text{in}(I)) = \text{reg}(I)$ for the reverse lexicographic order, in contrast to the usual jump for other orders. These statements show that standard bases with respect to the reverse lexicographic order usually are of minimal degree.

Thus, the reverse lexicographic order appears to be the most efficient choice for constructing standard bases when one can make a free choice of order, as in the computation of syzygies or Hilbert functions. To compute a projection $I \cap k[x_1, \ldots, x_n]$, one is not free to choose the reverse lexicographic order; the lexicographic order has commonly been used to date for this problem. However, the lexicographic order is capable of computing projections for each possible $i$; these capabilities are wasted if one only needs to compute a projection for a particular $i$.

Define $\text{deg}_{i-1}(x^A)$ to be the degree in $x_1, \ldots, x_{i-1}$ of the monomial $x^A$; define the nonstrict order $>_1$ by $x^A >_1 x^B$ if $\text{deg}_{i-1}(x^A) > \text{deg}_{i-1}(x^B)$. For any strict order $>_2$, the refinement of $>_1$ by $>_2$ can be used to compute the projection $I \cap k[x_1, \ldots, x_n]$. It is natural to ask which order $>_2$ can most efficiently be used for this purpose. Given the results in [BaSt87], an obvious guess is to use the reverse lexicographic order as $>_2$. In this paper, we confirm the optimality of this choice, by an extension of the analysis given in [BaSt87]. This result provides a framework by which one can begin to analyze the complexity of computing projections. As a practical consequence, the use of the lexicographic order in computations can unequivocally be discouraged, in favor of more carefully chosen orders.

More generally, we show that for any nonstrict order $>_1$ satisfying $x_1 >_1 \cdots >_1 x_n$, one optimally uses the reverse lexicographic order to refine $>_1$ to a strict order.
Let the general linear group \( \text{GL}(n) \) act on \( S \) via \( g \cdot f(x_1, \ldots, x_n) = f(g \cdot x_1, \ldots, g \cdot x_n) \). The maximal torus \( T(n) \subset \text{GL}(n) \) is the subgroup consisting of elements \( g \) of the form \( g \cdot x_i = a_ix_i \); \( T(n) \) is the set of diagonal matrices in \( \text{GL}(n) \). The Borel subgroup \( B(n) \subset \text{GL}(n) \) is generated by \( T(n) \) and elements \( g = g_{ij}(a), \ j < i, \) where \( g \cdot x_i = x_i + ax_j \) and \( g \cdot x_t = x_t \) for \( t \neq i \); \( B(n) \) is the set of upper triangular matrices in \( \text{GL}(n) \).

The following result is a generalization to all characteristics of a result first proved by Galligo in characteristic zero ([Gal74]; see also [Bay82]).

**Proposition 1.** Let \( > \) be a strict order satisfying \( x_1 > \cdots > x_n \), and let \( I \subset S \) be a homogeneous ideal. There exists a Zariski open subset \( U \subset \text{GL}(n) \) such that for any \( g \in U \), \( \text{in}(g \cdot I) \) is invariant under the action of the Borel subgroup \( B(n) \).

**Proof.** Fix a degree \( d \). By Proposition (1,1.8) of [Bay82], we can choose a weight vector \( w = (w_1, \ldots, w_n) \in \mathbb{Z}^n \) with \( w_1 > \cdots > w_n \) so that for each \( x^A, x^B \in S_d, \) \( x^A > x^B \) if and only if \( A \cdot w > B \cdot w \). Suppose that \( q = \text{dim}(I_d) \). Choose a basis \( f_1, \ldots, f_q \) of \( I_d \) so \( \text{in}(f_1), \ldots, \text{in}(f_q) \) is a monomial basis for \( \text{in}(I)_d \). Then the vector \( \wedge f_i = f_1 \wedge \cdots \wedge f_q \in \wedge^q S_d \) determines the subspace \( I_d \subset S_d \).

