Homework 12: Math 6710 Fall 2012

Due in class on Thursday, November 15.

1. Hat guessing! A countably infinite number of prisoners are forced by an evil warden to play the following game. Each prisoner will be randomly assigned a hat that is either black or white; he can see all the other prisoners' hats but not his own. Then each prisoner must try to guess the color of his own hat. The prisoners may strategize before the game, but once the hats are assigned they may not communicate in any way, and they do not get to hear each others' guesses. If *all but finitely many* of the prisoners can guess correctly, all the prisoners will be freed; otherwise they will all be executed.

   To express this as a probability problem, let X_n be the hat assigned to prisoner n, 0 for black and 1 for white. Let's say the warden chooses hats randomly by flipping a fair coin, so let's say the X_n are iid with $P(X_n = 0) = P(X_n = 1) = 1/2$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ be the information in the first n hats, and also let $\mathcal{G}_n = \sigma(X_1, \dots, X_{n-1}, X_{n+1}, \dots)$ be the information about every hat except the nth. Let Y_n be the nth prisoner's guess. He can see every hat except his own, so we will require that Y_n is a $\mathcal{G}_n$-measurable random variable. (Note that the guesses Y_n need not be independent, since for instance prisoners 2 and 3 could both base their guesses on the color of prisoner 1's hat.)

   Let $A_n = \{Y_n = X_n\}$ be the event that prisoner n guesses right, and then $A = \liminf A_n$ is the event that all but finitely many prisoners guess right and they are allowed to go free. Show, unfortunately, that $P(A) = 0$ using the following outline.

   (a) Show A_n is independent of $\mathcal{G}_n$ (even though Y_n is not), and that $P(A_n) = 1/2$. Conclude that for $n > m$, $P(A_n \mid \mathcal{F}_m) = 1/2$ a.s.

   (b) Show that $P(A \mid \mathcal{F}_m) \le 1/2$ a.s.

   (c) Use the Lévy zero-one law to show $P(A) = 0$.

   (d) For a sense of why this is maybe not completely obvious, consider the case where there are only a finite number N of prisoners. Give an example of a strategy such that, with probability 1/2, all of them guess correctly. (That is, in our notation, find random variables $Y_n \in \mathcal{G}_n$ such that $P(\bigcap_{n=1}^{N} A_n) = 1/2$.)

   Remark. This problem actually has surprising connections to the foundations of mathematics. Assuming the axiom of choice, there exist functions $f_n : \{0,1\}^{\mathbb{N}} \to \{0,1\}, n \in \mathbb{N}$ such that for every $x = (x(1), x(2), \dots) \in \{0,1\}^{\mathbb{N}}$, we have $f_n(x(1), \dots, x(n-1), x(n+1), \dots) = x(n)$ for all but finitely many n. If these functions f_n were measurable, then we could let $Y_n = f(X_1, \dots, X_{n-1}, X_{n+1}, \dots)$ and have $Y_n \in \mathcal{G}_n$ for which, almost surely, $Y_n = X_n$ for all but finitely many n. Using this strategy, the prisoners always win! So our argument above shows that these functions f_n cannot be measurable; they would have to be pretty nasty, and this strategy must be rather impractical. (If this interests you, you might like to write out the details.)

   This example and its relation to the axiom of choice were originally discovered by Cornell grad students Yuval Gubay and Michael O'Connor in 2004. For more on this and other infinite hat puzzles, see: Hardin, Christopher S.; Taylor, Alan D. An introduction to infinite hat problems. *Math. Intelligencer* **30** (2008), no. 4, 20–25, also available online at `http://www.math.union.edu/~hardinc/pub/introinf.pdf`.

2. (Durrett 5.5.2) Let $Z_1, Z_2, \dots$ be iid integrable random variables **with** $E[Z_i] = 0$, let θ be an integrable random variable which is independent of all the Z_n, and let $Y_n = \theta + Z_n$. Show that $E[\theta | Y_1, \dots, Y_n] \to \theta$ a.s. and in L^1.

   Idea: we would like to observe θ but it comes with some noise Z_n every time we observe it. However, our best guess $E[\theta | Y_1, \dots, Y_n]$ will converge to the true value of θ.

3. Consider a random walk where $\xi_1, \xi_2, \dots$ are iid with some arbitrary, nonconstant, integrable distribution, and let $S_n = \xi_1 + \cdots + \xi_n$. Suppose there exists a number $\theta > 0$ such that $E[\exp(-\theta \xi_i)] = 1$.

   (a) Show this implies $E[\xi_i] > 0$, so the process is biased towards increasing.

   (b) Let $a > 0$ and let $\tau = \inf\{n : S_n \le -a\}$. Prove that $P(\tau < \infty) \le \exp(-a\theta)$. (Hint: Observe that $X_n = \exp(-\theta S_n)$ is a martingale.)

   (c) Show that $\liminf_{n \to \infty} S_n > -\infty$ a.s.

   (d) If ξ_i has a normal distribution $N(\mu, \sigma^2)$ with mean $\mu > 0$ and variance σ^2, find θ such that $E[\exp(-\theta \xi_i)] = 1$. (Hint: Use the normal density, and complete a square.)

   (e) An insurance company initially has $A_0 = 10$ million dollars in assets. Its net income (premiums minus claims) ξ_i in year i is normally distributed with mean $\mu = 1$ million dollars and standard deviation $\sigma = 2$ million dollars, and independent from year to year. Use the previous parts to bound the probability that the company ever goes bankrupt (sees its assets drop below 0).

4. (Durrett 6.2.7) Let ξ_i be iid with a uniform distribution on $\{1, 2, \dots, N\}$ (i.e. $P(\xi_i = k) = 1/N$ for $k = 1, \dots, N$). Let $X_n = |\{\xi_1, \dots, \xi_n\}|$ be the number of distinct values observed up to time n. Show that X_n is a Markov chain (with respect to the filtration $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$).

5. (Durrett 6.2.8) Let S_n be symmetric simple random walk (ξ_i are iid coin flips, $S_n = \xi_1 + \cdots + \xi_n$), and let $X_n = \max\{S_i : 0 \le i \le n\}$. Show that X_n is not a Markov chain (with respect to $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$).