Mathematics 414, Spring 2008

Solutions to assignment 5

Problem 10.1.6: Let $f : D \mapsto \mathbb{R}$ be differentiable at y, and suppose $\nabla f(y) \neq 0$. Show that $d_u f(y)$ as u varies over all unit vectors ($|u| = 1$) attains its maximum value when $u = \lambda \nabla f(y)$ for some $\lambda > 0$ and $d_u f(y) = |\nabla f(y)|$ for that choice of u.

Solution. Let $f : D \mapsto \mathbb{R}$ be differentiable at y. Then

$$d_u f(y) = df(y) \cdot u = \nabla f(y) \cdot u = \langle \nabla f(y), u \rangle .$$

Using Cauchy-Schwarz inequality we get

$$\langle \nabla f(y), u \rangle \leq |\nabla f(y)||u| = |\nabla f(y)|,$$

where u varies over all unit vectors, i.e. $|u| = 1$. Thus the required maximum is $|\nabla f(y)|$ and is attained when u and $\nabla f(y) \neq 0$ are collinear vectors, i.e. there is $\lambda \in \mathbb{R}$ such that $u = \lambda \nabla f(y)$. Since we are interested in a maximum value, we must have $\lambda > 0$. Also $|u| = 1 \Rightarrow \lambda = \frac{1}{|\nabla f(y)|}$. Therefore $u = \frac{\nabla f(y)}{|\nabla f(y)|}$ is a point of maximum for $d_u f(y)$, as u varies over all unit vectors. □

Problem 10.1.7: let $f : D \mapsto \mathbb{R}$ be differentiable at y, and suppose $\nabla f(y) \neq 0$. Show that $d_u f(y) = 0$ if u is orthogonal to $\nabla f(y)$.

Solution. let $f : D \mapsto \mathbb{R}$ be differentiable at y, with $\nabla f(y) \neq 0$. Then

$$d_u f(y) = df(y) \cdot u = \nabla f(y) \cdot u = \langle \nabla f(y), u \rangle .$$

If u is orthogonal to $\nabla f(y)$ then $\langle \nabla f(y), u \rangle = 0$ and so $d_u f(y) = 0$. □

Problem 10.1.13: Show that the following functions are differentiable and compute df:

a. $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $f(x_1, x_2) = x_1 e^{x_2}$.

b. $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$, $f(x_1, x_2, x_3) = (x_3, x_2)$.

c. $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$, $f(x_1, x_2) = (x_1, x_2, x_1 x_2)$.

Solution. It is straightforward to check that these functions are indeed differentiable and that:

a. $df(x_1, x_2) = \begin{pmatrix} e^{x_2} & x_1 e^{x_2} \end{pmatrix}$.

b. $df(x_1, x_2, x_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

c. $df(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_2 & x_1 \end{pmatrix}$.

$\square$

Problem 10.1.18: If $f : \mathbb{R} \mapsto \mathbb{R}$ is $\mathcal{C}^1$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is continuous and one of them has compact support, show that $f * g$ is $\mathcal{C}^1$, and $(f * g)' = f' * g$.

Solution. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be $\mathcal{C}^1$ and $g : \mathbb{R} \mapsto \mathbb{R}$ be continuous. Define $h : \mathbb{R}^2 \mapsto \mathbb{R}, h(x, y) = f(x - y)g(y)$. Then h is continuous and $\frac{\partial h}{\partial x}(x, y) = f'(x - y)g(y)$ exists and is continuous. Moreover, if $a(x)$ and $b(x)$ are $\mathcal{C}^1$ functions of one variable then the expression

$$H(x) = \int_{a(x)}^{b(x)} h(x, y)dx$$

is $\mathcal{C}^1$ and

$$H'(x) = b'(x)h(x, b(x)) - a'(x)g(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial h}{\partial x}(x, y)dx. \tag{1}$$

This follows directly from the discussion in Chapter 10.1.4, in particular Lemma 10.1.1.

Suppose first that f has compact support, i.e. $\operatorname{supp}(f) \subset [a, b]$ for some constants $a \leq b$. Then $a(x) = x - a, b(x) = x - b$ and

$$(f * g)(x) = \int_{a(x)}^{b(x)} f(x - y)g(y)dy = \int_{a(x)}^{b(x)} h(x, y)dy.$$

It follows from (1) that

$$(f * g)'(x) = f(a)g(x - a) - f(b)g(x - b) + \int_{a(x)}^{b(x)} f'(x - y)g(y)dy = (f' * g)(x).$$

Suppose now that g has compact support, i.e. $\operatorname{supp}(g) \subset [c, d]$ for some constants $c \leq d$. Then $a(x) = c, b(x) = d$ and

$$(f * g)(x) = \int_c^d f(x - y)g(y)dy = \int_c^d h(x, y)dy.$$

Applying (1) we deduce that

$$(f * g)'(x) = \int_c^d f'(x - y)g(y)dy = (f' * g)(x).$$

In conclusion $(f * g)' = f' * g$, as claimed. $\square$

Problem 10.2.1: Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be defined by $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Express f in polar coordinates. Show that $\partial f/\partial x$, $\partial f/\partial x$, $\partial^2 f/\partial x\partial y$, $\partial^2 f/\partial y\partial x$ exist for all (x, y) in $\mathbb{R}^2$ but $\partial^2 f/\partial x\partial y(0, 0) \neq \partial^2 f/\partial y\partial x(0, 0)$

Solution. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then f in polar coordinates is expressed as

$$f(r, \theta) = r^2 \cos\theta \sin\theta(\cos^2\theta - \sin^2\theta) = \frac{r^2}{2}\sin 2\theta \cos 2\theta = \frac{r^2 \sin 4\theta}{4}.$$

For all $(x,y) \neq (0,0)$ we obtain

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} \ ,$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} \ ,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{(x^2-y^2)(x^4+10x^2y^2+y^4)}{(x^2+y^2)^3} \ ,$$

and

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{(x^2-y^2)(x^4+10x^2y^2+y^4)}{(x^2+y^2)^3}.$$

Clearly all partial derivatives above exist for $(x,y) \neq (0,0)$. This is still true at $(x,y) = (0,0)$. For example,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0)-f(0,0)}{x} = 0,$$

and similarly $\frac{\partial f}{\partial y}(0,0) = 0$. However

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1,$$

while

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \lim_{y \to 0} \frac{-y}{y} = -1.$$

Therefore $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

□