Mathematics 414, Spring 2008

Solutions to assignment 4

Problem 9.3.17: Let $T : \mathcal{C}([0,1]) \mapsto \mathcal{C}([0,1])$ be defined by $Tf(x) = x + \int_0^x tf(t)dt$. Prove that T satisfies the hypothesis of the contractive mapping principle. Show that the fixed point is a solution to the differential equation $f'(x) = xf(x) + 1$.

SOLUTION. Consider the metric on $\mathcal{C}([0,1])$ to be $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. In order to prove that T is a contraction, we need to show that there is a constant $0 < K < 1$ such that $d(Tf, Tg) \leq Kd(f,g)$, for all $f, g \in \mathcal{C}([0,1])$. We claim that $K = \frac{1}{2}$ suffices. Indeed,

$$d(Tf,Tg) = \sup_{x\in[0,1]} |\int_0^x t(f(t) - g(t))dt| \leq \sup_{x\in[0,1]} \int_0^1 t|f(t) - g(t)|dt$$

and since $|f(t) - g(t)| \leq \sup_{x\in[0,1]} |f(x) - g(x)|$ we get

$$d(Tf,Tg) \leq \sup_{x\in[0,1]} |f(x) - g(x)| \sup_{x\in[0,1]} \int_0^1 tdt \leq \frac{1}{2} \sup_{x\in[0,1]} |f(x) - g(x)| = \frac{1}{2} d(f,g).$$

If f is a fixed point of T then $Tf = f$ and the equation becomes

$$f(x) = x + \int_0^x tf(t)dt \Rightarrow f'(x) = 1 + xf(x),$$

as claimed. □

Problem 10.1.1: If A is a $m \times n$ matrix prove that there exists a constant M such that $|Ax| \leq M|x|$ for every x in $\mathbb{R}^n$.

SOLUTION. Recall that the transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$, $T(x) = Ax$ is a linear transformation. Let $x = \sum_{i=1}^n x_i e_i$ be the representation of x in the standard basis of R^n. Then $|Ax| = |A(\sum_{i=1}^n x_i e_i)| = |\sum_{i=1}^n x_i Ae_i| \leq \sum_{i=1}^n |x_i||Ae_i| \leq (\sum_{i=1}^n |x_i|^2)^{1/2} (\sum_{i=1}^n |Ae_i|)^{1/2}$. The last inequality follows from the Cauchy-Schwarz inequality. Setting $M = (\sum_{i=1}^n |Ae_i|)^{1/2}$ we get that $|Ax| \leq M|x|$ for every x in $\mathbb{R}^n$, hence the conclusion. □

Problem 10.1.2: Prove that $f : D \mapsto R^m$ is differentiable at a point if and only if each of the coordinate functions $f_k : D \mapsto \mathbb{R}$ is differentiable at that point.

SOLUTION. For the direct implication consider $\pi_k : \mathbb{R}^m \mapsto R$, $\pi_k(x_1, x_2, \ldots, x_m) = x_k$, for all $1 \leq k \leq m$. One can easily check that these functions are differentiable at every point of $\mathbb{R}^m$. Let $f : D \mapsto \mathbb{R}^m$, $f = (f_1, f_2, \ldots, f_m)$. Then $\pi_k \circ f = f_k$, for all $1 \leq k \leq m$. By the chain rule, if f is differentiable at $a \in D$ then $\pi_k \circ f$ is differentiable at a, since π_k is differentiable at $f(a)$.