A basis for \( \wedge^q S_d \) is given by elements of the form \( x^{A_1} \wedge \cdots \wedge x^{A_q} \), where \( x^{A_1}, \ldots, x^{A_q} \) are distinct monomials in \( S_d \). Define the weight of a basis element by \( w \cdot (x^{A_1} \wedge \cdots \wedge x^{A_q}) = \sum w \cdot A_j \). In terms of this basis, \( \wedge \text{in}(f_i) = \wedge \text{in}(f_j) \wedge \cdots \wedge \text{in}(f_q) \) is the unique component of \( \wedge f_i \) of highest weight, and \( \wedge \text{in}(f_i) \) determines the subspace \( \text{in}(I)_d \subset S_d \). Define the weight of \( \text{in}(I)_d \) to be the weight of \( \wedge \text{in}(f_i) \); this definition is independent of the choice of the \( f_i \).

If we choose \( g \in \text{GL}(n) \) to have indeterminate coefficients \( a_{ij} \), so \( g \cdot x_i = \sum a_{ij} x_j \), then \( \wedge (g \cdot f_i) \) has a unique highest weight component of the form \( h_d(a_{ij}) (x^{A_1} \wedge \cdots \wedge x^{A_q}) \). In other words, \( h_d(a_{ij}) \) is the \( (A_1, \ldots, A_q) \)-minor of \( g \cdot I_d \subset S_d \), which is the unique highest weight nonzero \( q \times q \) minor of \( g \cdot I_d \). Let \( J \subset S \) denote the monomial ideal \( J \subset S \) generated by each such set of monomials \( x^{A_1}, \ldots, x^{A_q} \), as \( d \) varies. Let \( m \) be the highest degree of a minimal generator of \( J \). For any choice of the coefficients \( a_{ij} \) of \( g \) so \( h_d(a_{ij}) \neq 0 \) for each \( d \) up to \( m \), we have \( J_d = \text{in}(g \cdot I_d), \ d \leq m \). Since \( \dim(J_d) = \dim(\text{in}(g \cdot I_d)) \) for all \( d \) [Mac27], it follows that \( J = \text{in}(g \cdot I) \). This defines a Zariski open set \( U \subset \text{GL}(n) \) so \( \text{in}(g \cdot I) \) is independent of the choice of \( g \in U \), and \( \text{in}(g \cdot I)_d \) has maximal possible weight for each degree \( d \).

By translation, we can assume that the identity transformation \( 1 \in U \). Thus, \( \text{in}(I)_d \) has maximal possible weight for each degree \( d \). We show that for each degree \( d \), the subspace \( \text{in}(I)_d \subset S_d \) is invariant under the action of \( B(n) \). Since \( \text{in}(I_d) \) is a monomial ideal, it is automatically invariant under the action of the maximal torus \( T(n) \). We need only check that for \( g = g_{ij}(a) \in B(n), \ j < i, \) where \( g \cdot x_i = x_i + ax_j \) and \( g \cdot x_t = x_t \) for \( t \neq i \), that \( g \cdot \text{in}(I_d) = \text{in}(I_d) \). To check this, it suffices to check that for \( x^{A_i} \in \text{in}(I_d) \), any nonzero term of \( g \cdot x^{A_i} \) belongs to \( \text{in}(I_d) \). The description of the nonzero terms of \( g \cdot x^{A_i} \) is slightly more
complicated in characteristic $p$ than in characteristic zero. The proof given here does not need this description, so we omit it.

Suppose that $ca^mx^B$ is a nonzero term of $g \cdot x^{A_1}$, with $x^B \neq x^{A_1}$, where $m = (\omega \cdot B - \omega \cdot A_1)/(\omega - \omega)$. Observe that $x^B$ is of higher weight than $x^{A_1}$, and that the exponent $m$ measures this difference in weight. Let $x^{A_1}, \ldots, x^{A_s}$ be a monomial basis for $\text{in}(I_d)$. We must show that $x^B = x^{A_i}$ for some $i > 1$.

Suppose otherwise. Observe that $x^B \lvert x^{A_1} \lvert x^{A_2} \lvert \cdots \lvert x^{A_s} \cdot g \cdot (x^{A_1} \lvert x^{A_2} \lvert x^{A_3} \lvert \cdots \lvert x^{A_s})$ contributes $ca^m$ to the coefficient of the term $h(a)x^B \lvert x^{A_2} \lvert x^{A_3} \lvert \cdots \lvert x^{A_s}$ of $\lvert g \cdot f_i \lvert$. Since $x^{A_1} \lvert \cdots \lvert x^{A_s}$ is the unique term of highest weight of $\lvert f_i \lvert$, the other terms of $\lvert f_i \lvert$ can only contribute higher powers of $a$ to $h(a)$ under the action of $g$, since they all have a greater difference of weights to reach over. Thus $h(a)$ is not identically zero, and there exist $g \in \text{GL}(n)$ so $\text{in}(g \cdot I_d)$ has higher weight than $\text{in}(I_d)$. This is a contradiction, proving the proposition.

