Mathematics 414, Spring 2008

Solutions to assignment 2

Problem 9.1.4: Prove that $||x||_{\sup} = \lim_{p\to\infty} ||x||_p$ on $\mathbb{R}^n$.

SOLUTION. Let $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$. Recall that $||x||_{\sup} = \max_{1\le j\le n}(|x_j|)$ and that $||x||_p = (\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}}$. Then $||x||_p \ge (|x_j|^p)^{\frac{1}{p}} = |x_j|$ for all $1 \le j \le n$. Taking the sup over all such j we have that $||x||_p \ge ||x||_{\sup}$ and passing to the limit $\lim_{p\to\infty} ||x||_p \ge ||x||_{\sup}$. To see the reverse inequality note that $|x_j| \le ||x||_{\sup}$ for all $j = \overline{1, n}$. Hence $(\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}} \le n^{\frac{1}{p}} ||x||_{\sup} \Rightarrow$ $||x||_p \le n^{\frac{1}{p}} ||x||_{\sup}$. Letting $p \to \infty$ we get $\lim_{p\to\infty} ||x||_p \le ||x||_{\sup}$ and in fact equality. □

Problem 9.1.9: Prove that if $||x||$ is any norm on $\mathbb{R}^n$, then there exists a positive constant M such that $||x|| \le M|x|$ for all x in $\mathbb{R}^n$ where $|x|$ is the Euclidean norm. (**Hint:** $M = (\sum_{j=1}^{n} ||e_j||^2)^{1/2}$ will do.)

SOLUTION. Let $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. The Cauchy-Schwarz inequality can be rewritten as $\sum_{j=1}^{n} x_j y_j \le (\sum_{j=1}^{n} x_j^2)^{\frac{1}{2}} (\sum_{j=1}^{n} y_j^2)^{\frac{1}{2}}$. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Then $x = \sum_{j=1}^{n} x_j e_j$ and using the norm properties we get $||x|| \le \sum_{j=1}^{n} |x_j| ||e_j|| \le (\sum_{j=1}^{n} |x_j|^2)^{\frac{1}{2}} (\sum_{j=1}^{n} ||e_j||^2)^{\frac{1}{2}} = |x|M$, where $|x|$ is the Euclidean norm of x and $M = (\sum_{j=1}^{n} ||e_j||^2)^{1/2}$ is a positive constant.

□

Problem 9.1.10: Prove that the norm $||x||_1$ on $\mathbb{R}^n$ for $n > 1$ is not associated with an inner product. (**Hint:** violate the parallelogram law.) Do the same for $||x||_{\sup}$.

SOLUTION. Let $x \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. Recall that $||x||_1 = \sum_{i=1}^{n} |x_i|$. If this norm is associated with an inner product, then it satisfies the parallelogram law

$$||x+y||_1^2 + ||x-y||_1^2 = 2(||x||_1^2 + ||y||_1^2),\ \forall x, y \in \mathbb{R}^n.$$

Let $x = (1, 0, \ldots, 0)$ and $y = (0, 2, 0, \ldots, 0)$. Then $||x||_1 = 1$, $||y||_1 = 2$, $||x+y||_1 = 3$ and $||x-y||_1 = 3$. In this case, the parallelogram law becomes $9 + 9 = 2(1 + 4)$; contradiction.

Similarly, $||x||_{\sup} = 1$, $||y||_{\sup} = 2$, $||x+y||_{\sup} = 2$ and $||x-y||_{\sup} = 2$. In this case, the parallelogram law becomes $4 + 4 = 2(1 + 4)$; contradiction. Thus in both cases the norm is not associated with an inner product. □

Problem 9.1.15: Verify that $d(x,y) = \frac{|x-y|}{1+|x-y|}$ defines a metric on $\mathbb{R}^n$, but this metric is not induced by any norm. (**Hint:** homogeneity fails.)

SOLUTION. Note that $d(x,y) = d(y,x)$, $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}^n$, with $d(x,y) = 0$ iff $x = y$. To show that d is a metric on $\mathbb{R}^n$, it is sufficient to prove the triangle inequality. Let $x, y, z \in \mathbb{R}^n$ and set $\eta(x,y) = |x-y|$. Clearly η is a metric on $\mathbb{R}^n$ and $d(x,y) = \frac{\eta(x,y)}{1+\eta(x,y)}$. Then

$$d(x,z) \leq d(x,y) + d(y,z) \Leftrightarrow \frac{\eta(x,z)}{1+\eta(x,z)} \leq \frac{\eta(x,y)}{1+\eta(x,y)} + \frac{\eta(y,z)}{1+\eta(y,z)}.$$

Set for simplicity $a = \eta(x,z), b = \eta(x,y)$ and $c = \eta(y,z)$. Then $a, b, c \geq 0$ and $a \leq b+c$, since η is a metric. The inequality above reduces to

$$\frac{b}{1+b} + \frac{c}{1+c} \geq \frac{a}{1+a} \Leftrightarrow b(1+a)(1+c) + c(1+a)(1+b) - a(1+b)(1+c) \geq 0.$$

Rearranging the terms, this is equivalent to

$$(b+c-a) + 2bc + abc \geq 0,$$

which is always true as $a, b, c \geq 0$ and $b + c \geq a$. This shows that d is indeed a metric.

