The Biachi Identity in Path Space

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Abstract

this is a test.

1 Two Geometric Problems

Throughout this paper, we will be interested in *local* problems; therefore, we will take the base manifold X to be contractible unless otherwise noted.

Given a connection $\nabla = d + \omega$ with corresponding curvature $F = d\omega + \omega \wedge \omega$, the Bianchi identity states that the covariant derivative of F vanishes:

$$0 = \nabla F = dF + \omega \wedge F$$

What does the Bianchi identity tell us about the 2-form F? In the abelian case of $G = \mathbb{C}^*$, the wedge products vanish and we find that

$$\nabla F = dF = 0$$

so that F is simply a closed $\mathbb{C}$-valued 2-form. We can then find an antiderivative $\omega \in \Omega^1(X; \mathbb{C})$ with

$$\operatorname{curv} \omega = d\omega = F$$

since the $\omega \wedge \omega$ term in the curvature is again zero. Finally, ω is unique up to the replacement $\omega \mapsto \omega + d \log \psi$ for any function $\psi \in C^\infty(X, \mathbb{C}^*)$.

Of course, this is nothing more than the classical Poincaré lemma, in the following form:

Lemma 1.1. *Let X be a manifold and A an abelian Lie group with corresponding Lie algebra $\mathfrak{a}$. Then*

$$\begin{tikzcd} 0 \arrow[r] & \underline{A} \arrow[r] & C^\infty_X(A) \arrow[r, "d\log"] & \Omega^1_X(\mathfrak{a}) \arrow[r, "\operatorname{curv}"] & \Omega^2_X(\mathfrak{a}) \arrow[r, "d"] & \cdots \end{tikzcd}$$

is an exact sequence of sheaves.

Note the slight abuse of notation: when the sheaf is of groups rather than rings, *exact* means that each sheaf acts on the next sheaf to the right and two sections have the same image if and only if they are in the same orbit of this action. In particular, the constants $\underline{\mathbb{C}}^*$ act on $\mathbb{C}^\infty_X(\mathbb{C}^*)$ by multiplication and the sections $f \in \mathbb{C}^\infty_X(\mathbb{C}^*)$ act on the connection forms $A \in \Omega^1_X(\mathbb{C})$ by

$$A \mapsto A + d \log f$$

****** pictures of abelian versus nonabelian bianchi identity? ******

In the nonabelian case, we still have a truncated version of the Poincaré lemma, roughly corresponding to the fundamental theorem of calculus on Lie groups.

Lemma 1.2. *Let X be a manifold and G a Lie group with Lie algebra $\mathfrak{g}$. Then*

$$\begin{tikzcd} 1 \arrow[r] & \underline{G} \arrow[r] & C^\infty_X(G) \arrow[r, "{}^*\theta"] & \Omega^1_X(\mathfrak{g}) \end{tikzcd}$$

is an exact sequence of sheaves, where ${}^*\theta$ denotes the pullback of the Maurer-Cartan form:

$$f^*\theta = f^{-1}df$$

Furthermore, if $0 = d\omega + \omega \wedge \omega = \operatorname{curv}\,\omega$ then $\omega = f^*\theta$, where f is unique up to left multiplication by a constant. However,

$$\begin{tikzcd} 1 \arrow[r] & \underline{G} \arrow[r] & C^\infty_X(G) \arrow[r, "{}^*\theta"] & \Omega^1_X(\mathfrak{g}) \arrow[r, "\operatorname{curv}"] & \Omega^2_X(\mathfrak{g}) \end{tikzcd}$$

is not *exact: there exist connections ω_1, ω_2 such that* $\operatorname{curv}\,\omega_1 = \operatorname{curv}\,\omega_2$ *but ω_1 and ω_2 are not gauge equivalent, even locally.*

Here, the action of constants on sections is by left multiplication, and sections act on connections by

$$\omega \mapsto f^{-1}\omega f + f^*\theta$$

Note the slight abuse of terminology: by *exact*, we will mean each sheaf acts on the next sheaf to the right, and that the level sets of the outgoing map are equal to the orbits of this action.