**Corollary 2.** Under the hypothesis of Proposition 1, the associated primes of $\text{in}(g \cdot I)$ are all of the form $(x_1, \ldots, x_i)$.

**Proof.** Since $\text{in}(g \cdot I)$ is invariant under the action of $B(n)$, its associated primes are also invariant under the action of $B(n)$. The ideals of the form $(x_1, \ldots, x_i)$ are the only prime ideals of $S$ which are invariant under the action of $B(n)$.

Recall that two ideals $I, J \subset S$ define the same subscheme of $\mathbb{P}^{n-1}$ if $I_d = J_d$ for all degrees $d \gg 0$; the saturation $I^{\text{sat}}$ of $I$ is the largest ideal in this equivalence class. The vertex of the affine cone in $\mathbb{A}^n$ defined by $I$ is an associated prime of $I$ if and only if $I$ is not saturated.

We recall a definition and a theorem from [BaSt87].

**Definition 3** (Definition (1.5), [BaSt87]). Call $h \in S$ generic for $I$, if $h$ is not a zero-divisor on $S/I^{\text{sat}}$. If $\dim(S/I) = 0$, interpret this to mean every $h \in S$ is generic for $I$.

For $j > 0$, define $U_j(I)$ to be the subset

$$\{(h_1, \ldots, h_j) \in S_\ell \mid h_i \text{ is generic for } (I, h_1, \ldots, h_{i-1}), 1 \leq i \leq j\}$$

of $S_\ell$.

**Theorem 4** (Theorem (2.4), [BaSt87]). Let $I \subset S$ be a homogeneous ideal, let $>$ be the reverse lexicographic order, and let $r = \dim(S/I)$.

(a) $(x_n, \ldots, x_{n-r+1}) \in U_r(I) \iff (x_n, \ldots, x_{n-r+1}) \in U_r(\text{in}(I))$.

(b) If $(x_n, \ldots, x_{n-r+1}) \in U_r(I)$, $I$ and $\text{in}(I)$ have the same regularity.

**Lemma 5.** Let $>$ be a strict order satisfying $x_1 > \cdots > x_n$, and let $I \subset S$ be a homogeneous ideal, with $r = \dim(S/I)$. Let $U \subset \text{GL}(n)$ be the open set given in Proposition 1 so that for any $g \in U$, $\text{in}(g \cdot I)$ is invariant under the action of the Borel subgroup $B(n)$. Then for any $g \in U$, $(x_n, \ldots, x_{n-r+1}) \in U_r(\text{in}(g \cdot I))$. 

Proof. For each $i$ in the range $1 < i < r$, it follows from Corollary 2 that $J_i = (\text{id}(g \cdot I), x_n, \ldots, x_{n-i+1})$ has associated primes of the form $p_i = (x_1, x_2, \ldots, x_n)$ with $t < n - i$. Since $x_{n-i}$ can only be contained in $p_{n-i}$, which is the associated prime corresponding to any nonsaturation of $J_i$, $x_{n-i}$ is not a zero-divisor on $S/J_i$. Thus, by definition 3, $(x_n, \ldots, x_{n-r+1}) \in U_i(\text{id}(g \cdot I))$. \\

Theorem 6. Let $>_1$ be a nonstrict order satisfying $x_1 >_1 \cdots >_1 x_n$, and let $>_\text{rlex}$ be the reverse lexicographic order. Define $>_3$ to be the strict order which is the refinement of $>_1$ by $>_\text{rlex}$; we have $x_1 >_3 \cdots >_3 x_n$. Let $\text{id}_{>_1}, \text{id}_{>_\text{rlex}}, \text{id}_{>_3}$ correspond to $>_1, >_\text{rlex}, >_3$. There exists a Zariski open subset $U \subset \text{GL}(n)$ such that for any $g \in U$, $\text{id}_{>_1}(g \cdot I)$ and $\text{id}_{>_3}(g \cdot I)$ have the same regularity. \\

Proof. We first show that $\text{id}_{>_3}(g \cdot I) = \text{id}_{>_\text{rlex}}(\text{id}_{>_1}(g \cdot I))$. Since the order $>_3$ is the refinement of $>_1$ by $>_\text{rlex}$, $\text{id}_{>_3}(f) = \text{id}_{>_\text{rlex}}(\text{id}_{>_1}(f))$ for any $f \in g \cdot I$. The statement follows immediately.

