Mathematics 414, Spring 2008

Solutions to assignment 6

Problem 11.1.3: Verify that $x(t)$ is a solution of the mth order o.d.e. $x^{(m)}(t) = G(t, x(t), \ldots, x^{(m-1)}(t))$ if and only if $(x_0, x_1, \ldots, x_{m-1}) = (x, x', \ldots, x^{(m-1)})$ is a solution of the first-order system

$$\begin{aligned} x'_{m-1}(t) &= G(t, x_0(t), \ldots, x_{m-1}(t)), \\ x'_k(t) &= x_{k+1}(t) \quad k = 0, 1, \ldots, m-2. \end{aligned}$$

Also verify that $x(t)$ satisfies the Cauchy initial conditions $x^{(k)}(t_0) = a^{(k)}$ for $k = 0, \ldots, m-1$ if and only if $(x_0, \ldots, x_{m-1})$ satisfies the Cauchy initial conditions $(x_0(t_0), \ldots, x_{m-1}(t_0)) = (a^{(0)}, \ldots, a^{(m-1)})$.

Solution. If $x(t)$ is a solution of the mth order o.d.e. $x^{(m)}(t) = G(t, x(t), \ldots, x^{(m-1)}(t))$, then set $x_k(t) = x^{(k)}(t)$ for $k = 0, 1, \ldots, m-1$. This gives directly

$$x'_{m-1}(t) = x^{(m)}(t) = G(t, x(t), \ldots, x^{(m-1)}(t)) = G(t, x_0(t), \ldots, x_{m-1}(t)),$$

and $x'_k(t) = x^{(k+1)}(t) = x_{k+1}(t)$, for all $k = 0, 1, \ldots, m-2$. Hence $(x_0, x_1, \ldots, x_{m-1}) = (x, x', \ldots, x^{(m-1)})$ is a solution of the first-order system.

Conversely, if $(x_0, x_1, \ldots, x_{m-1})$ is a solution of the first-order system, then set $x_0(t) = x(t)$. Since $x'_k(t) = x_{k+1}(t)$ for $k = 0, 1, \ldots, m-2$ we get $(x_0, x_1, \ldots, x_{m-1}) = (x, x', \ldots, x^{(m-1)})$, in particular $x^{(m)}(t) = x'_{m-1}(t) = G(t, x(t), \ldots, x^{(m-1)}(t))$. Thus $x(t)$ is a solution of the mth order o.d.e.

If $x^{(k)}(t_0) = a^{(k)}$ for $k = 0, \ldots, m-1$ then $(x_0(t_0), \ldots, x_{m-1}(t_0)) = (a^{(0)}, \ldots, a^{(m-1)})$, by simple substitutions. The converse is also immediate. □

Problem 11.1.4: Show that all solutions of $x''(t) = -x(t)$ are of the form $x(t) = A\cos(t) + B\sin(t)$. Using this, decide for which values of t_1 and t_2 the o.d.e. $x''(t) = -x(t)$ with boundary conditions $x(t_1) = a_1, x(t_2) = a_2$ has a unique solution on $[t_1, t_2]$, for any choice of a_1, a_2.

Solution. Let $m = 2$ and $G(t, x) = -x$. Then by the previous problem, $x(t)$ is a solution of $x''(t) = -x(t)$ if and only if $(x_0(t), x_1(t)) = (x(t), x'(t))$ is a solution to the first-order linear system

$$\begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_0(t) \\ x_1(t) \end{pmatrix}.$$

Let I be some interval and $t_0 \in I$. By specifying $x_0(t_0)$ and $x_1(t_0)$, then by Corollary 11.1.1 this solution to the linear system is unique on I. Thus $x(t)$ is unique if we specify $x(t_0)$ and $x'(t_0)$. It is easy to check that $x(t) = A\cos(t) + B\sin(t)$ is a solution to $x''(t) = -x(t)$. We determine A and B uniquely from the values of $x(t_0)$ and $x'(t_0)$. Therefore all solutions to $x''(t) = -x(t)$ are of the form $x(t) = A\cos(t) + B\sin(t)$, for some constants A and B.

Consider now $x''(t) = -x(t)$ on $[t_1, t_2]$, with boundary conditions $x(t_1) = a_1, x(t_2) = a_2$. The solution is $x(t) = A\cos(t) + B\sin(t)$, where A and B verify the system

$$\begin{aligned} A\cos t_1 + B\sin t_1 &= a_1 \\ A\cos t_2 + B\sin t_2 &= a_2. \end{aligned}$$

This system has a unique solution (A, B) if the determinant of the associated matrix is nonzero, i.e. $\cos t_1 \sin t_2 - \cos t_2 \sin t_1 = \sin(t_1 - t_2) \neq 0$. Notice that $\sin(t_1 - t_2) \neq 0$ if and only if $(t_1 - t_2)$ is not a multiple of π. In conclusion, the equation $x''(t) = -x(t)$ on $[t_1, t_2]$, with boundary conditions $x(t_1) = a_1, x(t_2) = a_2$ has a unique solution if and only if $(t_1 - t_2)$ is not a multiple of π. □

Problem 12.1.2: Show that it is impossible to have $\sin x = \sum_{k=2}^{\infty} a_k \sin kx$ on $0 \leq x \leq \pi$ with the series converging uniformly, for any choice of the a_k, even though there are an infinite number of parameters in the problem. (**Hint:** multiply by $\sin x$ and integrate.)

SOLUTION. Suppose that the sum $\sum_{k=2}^{\infty} a_k \sin kx$ converges uniformly to $\sin x$ on $[0, \pi]$. Then $\sum_{k=2}^{\infty} a_k \sin x \sin kx$ converges uniformly to $\sin^2 x$ on $[0, \pi]$ and we can write

$$\int_0^{\pi} \sin^2 x = \int_0^{\pi} \sum_{k=2}^{\infty} a_k \sin x \sin kx = \sum_{k=2}^{\infty} a_k \int_0^{\pi} \sin x \sin kx.$$

Since $k \geq 2$, $\int_0^{\pi} \sin x \sin kx = \frac{\cos x \sin kx - k\cos kx \sin x}{k^2-1}\Big|_0^{\pi} = 0$, and this does not depend on a_k. We then get

$$\int_0^{\pi} \sin^2 x = \frac{x - \sin x \cos x}{2}\Big|_0^{\pi} = \frac{\pi}{2} = 0;$$

contradiction. Hence it is impossible to have $\sin x = \sum_{k=2}^{\infty} a_k \sin kx$ on $[0, \pi]$, with the series converging uniformly. □