This paper is concerned with the problem of extending this sequence one more term to the right. Unfortunately, the obvious necessary condition for F to be the curvature of some connection $\nabla = d + \omega$ is the Bianchi identity

$$0 = d^\nabla F = dF + \omega \wedge F$$

which cannot even be formulated without already knowing the antiderivative ω for F.

2 Lifting to Path Space

Recall that connections are in 1-1 correspondence with parallel transport operators. Formally, this gives us a way of replacing a degree-1 object on X (the connection form) with a degree-0 object on $\mathsf{P}X$ (the transport). By systematically lifting our geometric objects into path space, we might hope to turn the curvature form into a degree-1 object and deal with it in simpler terms.

Let us first analyze the abelian case of $G = \mathbb{C}^*$ to pinpoint the features we would like to find in the nonabelian case.

We wish to single out a subset of the forms on path space which behave nicely with respect to the underlying path structure. There are source and target maps $s, t : \mathsf{P}X \rightrightarrows X$, so given compose-able paths α, β with vectorfields $V \in T_\alpha \mathsf{P}X$, $W \in T_\beta \mathsf{P}X$ such that

$$dt(W) = ds(V)$$

we can define the composition vectorfield $V \circ W \in T_{\alpha\circ\beta}\mathsf{P}X$. A k-form ω on $\mathsf{P}X$ is called *functorial* if, for all compose-able vectorfields $V_i \in T_\alpha \mathsf{P}X$, $W_i \in T_\beta \mathsf{P}X$ we have

$$\omega_{\alpha\circ\beta}(V_1 \circ W_1, \ldots, V_k \circ W_k) = \omega_\alpha(V_1, \ldots, V_k) + \omega_\beta(W_1, \ldots, W_k)$$

Since we will mainly be concerned with this functorial property, $\Omega^k_{\mathsf{P}X}$ will refer to the sheaf of functorial k-forms on $\mathsf{P}X$, while $\mathcal{A}^k_{\mathsf{P}X}$ will be used for the sheaf of general k-forms.

To clarify notation, lowercase roman letters (v_i, w_i, ...) will be used for vectorfields on X, while uppercase roman letters (V_i, W_i, ...) will be reserved for vectorfields on $\mathsf{P}X$.

Given a k-form on X, there is the *transgression map* $\tau : \Omega^{k+1}_X(\mathbb{C}) \longrightarrow \Omega^k_{\mathsf{P}X}(\mathbb{C})$ given by

$$(\tau\omega)_\gamma(v_1, \ldots, v_k) = \int_\gamma i_{v_1\wedge\cdots\wedge v_k}\omega$$

The transgression map has an inverse $\varepsilon : \Omega^k_{\mathsf{P}X} \longrightarrow \Omega^{k+1}_X$ called *infinitesimal evaluation* which works as follows: to compute

$$(\varepsilon\omega)_p(v_0, \ldots, v_k)$$

we find a parametrized path $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = p$ and $\gamma'(0) = v_0$. Then, defining $\gamma^t = \gamma|_{[0,t]}$ we have

$$(\varepsilon\omega)_p(v_0, \ldots, v_k) = \lim_{t\to 0} \frac{1}{t} \cdot \omega_{\gamma^t}(v_1, \ldots, v_k)$$

Theorem 2.1. *ε is a well-defined map from $\Omega^k_{\mathsf{P}X}$ to Ω^{k+1}_X which is a two-sided inverse to τ.*

Proof. Throughout this proof, let $\tilde{\omega}$ be any lift of ω into the sheaf of k-forms on the *parametrized* path space of X.

It follows from functorality $\omega_{\underline{p}} = 0$, where $\underline{p}$ is the constant path at p. We now show that $\varepsilon\omega$ does not depend on the choice of γ. Define the function $f : \mathsf{P}X \longrightarrow \mathbb{R}$ by

$$f(\gamma) = i_{v_1 \wedge \cdots \wedge v_k} \omega_\gamma = \omega_\gamma(v_1, \ldots, v_k)$$

Then

$$(\varepsilon\omega)_p(v_0, \ldots, v_k) = df_{\underline{p}}(v_0)$$

which does not involve any arbitrary choices.