Let $U \subset \text{GL}(n)$ be the open set given in Proposition 1 so that for any $g \in U$, $\text{id}_{>_3}(g \cdot I)$ is invariant under the action of the Borel subgroup $B(n)$. By Lemma 5, $(x_n, \ldots, x_{n-r+1}) \in U_i(\text{id}_{>_1}(g \cdot I))$ for any $g \in U$. Since $\text{id}_{>_3}(g \cdot I) = \text{id}_{>_\text{rlex}}(\text{id}_{>_1}(g \cdot I)), (x_n, \ldots, x_{n-r+1}) \in U_i(\text{id}_{>_\text{rlex}}(\text{id}_{>_1}(g \cdot I)))$. Since $>_\text{rlex}$ is the reverse lexicographic order, by Theorem 4 $(x_n, \ldots, x_{n-r+1}) \in U_i(\text{id}_{>_1}(g \cdot I))$, and $\text{id}_{>_1}(g \cdot I)$ and $\text{id}_{>_3}(g \cdot I) = \text{id}_{>_\text{rlex}}(\text{id}_{>_1}(g \cdot I))$ have the same regularity. \\

Recall the nonstrict order $>_1$ defined previously by $x^A >_1 x^B$ if $\text{deg}_{i-1}(x^A) > \text{deg}_{i-1}(x^B)$. Given an ideal $I$, call $\text{id}_{>_1}(I)$ the flat projection of $I$ to $k[x_j, \ldots, x_n]$. The flat projection $\text{id}_{>_1}(I)$ differs from the usual projection $I \cap k[x_j, \ldots, x_n]$ in that $\text{id}_{>_1}(I)$ and $I$ have the same Hilbert function: $\text{dim}(\text{id}_{>_1}(I_d)) = \text{dim}(I_d)$ for all $d$. For example, if $I$ defines the twisted cubic curve in $\mathbb{P}^3$, then the usual projection of $I$ to $\mathbb{P}^2$ defines a plane cubic curve, but the flat projection of $I$ to $\mathbb{P}^2$ defines a plane cubic with an embedded point sticking out of the plane.

It can be shown that in generic coordinates, the flat projection of a degree $d$ curve in $\mathbb{P}^n$ to $\mathbb{P}^2$ has regularity exactly $d$. It seems reasonable to conjecture the corresponding statement for projections of higher dimensional varieties to hypersurfaces. Since this would imply that the varieties themselves are of regularity $\leq d$, which is a major open question, it is unlikely that this conjecture will be easily settled.

If one understood the regularity of flat projections in general, Theorem 6 could be applied to understanding the complexity of computing projections. One can compute projections by refining $>_1$ by the reverse lexicographic order; Theorem 6 implies that in generic coordinates, the standard basis computed is bounded in degree by the regularity of the flat projection. Applying the results of [BaSt87], in characteristic zero we see that the highest degree of a standard basis element is exactly given by the regularity of the flat projection.
In the hypothesis of Theorem 6, it is crucial that the orders induced on the variables \(x_1, \ldots, x_n\) by \(>_{1}\) and \(>_{\text{lex}}\) be compatible. We give an example which illustrates this point.

We will construct a convex polytope whose vertices correspond to the possible \(\text{in}(I)\) for a given ideal \(I\), as the choice of an order varies. Following the correspondence between possible orders and weight vectors that was used in proposition 1, orders will correspond to directions in the space in which this polytope sits; the most extreme vertex for a given direction corresponds to the \(\text{in}(I)\) chosen by this order.

In fact, this polytope can be viewed as a model for the closure of the orbit of \(I\) in the Hilbert scheme under the action of the maximal torus \(T(n)\); see [BaMo87].

The concept of weight used in the proof of Proposition 1 can be generalized to a multigrading, by using weight vectors \((1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\).

Then \(x^{A_1} \land \cdots \land x^{A_q}\) has weight \(A = A_1 + \cdots + A_q\), where the coordinates of \(A\) sum to \(qd\). Given any weight vector \(w\) representing an order in degree \(d\), \(w \cdot A\) computes the order as before.

