Mathematics 414, Spring 2008

Solutions to assignment 9

Problem 14.1.3: Prove that the intersection of all σ-fields containing a field $\mathcal{F}$ is a σ-field and that it is the smallest σ-field containing $\mathcal{F}$.

Solution. Let $\mathcal{F}'$ be the intersection of all σ-fields containing a field $\mathcal{F}$, i.e.

$$\mathcal{F}' = \bigcap_{\mathcal{F}\subset\mathcal{G}} \mathcal{G}.$$

Clearly the empty set is in $\mathcal{F}'$. If A is in $\mathcal{F}'$, then A is in $\mathcal{G}$, for every σ-field $\mathcal{G}$ containing $\mathcal{F}$. Since $\mathcal{G}$ is a σ-field, it also contains A^c. Hence A^c is in $\mathcal{F}'$. If $A_1, A_2, \ldots$ are in $\mathcal{F}'$, then they are in $\mathcal{G}$, for every σ-field $\mathcal{G}$ containing $\mathcal{F}$. Since $\mathcal{G}$ is a σ-field, it also contains $\bigcup_{j=1}^{\infty} A_j$. Hence $\bigcup_{j=1}^{\infty} A_j$ is in $\mathcal{F}'$, which in consequence is a σ-field. If $\mathcal{F}''$ is the smallest σ-field containing $\mathcal{F}$, then

$$\bigcap_{\mathcal{F}\subset\mathcal{G}} \mathcal{G} \subset \mathcal{F}'',$$

so $\mathcal{F}' \subseteq \mathcal{F}''$. Since $\mathcal{F}'$ is a σ-field, it follows that $\mathcal{F}' = \mathcal{F}''$ and $\mathcal{F}'$ is indeed the smallest σ-field containing $\mathcal{F}$. $\square$

Problem 14.1.4: Prove that if $A = \bigcup_{j=1}^{n} I_j$, a disjoint union of intervals, then $\sum_{j=1}^{n} |I_j|$ is independent of the particular decomposition. Show that if we define $|A| = \sum_{j=1}^{n} |I_j|$, then all the axioms for a measure are satisfied on the field of finite unions of intervals, where σ-additivity means $|\bigcup_{j=1}^{\infty} A_j| = \sum_{j=1}^{\infty} |A_j|$ if $A_1, A_2, \ldots,$ and $\bigcup_{j=1}^{\infty} A_j$ are all in the field. (**Hint:** use the σ-additivity of the intervals proved in the text.) Why doesn't this argument establish the existence of Lebesgue measure?

Solution. Let $A = \bigcup_{j=1}^{n} I_j = \bigcup_{k=1}^{m} J_k$ be two decompositions of A as disjoint unions of intervals. Then $J_k \subset A$ and $J_k = J_k \cap \bigcup_{j=1}^{n} I_j = \bigcup_{j=1}^{n} (I_j \cap J_k)$ is a decomposition of J_k as disjoint union of intervals, some of them possibly empty. By Lemma 14.1.1 we have

$$|J_k| = \sum_{j=1}^{n} |I_j \cap J_k|, \text{ so } \sum_{k=1}^{m} |J_k| = \sum_{k=1}^{m}\sum_{j=1}^{n} |I_j \cap J_k| = \sum_{j=1}^{n}\sum_{k=1}^{m} |J_k \cap I_j| = \sum_{j=1}^{n} |I_j|.$$

The last equality follows from the fact that $I_j = I_j \cap \bigcup_{k=1}^{m} J_k = \bigcup_{k=1}^{m} (J_k \cap I_j)$ is a decomposition of I_j as disjoint union of intervals. Therefore $\sum_{j=1}^{n} |I_j| = \sum_{k=1}^{m} |J_k|$ is independent of the particular decomposition of A. Define $|A| = \sum_{j=1}^{n} |I_j|$.

Let $\{A_n\}_n$ be a sequence of disjoint sets such that $A_n = \bigcup_{j=1}^{m_n} I_{nj}$ are all finite union of disjoint intervals and $A = \bigcup_{n=1}^{\infty} A_n$ is a finite union of disjoint intervals. Since A_n are all disjoint, then I_{nj} are all disjoint intervals, for all n and $1 \leq j \leq m_n$. Notice that $A = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} I_{nj}$, but A is a finite union of disjoint intervals. Thus we can write

$$A = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n} I_{nj} = \bigcup_{k=1}^{m} J_k,$$

as a finite union of are disjoint intervals J_k. Assume without loss of generality that each interval I_{nj} is contained in one of the intervals J_k. Then let $I_{nj}^{(k)}$ denote those intervals that are contained in J_k. Then the disjoint union $\bigcup_{n,j} I_{nj}^{(k)} = J_k$, and in view of Lemma 14.1.1, $|J_k| = \sum\limits_{n,j} |I_{nj}^{(k)}|$. It follows that

$$|A| = \sum_{k=1}^{m} |J_k| = \sum_{k=1}^{m} \sum_{n,j} |I_{nj}^{(k)}| = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} |I_{nj}| = \sum_{n=1}^{\infty} |A_n|,$$

which shows σ-additivity. Hence $|A|$, as defined above, is a measure. This argument does not establish the existence of Lebesgue measure because it is defined only on a field of finite unions of intervals, while Lebesgue measure is defined on a σ-field, thus also for countable union of intervals. □

Problem 14.1.5: Prove that the Cantor set (delete middle thirds) has Lebesgue measure zero.

Solution. Consider the interval $I = [0, 1]$. At step 1, we split I into three equal intervals and delete the middle one, say J_{01}. It has length $\frac{1}{3}$. We are left with two intervals I_{11} and I_{12} of length $\frac{1}{3}$. At step 2, split I_{11} into three equal parts and remove the middle interval, J_{11}, of length $\frac{1}{9}$. We do the same with I_{12} and remove J_{12}, of length $\frac{1}{9}$. We are left with 4 intervals $I_{21}, I_{22}, I_{23}, I_{24}$, each of length $\frac{1}{9}$. We continue this process inductively and at step n, we remove intervals $J_{(n-1)1}, J_{(n-1)2}, \dots, J_{(n-1)2^{n-1}}$, each of length $\frac{1}{3^n}$, and remain with intervals $I_{n1}, I_{n2}, \dots, I_{n2^n}$, each of length $\frac{1}{3^n}$. The standard Cantor set is $C = \bigcap_{n=0}^{\infty} \bigcap_{j=1}^{2^n} I_{nj}$, while its complement is $I \setminus C = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} J_{(n-1)j}$. Then the Lebesgue measure of $I \setminus C$ is

$$|I \setminus C| = \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n-1}} |J_{(n-1)j}| = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} (\frac{2}{3})^n = 1.$$

Hence the Lebesgue measure of the Cantor set C is zero. □