The preceding argument proves that ε is well-defined, but it remains to be shown that $\varepsilon\omega$ is antisymmetric. It suffices to verify that

$$(\varepsilon\omega)_p(v_1, v_1, v_2, \ldots, v_k) = 0$$

First, extend v_1 to a parametrized curve $\gamma : [0,1] \longrightarrow X$, extend the vectors v_i to vectorfields $\tilde{v}_i$ along γ and use the previous computation to write

$$\omega_\gamma(\tilde{v}_1, \ldots, \tilde{v}_k) = \int_0^1 \gamma' \cdot (\varepsilon\omega)_{\gamma(t)}(\gamma', \tilde{v}_2, \ldots, \tilde{v}_k)\ dt$$

***** ☐

Corollary 2.2. *By writing* $\omega \in \Omega^k_{\mathsf{P}X}$ *as* $\omega = \tau\varepsilon\omega$*, any functorial k-form may be realized as*

$$\omega_\gamma(v_1, \ldots, v_k) = \int_0^1 \kappa_{\gamma(t)}(\gamma', v_1, \ldots, v_k)\ dt$$

where $\kappa = \varepsilon\omega$*.*

Transgression and infinitesimal evaluation allow us to move forms between X and $\mathsf{P}X$. It might be expected that they intertwine the exterior derivatives on X and $\mathsf{P}X$, but this is not quite true. By expanding the composition

$$\Omega^k_{\mathsf{P}X} \xrightarrow{\varepsilon} \Omega^{k+1}_X \xrightarrow{d} \Omega^{k+2}_X \xrightarrow{\tau} \Omega^{k+1}_{\mathsf{P}X}$$

we may derive the formula

$$\begin{aligned} (\tau d\varepsilon\omega)_\gamma &= \int_0^1 \left(\mathcal{L}_{\gamma'}(\varepsilon\omega)\right)_{\gamma(t)} dt - (d\omega)_\gamma \\ &= (\varepsilon\omega)_{t(\gamma)} - (\varepsilon\omega)_{s(\gamma)} - (d\omega)_\gamma \end{aligned}$$

This is another sort of exterior derivative, applicable only to *functorial* forms, which we will denote

$$\mathsf{d} = \tau d\epsilon : \Omega^k_{\mathsf{P}X}(\mathbb{C}) \longrightarrow \Omega^{k+1}_{\mathsf{P}X}(\mathbb{C})$$

It follows immediately that $\mathsf{d}^2 = 0$, leading to the *functorial cohomology*

$$H^n_{\mathsf{fun}}(\mathsf{P}X;\mathbb{C}) = \ker(\Omega^k_{\mathsf{P}X}(\mathbb{C})\xrightarrow{\mathsf{d}}\Omega^{k+1}_{\mathsf{P}X}(\mathbb{C}))/\mathrm{im}\ (\Omega^{k-1}_{\mathsf{P}X}(\mathbb{C})\xrightarrow{\mathsf{d}}\Omega^k_{\mathsf{P}X}(\mathbb{C}))$$

Lemma 2.3. *Transgression is a morphism of complexes of sheaves:*

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{C}}^*_X \arrow[r, "\iota"] \arrow[d, "\tau"] & C^\infty_X(\mathbb{C}^*) \arrow[r, "d\log"] \arrow[d, "\tau"] & \Omega^1_X(\mathbb{C}) \arrow[r, "d"] \arrow[d, "\exp\tau"] & \Omega^2_X(\mathbb{C}) \arrow[r, "d"] \arrow[d, "\tau"] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & \underline{\mathbb{C}}^*_{\mathsf{P}X} \arrow[r, "\iota"'] & C^\infty_{\mathsf{P}X}(\mathbb{C}^*) \arrow[r, "\mathsf{d}\log"'] & \Omega^1_{\mathsf{P}X}(\mathbb{C}) \arrow[r, "\mathsf{d}"'] & \cdots
\end{tikzcd}$$

Stated in more geometric language, this lemma means that flat $\mathbb{C}^*$-connections on $\mathsf{P}X$ are equivalent to *arbitrary* $\mathbb{C}^*$-connections on X. In the more general case of X not contractible, the holonomy of this flat connection corresponds to the monopole charges of the related 2-form on X.

