Homework 2
Math 6710 Fall 2012

Due in class on Thursday, September 6.

1. (Durrett 1.6.1) Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is *strictly* convex, i.e.

$$\varphi(tx + (1-t)y) < t\varphi(x) + (1-t)\varphi(y)$$

   for all $x \neq y$ and $0 < t < 1$. ("Convex" only requires $\leq$ in the above inequality.) Show that under this assumption, equality holds in Jensen's inequality only in the trivial case that X is a.s. constant. That is, if X and $\varphi(X)$ are integrable and $E[\varphi(X)] = \varphi(EX)$ then $X = EX$ a.s.

2. Suppose $X_n \to X$ in probability and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $f(X_n) \to f(X)$ in probability.

   (Durrett has a proof at Theorem 2.3.4 using the "double subsequence" trick, but for this problem, please prove it directly from the definition of convergence i.p. Hint: Break up the event $\{|f(X_n) - f(X)| > \epsilon\}$ according to whether $|X| \leq M$ or $|X| > M$ for some large M. Also, remember that f is *uniformly continuous* on compact intervals.)

3. Suppose $X_n \to X$ in probability. Show that, almost surely,

$$\liminf_{n\to\infty} X_n \leq X \leq \limsup_{n\to\infty} X_n.$$

   (Either work directly or use the double subsequence trick.)

4. A set $\mathcal{S}$ of random variables is said to be **uniformly integrable** or ui if for every $\epsilon > 0$ there exists $M > 0$ such that for all $X \in \mathcal{S}$,

$$E[|X| 1_{\{|X| \geq M\}}] < \epsilon.$$

   Prove the "crystal ball condition": Let $\mathcal{S}$ be a set of random variables. If for some $p > 1$ we have $\sup_{X \in \mathcal{S}} E[|X|^p] < \infty$ (i.e. $\mathcal{S}$ is bounded in L^p norm) then $\mathcal{S}$ is uniformly integrable.

5. Let $\mathcal{S}, \mathcal{S}'$ be two sets of random variables. Suppose that $\mathcal{S}'$ is ui, and for every $X \in \mathcal{S}$ there exists $Y \in \mathcal{S}'$ with $|X| \leq |Y|$ a.s. Show that $\mathcal{S}$ is also ui.

   (In particular, the Vitali convergence theorem implies the dominated convergence theorem. Note, however, that we used dominated convergence in our proof of Vitali.)