Mathematics 414, Spring 2008

Solutions to assignment 7

Problem 13.1.3: Write out the proof of the inverse function theorem by modifying the given proof of the implicit function theorem.

Solution. As indicated in the book, the proof of the inverse function theorem is the same as the one for implicit function theorem. Consider f a $\mathcal{C}^1$ function defined in a neighborhood of $\tilde{y}$ in $\mathbb{R}^n$ with values in $\mathbb{R}^n$ and let $F(x, y)$ be another $\mathcal{C}^1$ function defined in a neighborhood of $\tilde{x}$ and $\tilde{y}$ in $\mathbb{R}^n$ with values in $\mathbb{R}^n$, where $F(x, y) = f(y) - x$. We take $c = 0$ and assume $F(\tilde{x}, \tilde{y}) = c$ and that $F_y(\tilde{x}, \tilde{y}) = df(\tilde{y})$ is invertible. Then follow the proof from the book. □

Problem 13.1.4: In the following examples, decide at which points $(\tilde{x}, \tilde{y})$ the hypothesis of the implicit function theorem are satisfied:

a. $x^4 + xy^6 - 3y^4 = c$.

b. $\begin{cases} \sin(x + y_1) + y_2^2 = c_1, \\ y_1^2 + xy_2^2 = c_2. \end{cases}$

c. $\begin{cases} y_1^5 + 3y_1y_2 = x_1, \\ y_2^6 + 4y_1^2y_2^2 = x_2. \end{cases}$

Solution.

a. Consider $F(x, y) = x^4 + xy^6 - 3y^4$. Clearly it is a $\mathcal{C}^1$ function everywhere on $\mathbb{R}^2$ and $F_y(x, y) = 6y^3(xy^2 - 2)$. The hypothesis of the implicit function theorem are satisfied at all points (x, y) for which $F(x, y) = c$, with $y \neq 0$ and $xy^2 \neq 2$.

b. Consider $F(x, y_1, y_2) = (\sin(x + y_1) + y_2^2, y_1^2 + xy_2^2)$. It is a $\mathcal{C}^1$ function everywhere and

$$F_y(x, y_1, y_2) = \begin{pmatrix} \cos(x + y_1) & 2y_2 \\ 2y_1 & 2xy_2 \end{pmatrix}.$$

It is invertible whenever the determinant is nonzero, i.e. $2y_2(x\cos(x + y_1) - 2y_1) \neq 0$. The hypothesis of the implicit function theorem are satisfied at all points (x, y_1, y_2) for which $F(x, y_1, y_2) = (c_1, c_2)$, with $y_2 \neq 0$ and $x\cos(x + y_1) - 2y_1 \neq 0$.

c. Consider $F(x_1, x_2, y_1, y_2) = (y_1^5 + 3y_1y_2 - x_1, y_2^6 + 4y_1^2y_2^2 - x_2)$, which is again a $\mathcal{C}^1$ function everywhere and

$$F_y(x_1, x_2, y_1, y_2) = \begin{pmatrix} 5y_1^4 + 3y_2 & 3y_1 \\ 8y_1y_2^2 & 6y_2^5 + 8y_1^2y_2 \end{pmatrix}.$$

It is invertible whenever the determinant is nonzero, i.e. $(5y_1^4 + 3y_2)(6y_2^5 + 8y_1^2y_2) - 24y_1^2y_2^2 = 2y_2(15y_1^4y_2^4 + 20y_1^6y_2 + 9y_2^5) \neq 0$. The hypothesis of the implicit function theorem are satisfied at all points (x_1, x_2, y_1, y_2) for which $F(x_1, x_2, y_1, y_2) = (0, 0)$, with $y_2 \neq 0$ and $15y_1^4y_2^4 + 20y_1^6y_2 + 9y_2^5 \neq 0$.

$\square$

Problem 13.2.1: Decide which of the following maps are immersions, are for those that are decide which variables can be taken to be the independent variable(s) in the description of the image locally as a graph of a function:

a. $g(t) = (\cos t, \sin t, t)$ t in $\mathbb{R}^1$,

b. $g(t, s) = (\cos t, \sin t, s)$ (t, s) in $\mathbb{R}^2$,

c. $g(t, s) = (s \cos t, s \sin t, s)$ (t, s) in $\mathbb{R}^2$,

d. $g(t, s) = (s \cos t, s \sin t, s)$ $s > 1, t$ in $\mathbb{R}^1$.

