Homework 9: Math 6710 Fall 2012

Due in class on Thursday, October 15.
Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_n\}$.

1. Suppose $\{X_n\}$ is an adapted process with state space $(S, \mathcal{S})$. Fix a measurable $B \subset S$. In class we showed that the hitting time $\tau_B = \inf\{n \geq 0 : X_n \in B\}$ is a stopping time. Let $\tau_B^{(2)} = \inf\{n > \tau_B : X_n \in B\}$ be the *second* time that X_n hits B. Show that $\tau_B^{(2)}$ is a stopping time. (It follows by the same argument that $\tau_B^{(n)}$, the nth time that X_n hits B, is a stopping time for any n.)

2. (Durrett Exercise 4.1.7) Suppose L, M are stopping times with $L \leq M$, and $A \in \mathcal{F}_L$ is an event. Set $N = L1_A + M1_{A^c}$. Show that N is a stopping time. (Idea: Wait until time L and then ask whether the event A happened. If it did, stop immediately; otherwise, wait a bit longer until time M.)

3. Let X, Y be random variables with Y integrable. By the Doob–Dynkin lemma (Proposition 5.3 in the lecture notes), there exists a measurable $f : \mathbb{R} \to \mathbb{R}$ such that $E[Y \mid X] = f(X)$ a.s. (Recall that by definition, $E[Y \mid X] := E[Y \mid \sigma(X)]$.) Show that if (X', Y') has the same joint distribution as (X, Y), then $E[Y' \mid X'] = f(X')$ a.s. for the same function f.

4. (Like Durrett Exercise 5.1.3) Prove the conditional Markov inequality: if X is a nonnegative random variable and $a > 0$, and $\mathcal{G}$ is a σ-field, then then $P(X \geq a \mid \mathcal{G}) \leq \frac{1}{a} E[X \mid \mathcal{G}]$, almost surely.

5. (Durrett Exercise 5.1.4) Suppose X is a nonnegative random variable which is not necessarily integrable, and let $\mathcal{G}$ be a σ-field. Show that there exists a unique random variable Y with $Y \in \mathcal{G}$ and for every $A \in \mathcal{G}$, $E[Y; A] = E[X; A]$ (note that both expectations exist though they could be $+\infty$). (Hint: Let $X_M = X \wedge M$, $Y_M = E[X_M \mid \mathcal{G}]$, and let $M \uparrow \infty$.)

   This extends the notion of conditional expectation from integrable random variables to nonnegative random variables.