Homework 1
Math 6710 Fall 2012

Updated August 26, 4:45 pm.
Due in class on Thursday, August 30.

1. (Like Durrett 1.4.1). Let X be a random variable with $X \geq 0$ almost surely. Prove: if $EX = 0$ then $X = 0$ a.s.

2. Let $X \geq 0$, and let $f : [0, \infty) \to [0, \infty)$ be continuously differentiable and monotone increasing, with $f(0) = 0$.
   (a) Show that $Ef(X) = \int_0^\infty f'(t)P(X \geq t)\, dt$. (Hint: Write $P(X \geq t)$ as $E[1_{\{X \geq t\}}]$. Now you have two integrals, one over $[0, \infty)$ and the other over Ω. Use Fubini/Tonelli's theorem.)
   (b) In particular, $EX = \int_0^\infty P(X \geq t)\, dt$.

3. Let X be a random variable with cumulative distribution function F (i.e. $F(x) := P(X \leq x)$). Show that if F is continuous then $Y = F(X)$ has a uniform distribution on $[0, 1]$, i.e. $P(Y \leq t) = t$ for $t \in [0, 1]$. Give an example to show that this need not be true if F is not continuous. (This is a sort of converse to Theorem 1.2.2.)

4. Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$\limsup_{n \to \infty} A_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n.$$

   $\limsup A_n$ is also sometimes denoted $\{A_n \text{ i.o.}\}$ (for "infinitely often"); it is the event that "infinitely many of the events A_n occur".
   (a) Show that $1_{\limsup A_n} = \limsup 1_{A_n}$.
   (b) Show that $P(\limsup A_n) \geq \limsup P(A_n)$.
   (c) Give an example to show that equality need not hold in the previous part. Indeed, try to find an example where $P(\limsup A_n) = 1$ but $\limsup P(A_n) = 0$.
   (d) Define $\liminf A_n := \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty A_n$. This is also denoted $\{A_n \text{ a.a.}\}$ and is the event that "all but finitely many of the events A_n occur". By taking complements (or directly), show that $1_{\liminf A_n} = \liminf 1_{A_n}$, $P(\liminf A_n) \leq \liminf P(A_n)$, and that equality need not hold.

5. Let $1 \leq p < \infty$. Suppose X_n is a sequence of random variables, $X_n \to X$ a.s., and Y is another random variable such that $|X_n| \leq Y$ and $E[Y^p] < \infty$. Show that $E[|X|^p] < \infty$ and that $E[|X_n - X|^p] \to 0$. (This is a strengthening of the dominated convergence theorem.)