Let \(I \subset k[x, y, z]\) be the ideal of three points in general position in \(\mathbb{P}^2\). \(I\) has regularity 2. There are two possible Borel fixed ideals which can occur as \(\text{in}(I)\) for various orders in generic coordinates: \(J_1 = (x, y^3)\) and \(J_2 = (x^2, xy, y^2)\). \(J_1\) has regularity 3, and \(J_2\) has regularity 2. As we vary the choice of an order, six permutations of \(J_1\) and three permutations of \(J_2\) can occur as \(\text{in}(I)\) in generic coordinates, and these are the only possibilities.

In degree 3, \(J_1\) and \(J_2\) can be pictured as shown in diagram 1:

\[
\begin{align*}
\text{Diagram 1.} \\
\end{align*}
\]

The weight of \(J_1\) in degree 3 is therefore the exponent of \(x^{10}y^{10}z^{10}/z^3y^2z^2\), or \((10,7,4)\). Note that \(10 + 7 + 4 = 21 = 3 \cdot 7\). The weight of \(J_2\) in degree 3 is similarly the exponent of \(x^{10}y^{10}z^{10}/z^3y^2z^2\), or \((9,9,3)\).

We can plot the nonzero components of \(\land f_i\) for this example, according to their weights \(A\). The weight vector \(w\) determines a direction in this graph; the unique component of \(\land f_i\) furthest in this direction determines the ideal \(\text{in}(I)\) for the order given by \(w\). Since this holds for any direction \(w\) in the graph which induces a strict order, we see that the vertices of the convex hull of this diagram correspond to the possible \(\text{in}(I)\) as the order varies. Thus the vertices of the
convex hull for this example are (10, 7, 4), (9, 9, 3), and their various permutations, as shown in diagram 2.

The advantage of obtaining diagram 2 is that from it we can immediately determine what \( \in(I) \) will occur for any choice of order \( w \): \( \in(I) \) is given by the vertex of the hull which is extreme with respect to the vector \( w \).

(4, 1, 0) induces the lexicographic order in degree 3; (4, 3, 0) induces the reverse lexicographic order in degree 3. (4, 2, 0) induces a nonstrict order in degree 3, since \( xyz \) and \( y^3 \) have the same weight with respect to (4, 2, 0). Corresponding to this, an entire edge of the diagram is extreme with respect to (4, 2, 0).

In diagram 2, all permutations of \( J_1 \) and \( J_2 \) are plotted in barycentric coordinates. Projections of the three vectors (4, 1, 0), (4, 3, 0), and (4, 2, 0) are shown based at the center (7, 7, 7). The edge which is extreme for the order (4, 2, 0) is bold.

\( \in(I) \) with respect to the order (4, 2, 0) and generic coordinates is not a monomial ideal, and has regularity 2. If the weight vector (4, 2, 0) is perturbed even slightly, one or the other endpoint of this edge is chosen as \( \in(I) \). In this way, we can observe the effect of refining the order (4, 2, 0) by some other order \( w \): The projection of \( w \) to this edge points one way or the other along this edge, determining which \( \in(I) \) is chosen by the refinement. We want a systematic method of refining orders which will in this case choose the endpoint of regularity 2.

Refining by the lexicographic order (4, 1, 0), we obtain \( \in(I) = J_1 \) of regularity 3. Refining by the reverse lexicographic order (4, 3, 0), we obtain \( \in(I) = J_2 \) of regularity 2. If we consider orders so \( z > x > y \), however, we obtain \( \in(I) = J_1 \) by refining by either the lexicographic order (1, 0, 4) or the reverse lexicographic order (3, 0, 4). Similarly, we obtain \( \in(I) = J_2 \) by refining by either the lexicographic order (0, 4, 1) or the reverse lexicographic order (0, 4, 3). Thus the possible refinements of (4, 2, 0) are divided into two classes by this edge. One observes that by refining by the particular choice of reverse lexicographic order (4, 3, 0) which induces the same order \( x > y > z \) on the variables as is induced by the
nonstrict order (4, 2, 0), we get the desirable outcome of regularity 2. This example guided us to formulating and proving theorem 6 as stated.

REFERENCES


[Bay85] ________, An introduction to the division algorithm, informal notes.


