Mathematics 414, Spring 2008

Solutions to assignment 8

Problem 13.2.5: For each point $(\tilde{x}, \tilde{y}, \tilde{z})$ on the unit sphere $x^2 + y^2 + z^2 = 1$ and each vector $v = (v_1, v_2, v_3)$ in the tangent space of the sphere at $(\tilde{x}, \tilde{y}, \tilde{z})$, construct a $\mathcal{C}^1$ curve lying in the sphere whose tangent vector at the point $(\tilde{x}, \tilde{y}, \tilde{z})$ is (v_1, v_2, v_3).

Solution. Let $P = (\tilde{x}, \tilde{y}, \tilde{z})$ be a point on the unit sphere. The equation of the tangent plane at this point is $x\tilde{x} + y\tilde{y} + z\tilde{z} = 0$, so if (v_1, v_2, v_3) is a tangent vector at P then $v_1\tilde{x} + v_2\tilde{y} + v_3\tilde{z} = 0$. Let $f : (-\epsilon, \epsilon) \mapsto \mathbb{R}^3$ be a continuous function,

$$\begin{aligned} f(t) &= \cos(|v|t)P + \sin(|v|t)\frac{v}{|v|} \\ &= (\cos(|v|t)\tilde{x} + \sin(|v|t)\frac{v_1}{|v|}, \cos(|v|t)\tilde{y} + \sin(|v|t)\frac{v_2}{|v|}, \cos(|v|t)\tilde{z} + \sin(|v|t)\frac{v_3}{|v|}), \end{aligned}$$

where $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. Then the image of f is a $\mathcal{C}^1$ curve on the unit sphere. To see that it is well defined, write

$$\begin{aligned} |f(t)|^2 &= \sum (\cos(|v|t)\tilde{x} + \sin(|v|t)\frac{v_1}{|v|})^2 \\ &= \cos(|v|t)^2 + \sin(|v|t)^2 + 2\cos(|v|t)\sin(|v|t)\frac{v_1\tilde{x} + v_2\tilde{y} + v_3\tilde{z}}{|v|} = 1. \end{aligned}$$

We have used the fact that $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1$ and that $v_1\tilde{x} + v_2\tilde{y} + v_3\tilde{z} = 0$, from above. Then $f(0) = (\tilde{x}, \tilde{y}, \tilde{z})$ and the tangent vector at $f(0)$ is $f'(0) = -|v|\sin(0)P + \cos(0)\frac{v|v|}{|v|} = v$. □

Problem 13.2.7: Let M_2 be any $\mathcal{C}^1$ two-dimensional surface in $\mathbb{R}^3$ that is compact. Show that for every two-dimensional vector space V of $\mathbb{R}^3$, there exists a point x on M_2 whose tangent vector space equals V. (**Hint:** if u is a vector perpendicular to V, what happens at points on M_2 where $x \cdot u$ achieves a maximum or a minimum?)

Solution. Let M_2 be a compact $\mathcal{C}^1$ two-dimensional surface in $\mathbb{R}^3$ and let V be a two-dimensional vector space of $\mathbb{R}^3$. Since V is only two-dimensional, there exist $u \neq 0$ a vector in $\mathbb{R}^3$ which is perpendicular to V. Define $f : M_2 \mapsto \mathbb{R}$, $f(x) = x \cdot u$. Then f is a continuous function on a compact space, hence it attains its maximum at a point x_0 in M_2.

As M_2 is a $\mathcal{C}^1$ surface, we can take $g : U \mapsto M_2 \cap V_0$, the embedding that defines M in a neighborhood V_0 of x_0, where U is an open subset of $\mathbb{R}^2$. Since g is one-to-one, there is a unique $y_0 \in U$ such that $g(y_0) = x_0$. Then y_0 is also a maximum for the function $y \mapsto g(y) \cdot u$. Hence the derivative of this function vanishes at y_0, i.e. $dg(y_0)u = 0$. However, g is an embedding so $dg(y_0)$ is a 2×3 matrix of rank 2. If we denote v_1 and v_2 to be the rows of $dg(y_0)$, then v_1 and v_2 are linearly independent and $v_1 \cdot u = v_2 \cdot u = 0$. Let W be the space spanned by v_1 and v_2. Then W has dimension two and is orthogonal to u. Since u was orthogonal to V, it follows that $V = W$ and is the tangent space of M_2 at the point $g(y_0) = x_0$. □

Problem 13.2.8: Let M_2 be the surface of revolution in $\mathbb{R}^3$ obtained by rotating a circle in the x-z-plane, that does not intersect the z-axis, about the z-axis. Show that M_2 is a $\mathcal{C}^1$ two-dimensional surface, and compute its tangent space at any point. (Note: this surface is called a *torus*.)

