Mathematics 414, Spring 2008

Solutions to assignment 3

Problem 9.3.7: Prove that if A is an open set in $\mathbb{R}^n$, then A is connected if and only if A is arcwise connected. (**Hint:** consider the set of points in A that can be joined to a given point x_0 by a curve.)

Solution. Suppose first that $A \subset \mathbb{R}^n$ is connected. Following the hint, for all $x \in A$ define $C_x = \{z \in A \mid z \text{ can be joined to } x \text{ by a curve contained in } A\}$. We claim that C_x is open. If $z \in \mathbb{C}_x$ then $z \in A$ and since A is open, there is $r > 0$ sufficiently small such that the ball $B(z, r) \subset A$. But the ball $B(z, r)$ is arcwise connected and z can be joined to x by a curve $\Rightarrow B(z, r) \subset C_x$, which proves the claim.

If A is not arcwise connected, then it has two points x and y that cannot be joined by a curve contained in A. This means that C_x and C_y are disjoint. Otherwise, there is $z \in C_x \cap C_y$ which can be joined to x by a curve γ_1, and to y by a curve γ_2. Then following the curve γ_1 in the opposite direction, i.e. from x to z, and then γ_2 we get a curve γ from x to y; contradiction.

Let $C = \bigcup_{z \in A \setminus C_x} C_z$. It is open as it is a union of open subsets of A. It is nonempty since $C_y \subset C$. Therefore, we can write A as a union of two disjoint non-empty open sets, $A = C_x \cup C$; contradiction, since A is connected. It follows that A is arcwise connected.

The converse implication is proved in Theorem 9.3.5, in the book. □

Problem 9.3.8: Give an example of a continuous mapping of a disconnected set onto a connected set.

Solution. Consider $f : (-1, 0) \cup (0, 1) \mapsto (0, 1)$, $f(x) = |x|$. It is onto since $f(x) = x$ for all $x \in (0, 1)$. Note that $||x| - |y|| \le |x - y|$, for all $x, y \in \mathbb{R}$. Hence for all converging sequences $x_n \in (-1, 0) \cup (0, 1)$ with $x_n \to x \Rightarrow |x_n| \to |x|$. This shows that f is continuous. □

Problem 9.3.9: Give an example of a continuous mapping of a noncompact set onto a compact set.

Solution. Consider the continuous function $f : [0, 2\pi) \mapsto [-1, 1]$, $f(x) = \sin(x)$. Recall that $\sin : [\frac{\pi}{2}, \frac{3\pi}{2}] \mapsto [-1, 1]$ is bijective, hence f is onto. The interval $[0, 2\pi)$ is not closed in $\mathbb{R}$, hence it is not compact. The interval $[-1, 1]$ is both closed and bounded in $\mathbb{R}$, hence it is compact. □

Problem 9.3.15: Give an example of a continuous mapping of $(0, 1)$ onto $(0, 1)$ with no fixed points.

Solution. Consider $f : (0, 1) \mapsto (0, 1)$, $f(x) = x^2$. Clearly this map is continuous. It has has no fixed points, since otherwise $f(x) = x \Rightarrow x = 0$ or $x = 1$, which are not in $(0, 1)$. It is onto since for all $y \in (0, 1)$ there is $x = \sqrt{y} \in (0, 1)$ such that $f(x) = y$. □