Suppose d is induced by the norm $||\cdot||$. Then $d(x,y) = ||x-y||$ and $d(\alpha x, \alpha y) = |\alpha| d(x,y)$, for all real α. Let for simplicity $\alpha = 2$, $x = 1$, and $y = 0 \Rightarrow d(2,0) = 2d(1,0) \Leftrightarrow \frac{2}{3} = 2\frac{1}{2} = 1$; contradiction. Hence d is not induced by any norm. $\square$

Problem 9.2.3: Prove that the metric $d(f,g) = \int_a^b |f(x) - g(x)| dx$ on $\mathcal{C}([a,b])$ is not complete. (**Hint:** consider the example of a sequence of continuous functions converging pointwise to a discontinuous function.)

SOLUTION. Let $a < b$ and define $f_n : [a,b] \mapsto \mathbb{R}$

$$f_n(x) = \begin{cases} n(x-a) & , \quad x \in [a, a+\frac{1}{n}) \\ 1 & , \quad x \in [a+\frac{1}{n}, b]. \end{cases}$$

$(f_n)_{n\geq 1}$ is a sequence of continuous functions that converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0 & , \quad x = a \\ 1 & , \quad x \in (a,b]. \end{cases}$$

We claim that $(f_n)_{n\geq 1}$ is Cauchy with respect to the given metric. Indeed, for $m > n$

$$d(f_n, f_m) = (m-n) \int_a^{a+\frac{1}{m}} (x-a)dx + \int_{a+\frac{1}{m}}^{a+\frac{1}{n}} (1 - n(x-a))dx = \frac{1}{2n} - \frac{1}{2m}.$$

Since $(\frac{1}{2n})_{n\geq 1}$ is Cauchy $\Rightarrow$ $(f_n)_{n\geq 1}$ is Cauchy too. Suppose $\mathcal{C}([a,b])$ is complete. Then $f_n \to f$ pointwise and $f \in \mathcal{C}([a,b])$; contradiction, since f is discontinuous at $x = 0$. Hence $\mathcal{C}([a,b])$ is not complete in the given metric. $\square$

Problem 9.2.4: Prove that the space of bounded sequences with metric $d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n|$ is complete, and the same is true on the subspace of sequences converging to zero.

SOLUTION. Let $(y_m)_{m\geq 1}$ be a Cauchy sequence of bounded sequences, i.e. $y_m = (x_n^{(m)})_{n\geq 1}$ are bounded sequences for all $m \geq 1$. Let $\epsilon > 0$. By definition, there exists a positive integer N such that $\forall m, k \geq N$,

$$d(y_m, y_k) = \sup_n |x_n^{(m)} - x_n^{(k)}| < \epsilon.$$

For all $n \geq 1$ this gives

$$|x_n^{(m)} - x_n^{(k)}| \leq \sup_n |x_n^{(m)} - x_n^{(k)}| < \epsilon, \ \forall m, k \geq N.$$

Thus the sequence $(x_n^{(m)})_{m\geq 1}$ is Cauchy in $\mathbb{R}$, hence convergent since $\mathbb{R}$ is complete. Let its limit be z_n. We obtain a new sequence $(z_n)_{n\geq 1}$ and we prove that this is the limit of $(y_m)_{m\geq 1}$. Letting $k \to \infty$ in the equation above we get that for all $n \geq 1$

$$|x_n^{(m)} - z_n| \leq \sup_n |x_n^{(m)} - z_n| < \epsilon, \ \forall m \geq N.$$

Then for some $m \geq N$

$$|z_n| \leq |x_n^{(m)} - z_n| + |x_n^{(m)}| \leq \sup_n |x_n^{(m)} - z_n| + \sup_n |x_n^{(m)}| < \epsilon + \sup_n |x_n^{(m)}|, \ \forall n \geq 1.$$

However, the sequence $(x_n^{(m)})_{n\geq 1}$ is bounded, i.e. $\sup_n |x_n^{(m)}| < \infty$, so $(z_n)_{n\geq 1}$ is bounded. To see that $z = (z_n)_{n\geq 1}$ is the limit of $(y_m)_{m\geq 1}$ it is sufficient to note that

$$d(y_m, z) = \sup_n |x_n^{(m)} - z_n| < \epsilon, \ \forall m \geq N.$$

Therefore $(y_m)_{m\geq 1}$ is convergent and so the space of all bounded sequences is complete in the metric d. The same is true for the subspace of sequences converging to zero. We have to modify the proof above for sequences $(x_n^{(m)})_{n\geq 1}$ converging to 0 as $n \to \infty$. It is sufficient to prove that $z_n \to 0$ as $n \to 0$.

