Hypoelliptic heat kernel inequalities on H-type groups

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**Question**

Let $L$ be a second-order differential operator (elliptic or hypoelliptic). When does it hold that

$$\left| \nabla e^{tL}f \right| \leq K(t)e^{tL} \left| \nabla f \right|$$

for all $f \in C_c^\infty$?
Consequences of gradient estimates $|\nabla e^{tL}f| \leq K(t)e^{tL} |\nabla f|$

- Logarithmic Sobolev inequalities
- Poincaré inequalities
- Isoperimetric inequalities
- Several other functional inequalities

[Bakry et. al. 2008]
The elliptic case: Ricci curvature

**Theorem (Bakry 1985 et seq)**

If $L = \Delta$ is the Laplace-Beltrami operator on a Riemannian manifold $M$, then

$$\left| \nabla e^{t\Delta} f \right| \leq e^{kt} e^{t\Delta} |\nabla f| \quad \forall f \in C^\infty_c(M)$$

(i.e. $K(t) = e^{kt}$) if and only if

$$\text{Ric}(v, v) \geq -k \, |v|^2 \quad \forall v \in TM.$$
The hypoelliptic case

- \{X_1, \ldots, X_k\} a bracket-generating set of vector fields on \(M\); i.e.
  \[
  \text{span}\{X_i(p), [X_i, X_j](p), [[X_i, X_j], X_k](p), \ldots\} = T_pM \text{ for all } p \in M
  \]

- \(L := X_1^2 + \cdots + X_k^2\) is hypoelliptic [Hörmander 1967]
- \(\nabla f := (X_1f, \ldots, X_kf)\)
- Prototypical example: \(M = G\) a nilpotent Lie group, \(\{X_i\}\) left-invariant and generate Lie algebra \(g\) [Rothschild, Stein 1976]
The Heisenberg group $\mathbb{H}_1$

- Let $G$ be the Heisenberg group $\mathbb{H}_1 \cong \mathbb{R}^3$, with group operation
  \[(x_1, x_2, z)(x'_1, x'_2, z') = \left(x_1 + x'_1, x_2 + x'_2, z + z' + \frac{1}{2}(x_1x'_2 - x_2x'_1)\right)\]

- Nilpotent, step 2, stratified Lie group; one-dimensional center
- Bracket generating vector fields
  \[X_1 := \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial z}, \quad X_2 := \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial z}\]

- $L := X_1^2 + X_2^2$ is hypoelliptic
- “Ric $= -\infty$”
Theorem (H.-Q. Li 2006; Bakry et al 2008)

For the Heisenberg group $\mathbb{H}_1$, there exists a constant $K$ such that

$$|\nabla e^{tL}f| \leq Ke^{tL} |\nabla f|.$$  \hspace{1cm} (1)

What does the proof use?
Heat kernel estimates on $\mathbb{H}_1$

Key ingredient: precise pointwise estimates on the heat kernel

$p_t := e^{tL}\delta_0$ (smooth function).

**Theorem (H.-Q. Li 2007)**

*For the Heisenberg group $\mathbb{H}_1$, there exists an explicit function $Q_t$ of polynomial growth such that*

$$C_1 Q_t(g) e^{-\frac{1}{4t}d(0,g)^2} \leq p_t(g) \leq C_2 Q_t(g) e^{-\frac{1}{4t}d(0,g)^2}$$

*for all $g \in \mathbb{H}_1$, where $d$ is the subriemannian or Carnot-Carathéodory distance on $\mathbb{H}_1$.\n
▶ Also shown for Heisenberg-Weyl groups $\mathbb{H}_n$ of dimension $2n + 1$ (one-dimensional center).
Subriemannian geometry on a manifold $M$

- Use $\{X_i\}$ as orthonormal frame for inner product (subriemannian metric) on a “horizontal” subbundle $\mathcal{H}$ of $TM$
- $M$ becomes a subriemannian manifold
- Chow’s theorem: bracket generating condition $\Rightarrow$ can join points by horizontal curves
- Carnot-Carathéodory distance $d(g, h)$ measures length of shortest curve joining points $g, h$; induces manifold topology

Carnot-Carathéodory unit ball in $\mathbb{H}_1$. 
H-type groups [Kaplan 1980]

- Generalization of Heisenberg groups $\mathbb{H}_n$
- Step 2 stratified nilpotent Lie group $G$
- Diffeomorphic to $\mathbb{R}^{2n+m}$
- Center may have any dimension $m$
- Vector fields $\{X_1, \ldots, X_{2n}\}$ are “strongly” bracket generating
- Correspond to representations of Clifford algebras $\mathcal{C}\ell(\mathbb{R}^m, Q)$, where $Q > 0$ (depending on sign convention)
- $m = 1$ gives Heisenberg-Weyl groups $\mathbb{H}_n$; $m = n = 1$ gives $\mathbb{H}_1$. 

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Hypoelliptic heat kernel inequalities on H-type groups
Heat kernel results for H-type groups

Theorem (E. 2009)

For an H-type group $G$, there exists an explicit function $Q_t$ of polynomial growth such that

$$C_1 Q_t(g) e^{-\frac{1}{4t} d(0,g)^2} \leq p_t(g) \leq C_2 Q_t(g) e^{-\frac{1}{4t} d(0,g)^2}$$

for all $g \in G$.

Theorem (E. 2009)

For an H-type group $G$, there exists a constant $K$ such that

$$|\nabla e^{tL} f| \leq K e^{tL} |\nabla f|.$$

(2)
Pointwise estimates on $p_t$: ideas in the proof

- Explicit integral formula for $p_t$:

$$p_t((x, z)) = (2\pi)^{-m}(4\pi)^{-n} \int_{\mathbb{R}^m} e^{i\lambda \cdot z - \frac{1}{4}|\lambda| \coth(t|\lambda|)|x|^2} \left( \frac{|\lambda|}{\sinh(t|\lambda|)} \right)^n d\lambda$$

- Handle oscillations by method of steepest descent [Gaveau 1977 and others]

- Care adds uniformity to steepest descent asymptotics

- Only covers certain regions of $G$ (far from center)
Pointwise estimates on $p_t$: ideas in the proof (continued)

- Arbitrary center dimension $m$ adds complexity
- If $m$ odd: Radon transform (polar coordinates), residue computations about essential singularity (if $m$ odd)
- Hadamard descent (if $m$ even):
  \[
p_1^{(n,m)}(x, z) = \int_{\mathbb{R}} p_1^{(n,m+1)}(x, (z, z_{m+1})) \, dz_{m+1}
\]
- An H-type group with dimensions $(n, m + 1)$ need not exist!
We follow [Bakry et. al. 2008]’s proof for $\mathbb{H}_1$. Proof ingredients:

- Precise heat kernel estimates
- Geodesic coordinates, with precise estimates for Jacobian
- Isoperimetric-type estimate, akin to

$$\mu_t(B(0, r)^C) \leq C \mu_t^{\text{surface}}(\partial B(0, r))$$

- $L^1$ Poincaré inequality on geodesic balls [see Maheux, Saloff-Coste 1995]:

$$\int_B \left| f - \frac{1}{|B|} \int_B f \right| \leq C \int_B |\nabla f|$$ (3)
Extensions to other Lie groups

- Reliance on “nice” formula for $p_t$ makes this approach difficult to extend
- Other approaches: Probabilistic methods, infinite dimensional calculus

Theorem (Melcher 2004)

For any Lie group $G$ and every $p > 1$, we have

$$
|\nabla e^{tL}f|^p \leq K_p(t)e^{tL}(|\nabla f|^p).
$$
The end

Thanks!