Homework 8: Math 6710 Fall 2012

Due in class on Thursday, October 18.

1. Let μ be a probability measure supported on the integers (i.e. $\mu(\mathbb{Z}) = 1$, so a random variable with distribution μ only takes integer values). Let φ be its characteristic function.

   (a) Show that φ is 2π-periodic, i.e. $\varphi(t + 2\pi) = \varphi(t)$ for all t.

   (b) Use the previous part to show that $\int_{-\infty}^{\infty} |\varphi(t)|\, dt = \infty$.

   (c) Prove the following "Fourier inversion formula": for any $k \in \mathbb{Z}$, we have

   $$\mu(\{k\}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi(t)\, dt.$$

   (d) Suppose $\mu_1, \mu_2, \dots$ are supported on the integers. Show that $\mu_n \to \mu$ weakly iff $\mu_n(\{k\}) \to \mu(\{k\})$ for every $k \in \mathbb{Z}$.

2. Let μ_n be a binomial distribution for n trials with success probability p_n. (That is, $\mu(\{k\}) = \binom{n}{k} p_n^k (1 - p_n)^{n-k}$ is the probability of getting k heads in n flips of a biased coin that comes up heads with probability p_n.) Suppose that $np_n \to \lambda$ as $n \to \infty$. Show that μ_n converges weakly to the Poisson distribution with rate parameter λ, i.e. the measure μ with $\mu(\{k\}) = e^{-\lambda} \frac{\lambda^k}{k!}$.

   This is sometimes called the **law of rare events**. It says if we make a large number n of trials of an experiment that succeeds with a proportionally small probability $p \approx \lambda/n$, the number of successes approximately follows a Poisson distribution. This is in contrast to the central limit theorem; if p is not small but instead of moderate size, then since the binomial distribution is a sum of n iid Bernoulli random variables with success probability p, the central limit theorem says the number of successes is approximately normally distributed (with mean np and variance $np(1 - p)$.

3. (Durrett 3.4.5) Let $X_1, X_2, \dots$ be iid with mean 0 and variance $\sigma^2 \in (0, \infty)$. Let $S_n = X_1 + \cdots + X_n$, and let $Q_n = X_1^2 + \cdots + X_n^2$. Show that $S_n / \sqrt{Q_n} \to N(0, 1)$ weakly.

4. Suppose X, Y are iid with mean 0 and variance 1. Show that X, Y are $N(0, 1)$ iff $\frac{X+Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y$. (Try using chfs for one direction, and the central limit theorem for the other.)

5. Let $X_1, X_2, \dots$ be iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Let $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$ (statisticians call this the **sample mean**). Let $g : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at μ and with $g'(\mu) \neq 0$. Show that:

   $$\sqrt{n} \left(\frac{g(\bar{X}_n) - g(\mu)}{\sigma g'(\mu)} \right) \to N(0, 1) \text{ weakly.}$$

   In other words, the distribution of $g(\bar{X}_n)$ is approximately $N(g(\mu), \sigma^2 g'(\mu)^2/n)$. (Notice that if $g(x) = x$, this is the central limit theorem.) This establishes that not only is $\bar{X}_n$ approximately normally distributed for large n ("**asymptotically normal**"), but so is any reasonable function of it. For reasons which I have never understood, statisticians call this fact the **delta method**.