Homework 4: Math 6710 Fall 2012

Due in class on Thursday, September 20.

1. Let μ be a probability measure on $\mathbb{R}$. The **support** of μ is the set S_μ (also denoted $\operatorname{supp}\mu$) defined by
$$S_\mu := \bigcap \{F \subset \mathbb{R} : F \text{ closed}, \mu(F) = 1\} \tag{1}$$
i.e. the intersection of all closed subsets of $\mathbb{R}$ which have measure 1. This is in some sense the set where the measure "lives".

   (a) Show that $x \in S_\mu$ iff every open set U containing x has $\mu(U) > 0$.

   (b) Show that $\mu(S_\mu) = 1$ (or equivalently, $\mu(S_\mu^C) = 0$, which may be easier to think about). (Caution: the definition (1) involves an *uncountable* intersection.)

   (c) Suppose $X_1, X_2, \ldots$ is an iid sequence of random variables with distribution μ. Show that, almost surely, the closure of the random set $\{X_1, X_2, \ldots\}$ is S_μ. That is, for P-almost every $\omega \in \Omega$, the countable set of real numbers $\{X_1(\omega), X_2(\omega), \ldots\}$ has S_μ as its closure.

   (d) Show that $\limsup_{n\to\infty} X_n = \sup S_\mu$ and $\liminf_{n\to\infty} X_n = \inf S_\mu$ almost surely.

   (e) We know from the Kolmogorov zero-one law that the event $\{\lim_{n\to\infty} X_n \text{ exists}\}$ has probability 0 or 1. Show that if it has probability 1, then there is a constant c such that $X_n = c$ almost surely. (Hint: Show that S_μ consists of one point.) So, except in trivial cases, an iid sequence of random variables never converges.

   Remark. Parts (c) and (d) are sort of a "Murphy's law" for iid sequences. S_μ is in some sense all the "possible" values for a random variable with distribution μ. So (c) says that if you try infinitely many times, any value that could "possibly" be obtained, will (approximately) be obtained. (We might not see $x \in S_\mu$ on our list $\{X_1, X_2, \ldots\}$, but we will come arbitrarily close.) In particular, according to (d), you will (approximately) obtain the most extreme "possible" values.

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2. (a) Let $x_1, x_2, \dots$ be a sequence of real numbers. Set $s_n = x_1 + \cdots + x_n$ and $s_{m,n} = x_m + \cdots + x_n$. Show that, for any fixed m, we have

$$\lim_{n\to\infty} \left| \frac{s_n - s_{m,n}}{n} \right| = 0.$$

Conclude that

$$\limsup_{n\to\infty} \frac{s_n}{n} = \limsup_{n\to\infty} \frac{s_{m,n}}{n}, \quad \liminf_{n\to\infty} \frac{s_n}{n} = \liminf_{n\to\infty} \frac{s_{m,n}}{n}.$$

(b) Let $X_1, X_2, \dots$ be a sequence of random variables, let $\mathcal{T} = \bigcap_{m=1}^{\infty} \sigma(X_m, X_{m+1}, \dots)$ be their tail σ-field, and let $S_n = X_1 + \cdots + X_n$. Show that $\limsup_{n\to\infty} \frac{S_n}{n}$ and $\liminf_{n\to\infty} \frac{S_n}{n}$ are $\mathcal{T}$-measurable random variables.

(c) If $X_1, X_2, \dots$ are *independent* random variables, then $\frac{S_n}{n}$ converges with probability 0 or 1, and if it converges, its limit is a constant. (If they are furthermore identically distributed and integrable, the SLLN tells us this limit is $E[X_1]$.)

3. Let μ be a probability measure on $\mathbb{R}^n$. A set $A \subset \mathbb{R}^n$ is said to be **μ-regular** if for every $\epsilon > 0$, there exist a closed set F and an open set U such that $F \subset A \subset U$ and $\mu(U \setminus F) < \epsilon$. (The idea is that if A is μ-regular we can approximate it from outside by open sets, and from inside by closed sets, missing only a small amount of mass.) Show that every Borel set is μ-regular. (Hint: Try showing that the μ-regular sets form a σ-field which contains every open ball.)

Remark. The statements of problems 1 a–c and 3 make sense if instead of a probability measure μ on $\mathbb{R}$ or $\mathbb{R}^n$, we use an arbitrary topological space X with its Borel σ-field. However, they may or may not actually be true. You might like to think about what topological properties of $\mathbb{R}^n$ you have actually used, and for which topological spaces your proof would still work. You could also try to think of counterexamples where things would break.