Mathematics 414, Spring 2008

Solutions to assignment 1

Problem 8.1.5: Show that $f' = \lambda f$ for a real constant λ has only $ce^{\lambda x}$ as solutions.

SOLUTION. Multiplying the relation with $e^{-\lambda x}$ we get $f'(x)e^{-\lambda x} - \lambda f(x)e^{-\lambda x} = (f(x)e^{-\lambda x})' = 0$. Thus $f(x)e^{-\lambda x} = c$, for some real constant c, and so $f(x) = ce^{\lambda x}$. $\square$

Problem 8.1.7: For which values of a is the improper integral

$$\int_2^\infty \frac{1}{x|\log x|^a} dx$$

finite?

SOLUTION. We make the change of variable $y = \log x$ and the integral becomes:

$$\int_{\log 2}^\infty y^{-a} dy = \lim_{y\to\infty} \frac{y^{1-a}}{1-a} - \frac{(\log 2)^{1-a}}{1-a}, \text{ if } a \neq 1.$$

If $a = 1$, then the integral is $\lim\limits_{y\to\infty} \log y - \log(\log 2) = \infty$. Evaluating the limit above we get that the integral is finite iff $a > 1$.

$\square$

Problem 8.1.11: Show that there is a $\mathcal{C}^\infty$ function on (a, b) having prescribed derivatives of all orders on any sequence of distinct points $x_1, x_2, \ldots$ with no limit point in (a, b).

SOLUTION. Since the sequence $(x_n)_{n\geq 1}$ does not have a limit point in (a, b), for every $i \geq 1$ there exists a neighborhood $U_i = (x_i - \epsilon_i, x_i + \epsilon_i)$ of x_i, for some small enough $\epsilon_i > 0$, such that $x_j \notin U_i$ for all $j \neq i$. Otherwise, we could find infinitely many $x_j's$ arbitrarily close to x_i, and then x_i would be a limit point; contradiction.

Using *Theorem* 8.1.7 *(Borel)*, there is a $\mathcal{C}^\infty$ function f_i vanishing outside U_i and having prescribed derivatives of all orders at x_i. Let then $f : (a, b) \mapsto \mathbb{R}$, $f(x) = f_i(x)$ for $x \in U_i$, for all $i \geq 1$, and $f(x) = 0$ otherwise. This is well defined since U_i are all disjoint with $\bigcup\limits_{i=1}^\infty U_i \subset (a, b)$, and $f^{(k)}(x_i) = f_i^{(k)}(x_i)$, for all $i \geq 1$ and $k \geq 0$. $\square$

Problem 8.2.4: Verify the identity $\sin^2\theta + \cos^2\theta = 1$ by rearranging the power series expansions.

SOLUTION. Recall that $\sin\theta = \sum\limits_{k=0}^\infty (-1)^k \frac{\theta^{2k+1}}{(2k+1)!}$ and $\cos\theta = \sum\limits_{k=0}^\infty (-1)^k \frac{\theta^{2k}}{(2k)!}$. Then $\sin^2\theta = \sum\limits_{k=0}^\infty \sum\limits_{i+j=k} (-1)^{i+j} \frac{\theta^{2(i+j+1)}}{(2i+1)!(2j+1)!} = \sum\limits_{k=0}^\infty \sum\limits_{i=0}^k (-1)^k \frac{\theta^{2(k+1)}}{(2i+1)!(2k-2i+1)!} = \sum\limits_{k=0}^\infty (-1)^k \frac{\theta^{2k+2}}{(2k+2)!} \sum\limits_{i=0}^k \begin{pmatrix} 2k+2 \\ 2i+1 \end{pmatrix}$.

From the binomial theorem we have that $\sum_{i=0}^{n} \binom{n}{2i+1} = \sum_{i=0}^{n} \binom{n}{2i} = 2^{n-1}$, for all $n \geq 1$. Thus $\sin^2 \theta = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1}\theta^{2k+2}}{(2k+2)!}$.

In a similar way, $\cos^2 \theta = \sum_{k=0}^{\infty} \sum_{i+j=k} (-1)^{i+j} \frac{\theta^{2(i+j)}}{(2i)!(2j)!} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^k \frac{\theta^{2k}}{(2i)!(2k-2i)!} =$
$= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} \sum_{i=0}^{k} \binom{2k}{2i} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1}\theta^{2k}}{(2k)!} = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1}\theta^{2k+2}}{(2k+2)!}$. From this, it follows that $\cos^2 \theta = 1 - \sin^2 \theta$. □

Problem 8.2.7: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be convergent in $|z| < R$, and suppose the coefficients a_n are all real. Show that $f(\overline{z}) = \overline{f(z)}$.

Solution. Recall that $\overline{z}^n = \overline{z^n}$ and $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ for any complex numbers z, z_1, z_2. Note that $|z| = |\overline{z}|$ so $f(z)$ is convergent in $|z| < R$ iff so is $f(\overline{z})$. We have $f(\overline{z}) = \sum_{n=0}^{\infty} a_n \overline{z}^n = \sum_{n=0}^{\infty} a_n \overline{z^n} = \sum_{n=0}^{\infty} \overline{a_n z^n} = \overline{\sum_{n=0}^{\infty} a_n z^n} = \overline{f(z)}$. The middle equality follows from the fact that all a_n are real, so that $a_n = \overline{a_n}$. □

Problem 8.2.11: Show $\exp(z)$ assumes every complex value except zero and that $\exp(z_1) = \exp(z_2)$ if and only if $z_1 - z_2 = 2\pi k i$ for some integer k.

Solution. Let $z \in \mathbb{C}$, $z \neq 0$. Then, in polar coordinates $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, for some real $r = |z| > 0$ and $\theta \in [0, 2\pi)$. Since $r > 0$ we can write $r = e^{\log r}$ and so $z = e^{\log r + i\theta}$. This shows that e^z assumes every nonzero complex value.

Let $z_1 = x_1 + iy_1$ and $z_1 = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$, such that $e^{z_1} = e^{z_2}$. Then $e^{z_1} = e^{x_1}(\cos(y_1) + i\sin(y_1))$ and $e^{z_2} = e^{x_2}(\cos(y_2) + i\sin(y_2))$, and also $|e^{z_1}| = e^{x_1} = |e^{z_2}| = e^{x_2} \Rightarrow x_1 = x_2$. Thus $\cos(y_1) + i\sin(y_1) = \cos(y_2) + i\sin(y_2) \Rightarrow \cos(y_1) = \cos(y_2)$, so there exists an integer k such that $y_1 = y_2 + 2\pi k$. It follows that $z_1 - z_2 = 2\pi k i$, as claimed. □