SOLUTION. Suppose C is the circle in the x-z-plane we wish to rotate around the z-axis. Let r be the radius of C and R be the distance from its center to the origin. Then $R > r$ as C does not intersect the z-axis. Assume also wlog that C has center on the x-axis. Let P and Q be the two points on C at height z, where clearly $-r \le z \le r$. Then the x-coordinate of P and Q are $R - \sqrt{r^2 - z^2}$ and $R + \sqrt{r^2 - z^2}$ respectively. We now rotate C as described and obtain M_2. Point P describes a circle of radius $R - \sqrt{r^2 - z^2}$, while point Q describes a circle of radius $R + \sqrt{r^2 - z^2}$. The equations describing these circles in $\mathbb{R}^3$ are $x^2 + y^2 = (R \pm \sqrt{r^2 - z^2})^2$. As $R > r$ we are left with $\sqrt{x^2 + y^2} = R \pm \sqrt{r^2 - z^2}$, hence with $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$, which is the equation describing M_2.

Set $F(x, y, z) = (\sqrt{x^2 + y^2} - R)^2 + z^2 - r^2$. Then M_2 is described by $F(x, y, z) = 0$ and

$$dF(x, y, z) = \begin{pmatrix} 2x\frac{\sqrt{x^2+y^2}-R}{\sqrt{x^2+y^2}} & 2y\frac{\sqrt{x^2+y^2}-R}{\sqrt{x^2+y^2}} & 2z \end{pmatrix}.$$

Clearly x, y and z cannot vanish simultaneously because $(0, 0, 0) \notin M_2$. If $x^2 + y^2 = R^2$ then $z = \pm r \neq 0$ and so dF has rank 1 everywhere. It follows that M_2 is a two-dimensional $\mathcal{C}^1$ surface. Let $v = (v_1, v_2, v_3)$ be a point in the tangent space at $(x, y, z) \in M_2$. Then

$$2xv_1\frac{\sqrt{x^2 + y^2} - R}{\sqrt{x^2 + y^2}} + 2yv_2\frac{\sqrt{x^2 + y^2} - R}{\sqrt{x^2 + y^2}} + 2zv_3 = 0.$$

Thus the equation of the tangent space at (x, y, z) is given by

$$v_1x(\sqrt{x^2 + y^2} - R) + v_2y(\sqrt{x^2 + y^2} - R) + v_3z\sqrt{x^2 + y^2} = 0.$$

or equivalently by

$$\sqrt{x^2 + y^2}(xv_1 + yv_2 + zv_3) = R(xv_1 + yv_2).$$

□

Problem 13.3.4b: Use the method of Lagrange multipliers to locate possible maxima and minima of the function f subject to the conditions $G = 0$ in the following:

b. $f(x, y, z) = zx + 2y$, $G_1(x, y, z) = x^2 + y^2 + 2z^2 - 1$, $G_2(z, y, z) = x^2 + y + z$. (Set up the equations to be solved.)

SOLUTION. Let $G(x, y, z) = (G_1(x, y, z), G_2(x, y, z))$ and let (x, y, z) be a point where $G(x, y, z) = 0$. We first need to check that $dG(x, y, z)$ has rank 2, where

$$dG(x, y, z) = \begin{pmatrix} 2x & 2y & 4z \\ 2x & 1 & 1 \end{pmatrix}.$$

It is clear that the matrix above does not have rank 2 only when $(x, y, z) = (0, t, \frac{t}{2})$, $t \in \mathbb{R}$. But then $G_2(x, y, z) = 0$ so $t = 0$ is the only possibility, which does not satisfy $G_1(x, y, z) = 0$. Hence dG has rank 2 whenever $G(x, y, z) = 0$. Define

$$H(x, y, z, \lambda_1, \lambda_2) = xz + 2y + \lambda_1(x^2 + y^2 + 2z^2 - 1) + \lambda_2(x^2 + y + z).$$

If $f(x, y, z)$ is a point of maximum or minimum subject to $G(x, y, z) = 0$, then $(x, y, z, \lambda_1, \lambda_2)$ are solutions to the system

$$\begin{cases} \frac{\partial H}{\partial x} = 2(\lambda_1 + \lambda_2)x + z = 0 \\ \frac{\partial H}{\partial y} = 2\lambda_1 y + \lambda_2 + 2 = 0 \\ \frac{\partial H}{\partial z} = 4\lambda_1 z + \lambda_2 + x = 0 \\ \frac{\partial H}{\partial \lambda_1} = x^2 + y^2 + 2z^2 - 1 = 0 \\ \frac{\partial H}{\partial \lambda_2} = x^2 + y + z = 0 \end{cases}$$

If $\lambda_1 = 0$ then $\lambda_2 = -2$ and so $x = 2$. From the first equation we find $z = 8$. However, substituting in the fourth equation gives $4 + y^2 + 127 = 0$, which is impossible. Hence $\lambda_1 \neq 0$. Then the second equation gives $y = -\frac{2+\lambda_2}{2\lambda_1}$. The third equation gives $z = -\frac{x+\lambda_2}{4\lambda_1}$ and plugging in the first equation we get $2(\lambda_1 + \lambda_2)x = \frac{x+\lambda_2}{4\lambda_1}$ or $(8\lambda_1^2 + 8\lambda_1\lambda_2 - 1)x = \lambda_2$. The discussion continues until all cases are exhausted. □