9.4.1

(a) If $f$ coincides with its inverse over an interval then the graph of $f$ is symmetric about $y=x$, over this interval.

(b) i) $\frac{1}{x}$ is its own inverse where it is defined on $(-\infty, 0) \cup (0, \infty)$.

\[
y = \frac{1}{x}
\]

So $x = \frac{1}{y}$, switching $y$ for $x$ gives the original function. Also, $f^2 = I$, hence $f$ is its own inverse.

ii) $y = \sqrt{1 - x^2}$ for $x \in [-1, 1]$

\[
y^2 = 1 - x^2
\]

So $x^2 = 1 - y^2$

and $x = \sqrt{1 - y^2}$ provided $y \in [-1, 1]$

Switching $x, y$ gives $y = \sqrt{1 - x^2}$, the original function.

Specifically, for $x \in [0, 1]$, we have

\[
f^2(x) = \sqrt{1 - (\sqrt{1 - x^2})^2} = \sqrt{1 - (1 - x^2)} = \sqrt{x^2} = x
\]

So $f = f^{-1}$ on $[0, 1]$

c) \( f(x) = \frac{ax+b}{cx+d} = y \)

then \( ax - cy = dy - b \)

so \( (a - cy)x = dy - b \)

and \( x = \frac{dy - b}{-cy + a} \)

wherever this is defined.

Set

\[
\frac{ax+b}{cx+d} = \frac{dx-b}{-cx+a}
\]

so

\[-ca x^2 + (a^2 - bc)x + bn = dc x^2 + (d^2 - cb)x - db.\]

If \( c = 0 \) then \( a \neq 0, d \neq 0 \) must be true for \( f = f^{-1} \) (the graph here is a non-vertical, non-horizontal line). Otherwise

\[ a = -d \]

must be true, and \( c, b \in \mathbb{R} \) can be any reals.
10.1.6

(a) \[ f(x) = \frac{1}{2 + 5 \sin x} \]

\[ I = [0, \pi] \quad \text{then} \quad f(I) = [\frac{1}{3}, \frac{1}{2}] \]

So \( \min f = \inf f = \frac{1}{3} \) and \( \max f = \sup f = \frac{1}{2} \)

\[ I = (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \text{then} \quad f(I) = (\frac{1}{3}, 1) \]

So \( \inf f = \frac{1}{3} \), \( \sup f = 1 \), \( \min f, \max f \) D.N.E.

\[ I = (-\infty, \infty) \quad \text{then} \quad f(I) = [\frac{1}{3}, 1] \]

So \( \min f = \inf f = \frac{1}{3} \) and \( \max f = \sup f = 1 \).

(b) \[ f(x) = xe^{-x} \quad f'(x) = (1-x)e^{-x} = 0 \quad \text{only at} \quad x = 1, \quad \text{where} \quad f'' \quad \text{is} \]

\[ (x-2)e^{-x} \bigg|_{x=1} = -e^{-1} < 0 \]

For \( I = [0, \infty) \)

\[ f(I) = [0, e] \]

So \( \min f = \inf f = 0 \) and \( \max f = \sup f = e \)

For \( I = (0, \infty) \)

\[ f(I) = (0, e] \]

So \( \inf f = 0 \), \( \max f = \sup f = e \) and \( \min f \) D.N.E.

\[ I = (\infty, \infty) \quad f(I) = (-\infty, e] \]

So \( \max f = \sup f = e \) and \( \inf f, \min f \) D.N.E.
\[ \frac{x^4}{1 + x^6} \leq x^4 \text{ on } [0, 1] \]

so \[ \int_0^1 \frac{x^4}{1 + x^6} \, dx \leq \int_0^1 x^4 \, dx = \frac{1}{5} x^5 \bigg|_0^1 = \frac{1}{5}. \]

Indeed a lower bound is given by
\[ \int_0^1 \frac{x^4}{2} \, dx = \frac{1}{10}, \text{ since } \frac{x^4}{1 + x^6} \geq \frac{1}{2} x^4 \text{ on } [0, 1] \]

In fact \( \frac{1}{5} \) is much closer to the actual integral.

10.1.8

(a) \( f \) is defined on \( I \) if it's defined at each point on \( I \), hence pointwise.

(b) \( \sup f(I) = 2 \) if \( f(I) \) is bounded above by 2 and \( \forall \varepsilon > 0 \), some \( x \in I \) has \( 2 - f(x) < \varepsilon \). Where might this happen? We don't know without looking globally.

(c) \( f(x) < \frac{1}{x} \) on \( (0, 1) \) if \( f(c) < \frac{1}{c} \) for each point \( c \in (0, 1) \), hence pointwise.

(d) \( f(x) \) attains a minimum on \( I \). Global. See (b) or write-up on pointwise/local/global.
10.4.1

(e) \( f \) is constant on \( I \) if \( f \) is constant on every sufficiently small neighborhood of each \( x \in I \). (This is because intervals are connected). Hence local.

(f) \( f(x) \) is periodic with period 2 if the condition \( f(x+2) = f(x) \) holds at each point \( x \in I \), hence pointwise.

(g) \( f(x) \) is periodic if it has some period.

How can we know what that period is a priori? We cannot. Must look globally.

10.4.2

(a) \( f(x) \) can be expressed as \( \Sigma a_n x^n \) around 0 is local at 0. (Hint: "neighborhood at 0" = "local at 0")

(b) \( f(x) = \Sigma a_n x^n \) on \((-1,1)\), (for some unknown). It's both necessary and sufficient that this holds in every neighborhood \( U \) such that \( x \in U \subseteq (-1,1) \), for every \( x \in (-1,1) \). Hence local on \( I \).

(It's not enough to check \( f(x) = \Sigma a_n x^n \) at each pt, since we have no idea what an is.)
10.4.2

(c) \[ f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \text{ in } (-1, 1) \] if \[ f(c) = \sum_{n=1}^{\infty} \frac{c^n}{n!} \] at each point \( c \in (-1, 1) \), hence \text{pointwise}.

(d) "\[ \ldots \] Convergent in some neighborhood of \( x_0 \)." is clearly \text{local on I}. It would be "local at \( x_0 \)" except that \( x_0 \) ranges over the whole interval, which is the meaning of "local on I."