Conversely, if $f_k : D \subset \mathbb{R}^n \mapsto \mathbb{R}$ is differentiable at a then

$$\lim_{h \to 0} \frac{|f_k(a+h) - f_k(a) - Df_k(a)h|}{|h|} = 0,$$

where $Df_k(a)$ is an $1 \times n$ matrix. Let $T(a) = (Df_1(a), Df_2(a), \ldots, Df_m(a))^t$ be an $m \times n$ matrix. Squaring the equations above and summing over all k gives

$$\lim_{h \to 0} \frac{\sum_{k=1}^{m} |f_k(a+h) - f_k(a) - Df_k(a)h|^2}{|h|^2} = \lim_{h \to 0} \frac{|f(a+h) - f(a) - T(a)h|^2}{|h|^2} = 0,$$

and so

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - T(a)h|}{|h|} = 0.$$

This in fact shows that f is differentiable at $a \in D$. □

Problem 10.1.4: Let $f : D \mapsto \mathbb{R}^m$ and $g : D \mapsto \mathbb{R}^m$ be differentiable at y. Let $f \cdot g : D \mapsto \mathbb{R}$ be defined by the dot product in R^m. Prove that $f \cdot g$ is differentiable at y, and find a formula for the differential $d(f \cdot g)(y)$.

Solution. Let $f = (f_1, f_2, \ldots, f_m)$, and $g = (g_1, g_2, \ldots, g_m)$, where $f_k, g_k : D \mapsto \mathbb{R}$, for all $1 \le k \le m$. Then $f \cdot g = \sum_{k=1}^{m} f_k g_k$. Since f and g are differentiable at y, it follows from the previous problem that f_k and g_k are all differentiable at y. We claim that $f_k g_k$ is also differentiable at y with $Df_k g_k(y) = g_k(y)Df_k(y) + f_k(y)Dg_k(y)$. Notice that

$$\lim_{h \to 0} \frac{f_k(y+h)g_k(y+h) - f_k(y)g_k(y) - g_k(y)Df_k(y)h - f_k(y)Dg_k(y)h}{|h|} =$$

$$= g_k(y) \lim_{h \to 0} \frac{f_k(y+h) - f_k(y) - Df_k(y)h}{|h|} + f_k(y) \lim_{h \to 0} \frac{g_k(y+h) - g_k(y) - Dg_k(y)h}{|h|} +$$

$$+ \lim_{h \to 0} \frac{(f_k(y+h) - f_k(y))(g_k(y+h) - g_k(y))}{|h|} = 0.$$

The last limit reduces to $\lim_{h \to 0} Df_k(y)(g_k(y+h) - g_k(y)) = 0$, since g_k is continuous as well. Hence $f_k g_k$ is differentiable at y and so $\sum_{k=1}^{m} f_k g_k = f \cdot g$ is differentiable at y, with

$$D(f \cdot g)(y) = \sum_{k=1}^{m} Df_k g_k(y) = \sum_{k=1}^{m} g_k(y)Df_k(y) + f_k(y)Dg_k(y).$$

□

Extra Problem 1: Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that f is not differentiable at $(0,0)$.

SOLUTION. Suppose f is differentiable at $0 = (0,0)$ and let $T = Df(0)$. Since f is differentiable, the partial derivatives $\frac{\partial f}{\partial x}(0)$ and $\frac{\partial f}{\partial y}(0)$ exist and we have $T = (\frac{\partial f}{\partial x}(0), \frac{\partial f}{\partial y}(0))$. But

$$\frac{\partial f}{\partial x}(0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0,$$

and similarly $\frac{\partial f}{\partial y}(0) = 0$. Thus $T = 0$. Then, by definition, f is differentiable at 0 if the limit

$$\lim_{(h,k)\to 0} \frac{|f(h,k)|}{\sqrt{h^2+k^2}} = 0 \Leftrightarrow \lim_{(h,k)\to 0} \frac{|hk|}{h^2+k^2} = 0.$$

This is however not true as

$$\lim_{(h,h)\to 0} \frac{|h^2|}{2h^2} = \frac{1}{2} \neq 0.$$

Therefore f is not differentiable at 0. □

Extra Problem 2: Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is differentiable at 0.

SOLUTION. Notice first that $|f(0)| \leq 0 \Rightarrow f(0) = 0$. Then

$$\lim_{h\to 0} \frac{|f(h) - f(0)|}{|h|} \leq \lim_{h \to 0} \frac{|h^2|}{|h|} = 0.$$

Thus f is differentiable at 0 with differential $Df(0) = 0$. □