Mathematics 414, Spring 2008

Solutions to assignment 10

Problem 14.1.14: Prove the following *inner regularity* for Lebesgue measure: $|B| = \sup\{|F| : F \subseteq B \text{ is closed}\}$, for all Borel sets B. (**Hint:** if B is contained in $(-N, N)$ use the outer regularity for $(-N, N) \setminus B$.)

Solution. Let B be a Borel set and define $M = \sup\{|F| : F \subseteq B \text{ is closed}\}$. Then $M \leq |B|$ and so it remains to show $M \geq |B|$. If B is bounded then $B \subset (-N, N)$ for some N. Let $B' = (-N, N) \setminus B$. By additivity we get $|B'| = 2N - |B|$. By outer regularity we have,

$$|B'| = \inf\{|U| : B' \subseteq U,\ U \text{open}\}.$$

Let $\epsilon > 0$. There is an open set U such that $B' \subseteq U$ and $|U| \leq |B'| + \epsilon = 2N - |B| + \epsilon$. Consider $F = (-N, N) \setminus U \subseteq B$. Also, since U is open, F is closed and $|F| = 2N - |U| \geq 2N - (2N - |B| + \epsilon) = |B| - \epsilon$. Thus $|B| - \epsilon \leq |F| \leq M$. Since ϵ was arbitrary we get $M \geq |B|$ and in fact equality holds.

If B is not bounded the define $B_N = B \cap (-N, N)$ and $B'_N = (-N, N) \setminus B_N$. Then $B = \bigcup_{N=1}^{\infty} B_N$. Notice that B_N are Borel sets and they form an increasing sequence, i.e. $B_1 \subset B_2 \subset \ldots$, so $|B| = \lim_{N\to\infty} |B_N|$. Apply the above result to get $|M| \geq |B_N| + \epsilon$ for every N and $\epsilon > 0$. This gives $|M| \geq \lim_{N\to\infty} |B_N| - \epsilon = |B| - \epsilon$. Hence $M \geq B$ also in this case. □

Problem 14.1.15: Prove that for compact sets A, the Lebesgue measure can be computed using only finite coverings,

$$|A| = \inf\left\{\sum_{j=1}^{N} |I_j| : A \subseteq \bigcup_{j=1}^{N} I_j\right\}.$$

Solution. By definition, for general sets A,

$$|A| = \inf\left\{\sum_{j=1}^{\infty} |I_j| : A \subseteq \bigcup_{j=1}^{\infty} I_j\right\}.$$

Let $\epsilon > 0$. Then there are intervals I_j such that $A \subseteq \bigcup_{j=1}^{\infty} I_j$ and $|A| < \sum_{j=1}^{\infty} |I_j| + \frac{\epsilon}{2}$. Slightly enlarge the I_j's by modifying the endpoints with $\epsilon 2^{-j-2}$. Let the new intervals be I'_j, with $|I'_j| = |I_j| + \epsilon 2^{-j-1}$. Since A is compact, by the Heine-Borel Theorem, there is N such that, after a possible re-indexing of the I'_j's, we have $A \subseteq \bigcup_{j=1}^{N} I'_j$. Clearly $\sum_{j=1}^{N} |I'_j| \leq \sum_{j=1}^{\infty} |I'_j| = \sum_{j=1}^{\infty} |I_j| + \sum_{j=1}^{\infty} \epsilon 2^{-j-1} < |A| + \epsilon$. Since ϵ was arbitrary, it follows that indeed

$$|A| = \inf\left\{\sum_{j=1}^{N} |I'_j| : A \subseteq \bigcup_{j=1}^{N} I'_j\right\}.$$

Problem 14.3.2: Prove that if $f : X \mapsto \mathbb{R}$ and $g : X \mapsto \mathbb{R}$ are measurable, then $\max(f, g)$ is measurable.

SOLUTION. Set $F_a = \{x : f(x) > a\}$, $G_a = \{x : g(x) > a\}$ and $H_a = \{x : h(x) > a\}$, where $h = \max(f, g)$. Notice that $F_a = f^{-1}(a, \infty)$ and similar for G_a and H_a. It is easy to check by double inclusion that $H_a = F_a \cup G_a$. Since F_a and G_a are measurable, it follows that H_a is measurable. Hence h is measurable. □

Problem 14.3.4: Prove that if $\sum_{k=1}^{N} a_k \chi_{A_k} = \sum_{j=1}^{M} b_j \chi_{B_j}$ for every x, then $\sum_{k=1}^{N} a_k \mu(A_k) = \sum_{j=1}^{M} b_j \mu(B_j)$.

SOLUTION. Let $f = \sum_{k=1}^{N} a_k \chi_{A_k}$ and $g = \sum_{j=1}^{M} b_j \chi_{B_j}$. Assume w.l.o.g that A_k's are disjoint and B_j's are also disjoint. Then $A_k \cap B_j$ are disjoint for all k and j. Moreover $A_k = \bigcup_{j=1}^{M} A_k \cap B_j$ and $B_j = \bigcup_{k=1}^{N} B_j \cap A_k$, so

$$f = \sum_{k=1}^{N} \sum_{j=1}^{M} a_k \chi_{A_k \cap B_j} = \sum_{j=1}^{M} \sum_{k=1}^{N} b_j \chi_{B_j \cap A_k} = g.$$

Evaluating f and g at a point $x \in A_k \cap B_j$, whenever this is nonempty gives $a_k = b_j$ and so $a_k \mu(A_k \cap B_j) = b_j \mu(B_j \cap A_k)$. Thus we get

$$\sum_{k=1}^{N} a_k \mu(A_k) = \sum_{k=1}^{N} \sum_{j=1}^{M} a_k \mu(A_k \cap B_j) = \sum_{j=1}^{M} \sum_{k=1}^{N} b_j \mu(B_j \cap A_k) = \sum_{j=1}^{M} b_j \mu(B_j),$$

which proves the claim. □