Solution.

a. We have $dg(t) = (-\sin t, \cos t, 1)$, which has rank 1 at all points $t \in \mathbb{R}$. Thus g is an immersion. If $t \neq k\pi$ then x can be taken as independent variable. If $t \neq k\pi + \frac{\pi}{2}$ then y can be taken as independent variable and z can always be taken as independent variable in the description of the image locally as a graph of a function.

b. We have
$$dg(t, s) = \begin{pmatrix} -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
which has rank 2 at all points $(t, s) \in \mathbb{R}^2$. Clearly this is true as $\sin t$ and $\cos t$ cannot both be zero. Hence g is an immersion. If $t \neq k\pi$ then x and z can be taken as independent variables. If $t \neq k\pi + \frac{\pi}{2}$ then y and z can be taken as independent variables in the description of the image locally as a graph of a function.

c. We have
$$dg(t, s) = \begin{pmatrix} -s \sin t & s \cos t & 0 \\ \cos t & \sin t & 1 \end{pmatrix},$$
which does not have rank 2 at all points $(t, s) \in \mathbb{R}^2$. Indeed at $s = 0$, $dg(t, s)$ has rank 1 as the first row is 0. This shows that g is not an immersion.

d. We have
$$dg(t, s) = \begin{pmatrix} -s \sin t & s \cos t & 0 \\ \cos t & \sin t & 1 \end{pmatrix},$$
which has rank 2 at all points $(t, s) \in \mathbb{R}^2$ with $s > 1$. To see this simply notice that $-s \sin t$ and $s \cos t$ cannot both be zero on our domain. Hence g is an immersion. If $t \neq k\pi$ then x and y can be taken as independent variables. If $t \neq k\pi + \frac{\pi}{2}$ then y and z can be taken as independent variables and x and y can always be taken as independent variables in the description of the image locally as a graph of a function. $\square$

Problem 13.2.2: For which values of the constants do the following implicit equations define e $\mathcal{C}^1$ surface? For those that do, decide which variables can be taken to be the independent variables in the description of the surface locally as a graph of a function:

a. $x^2 + y^2 - z^2 = c$;

b. $x^2 + y^2 + z^2 = c_1$, $x^2 + y^2 - z^2 = c_2$;

c. $xyz = c$.

Solution.

a. Let $F(x, y, z) = x^2 + y^2 - z^2$. Then $dF(x, y, z) = (2x, 2y, -2z)$ has rank 1 provided $(z, y, z) \neq (0, 0, 0)$. Thus by Theorem 13.2.2 all surfaces $F(x, y, z) = c$ for $c \neq 0$ are $\mathcal{C}^1$ surfaces. For $c \neq 0$, any two variables can be taken to be the independent variables in the description of the surface locally as a graph of a function. However, if a coordinate vanishes at a point on the surface then the other two variables are considered.

b. Let $F(x, y, z) = (x^2 + y^2 + z^2, x^2 + y^2 - z^2)$. Then

$$dF(x, y, z) = \begin{pmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \end{pmatrix}.$$

In order for this matrix to have full rank, we must have $z \neq 0$ and either $x \neq 0$ or $y \neq 0$. First notice that $x^2 + y^2 + z^2 \geq 0$ so $c_1 \geq 0$. Also $4z^2 = c_1 - c_2 \geq 0$ and $2(x^2 + y^2) = c_1 + c_2 \geq 0$. So the matrix $dF(x, y, z)$ has rank 2 if $c_1 \pm c_2 > 0$. Hence if $c_1 \pm c_2 > 0$ then by Theorem 13.2.2 the surface $F(x, y, z) = (c_1, c_2)$ is a $\mathcal{C}^1$ surface of dimension 1. Clearly we cannot take z to be the independent variable. If $x \neq 0$ ($y \neq 0$) then we can take x (y) to be the independent variables in the description of the surface locally as a graph of a function.

Notice that if $z = 0$ occurs on the level curve $F(x, y, z) = (c_1, c_2)$ then $c_1 = c_2$, where clearly $c_1 > 0$. Then the surface becomes $x^2 + y^2 = c_1$ which is a $\mathcal{C}^1$ one dimensional surface in $\mathbb{R}^3$; in fact it is a circle.

c. Let $F(x, y, z) = xyz$. Then $dF(x, y, z) = (yz, xz, xy)$ which has full rank if two of the variables are nonzero. Thus if $c \neq 0$ then $dF(x, y, z)$ has rank 1 and by Theorem 13.2.2 we have a $\mathcal{C}^1$ surface of dimension 2. We can take any two variables to be the independent variables in the description of the surface.

$\square$