Fix $m > N$. Since $\lim_{n\to\infty} x_n^{(m)} = 0$ then, using the same ϵ as before, there is N_1 such that

$$|x_n^{(m)}| < \epsilon, \ \forall n \geq N_1.$$

We then have

$$|z_n| \leq |x_n^{(m)} - z_n| + |x_n^{(m)}| \leq \sup_n |x_n^{(m)} - z_n| + |x_n^{(m)}| < 2\epsilon, \ \forall n \geq N_1,$$

which shows that $(z_n)_{n\geq 1}$ converges to 0, and the conclusion follows. □

Problem 9.2.7: Prove that a metric space is compact if and only if it is bounded, complete, and given any $\frac{1}{m}$ there exists a finite subset $x_1, \ldots, x_n$ such that every point x in the space is within $\frac{1}{m}$ of one of them ($d(x, x_k) \leq \frac{1}{m}$ for some k, $1 \leq k \leq n$).

Solution. Let M be a metric space. Suppose first that M is compact. Then any Cauchy sequence $(x_n)_{n\geq 1}$ has a limit point in M. Since we are in a metric space, this limit point is actually a limit (see Problem 9.2.5) $\Rightarrow M$ is complete.

Let $m \geq 1$. Note that $\mathcal{B} = \{B(x, \frac{1}{m}) \mid x \in M\}$ is an open covering of M, where $B(x, \frac{1}{m})$ denotes the open ball of radius $\frac{1}{m}$ centered at x, i.e. $B(x, \frac{1}{m}) = \{y \in M \mid d(x, y) < \frac{1}{m}\}$. By the Heine-Borel Theorem, there is a finite subcovering of $\mathcal{B}$ that covers M, i.e. there exist $x_1, x_2, \ldots, x_k \in M$ such that $M \subset \bigcup_{i=1}^{k} B(x_i, \frac{1}{m})$. This also shows that M is bounded.

Conversely, suppose M is complete and bounded and given any $m \geq 1$ there exist $x_1, x_2, \ldots, x_k \in M$ such that $M \subset \bigcup_{i=1}^{k} B(x_i, \frac{1}{m})$. Let $(y_n)_{n\geq 1}$ be a sequence in M. We prove that it has a Cauchy subsequence.

Let $m = 1$. Then there are finitely many balls $B(x_i^{(1)}, 1)$, $x_i^{(1)} \in M$, that cover $M \Rightarrow$ wlog $B(x_1^{(1)}, 1)$ contains infinitely many terms of the sequence, i.e. there is a subsequence $(y_n)_{n\in I_1}$ of $(y_n)_{n\geq 1}$ contained in $B(x_1^{(1)}, 1)$. By I_1 we mean a subset of $\mathbb{N}$. For $m = 2$, we can find finitely many balls $B(x_i^{(2)}, \frac{1}{2})$, $x_i^{(2)} \in M$, that cover $M \cap B(x_1^{(1)}, 1) \Rightarrow$ wlog there is a subsequence $(y_n)_{n\in I_2}$ of $(y_n)_{n\in I_1}$ contained in $B(x_1^{(2)}, \frac{1}{2})$, where $I_2 \subset I_1$. In general, for $m \geq 2$, we can find finitely many balls $B(x_i^{(m)}, \frac{1}{m})$, $x_i^{(m)} \in M$, that cover $M \cap B(x_1^{(m-1)}, \frac{1}{m-1}) \Rightarrow$ wlog there is a subsequence $(y_n)_{n\in I_m}$ of $(y_n)_{n\in I_{m-1}}$ contained in $B(x_1^{(m)}, \frac{1}{m})$, where $I_m \subset I_{m-1}$.

Choose $i_1 \in I_1$ arbitrary and for $m \geq 2$ choose $i_m \in I_m$, with $i_{m-1} < i_m$. This is possible since all I_m are infinite. Set $z_m = y_{i_m}$. Then this is a Cauchy subsequence of $(y_n)_{n\geq 1}$. To verify it, let $m > 1$. Notice that $z_k \in B(x_1^{(2m)}, \frac{1}{2m})$ for all $k \geq 2m \Rightarrow d(z_k, x_1^{(2m)}) < \frac{1}{2m}$, for all $k \geq 2m$. Then $d(z_k, z_p) \leq d(z_k, x_1^{(2m)}) + d(z_p, x_1^{(2m)}) < \frac{1}{m}$, for all $k, p \geq 2m$ and so $(z_n)_{n\geq 1}$ is Cauchy.

Since M is complete, $(z_n)_{n\geq 1}$ converges to a point in M. Therefore M is compact. $\square$