The idea of relating curvature forms on X to connections on $\mathsf{P}X$ in the nonabelian case will be the focus of the rest of this paper. A connection on $\mathsf{P}X$ is closely related to the notion of *surface transport* — that is, transport of a quantity from one boundary curve of a surface to the other, preferably in a parametrization-invariant manner.

3 Nonabelian Surface Transport

Throughout this section, suppose that S is a surface in X bounded by two curves γ_0, γ_1 connecting x_0 to x_1.

We may think of a 2-form $F \in \Omega^2_X(\mathbb{C})$ as an assignment to each infinitessimal surface element an infinitessimal element of $\mathbb{C}^*$. Then we could compute a *surface-ordered product*

$$P_F(S) = \exp \int_S F$$

In the language of the last chapter, if F is the curvature of a connection A then $P_F = \exp\tau A$, so P_F is independent of the surface S connecting γ_0 to γ_1 exactly when $dF = 0$.

Now consider the nonabelian case $F \in \Omega^2_X(\mathfrak{g})$. We are unable to define a surface-ordered product easily due to the lack of natural ordering on S. Without this ordering, the nonabelian product of G prevents us from defining a surface-ordered product unambiguously.

As a first approximation to a solution, we might allow G to be a *bigroup*: a set carrying two operations $\cdot$, $\circ$ such that it becomes a group under each, with the same identity. Then if S is the image of a homotopy

$$h : [0,1]\times[0,1] \longrightarrow X \qquad h(s,0)=\gamma_0(s),\ h(s,1)=\gamma_1(s)$$

we could use one product for multiplication in the ∂_t direction, and the other product for multiplication in the ∂_s direction. This surface-ordered product is well-defined as long as the *exchange identity*

$$(g_4 \circ g_3) \cdot (g_2 \circ g_1) = (g_4 \cdot g_2) \circ (g_3 \cdot g_1)$$

holds in G.

Lemma 3.1. (Eckmann-Hilton) *If G is a bigroup which satisfies the exchange identity then $\cdot = \circ$ and both products are abelian.*

Proof. To see that the products agree,

$$x \cdot y = (x \circ 1) \cdot (1 \circ y) = (x \cdot 1) \circ (1 \cdot y) = x \circ y$$

Knowing that $\cdot = \circ$,

$$x \cdot y = (1 \cdot x) \cdot (y \cdot 1) = (1 \cdot y) \cdot (x \cdot 1) = y \cdot x$$

$\square$

Rather than flail around for a weakened notion of bigroup which will work for surface transport, we take a detour into abstraction and let categories do the work for us.

4 Categorified Groups

In order to have a surface-ordered product, we need something which acts like a group (for horizontal multiplication of surface elements) but also supports a notion of composition (for vertical multiplication). There is a general procedure called *internalization* in category theory for doing just this sort of hybridization.

Given two kinds of mathematical objects K_1, K_2, let C be the category of K_1s and let $\mathcal{D}$ be a set of diagrams which define K_2 axiomatically. Then a K_2 *internalized in* K_1 is a realization of $\mathcal{D}$ as diagrams in C. Rather than dwelling on the details of this abstraction, we will simply present two examples before continuing.

Example 4.1. Let $\mathcal{D}$ be the diagram defining a direct sum ($\oplus$). Then internalizing $\oplus$ in Groups gives the direct sum of groups, while internalizing $\oplus$ in the divisibility poset of $\mathbb{N}$ gives the notion of least common multiple.

Example 4.2. Groups may be defined by diagrams, so we can internalize the notion of "group". A group internalized in $\mathsf{FinSets}$ is a finite group, and a group internalized in Top is a topological group.

Now, we are after a group which has a notion of composition. Composition is codified in the notion of a category, and categories may themselves be defined entirely by diagrams. So the type of object we are after is a *2-group*[1]: a category internalized in Groups. Such a 2-group G is a category with a *group* of objects, a *group* of arrows, and all maps (source, target, identity, composition) are group homomorphisms.

Structurally, such a 2-group G consists of a group H of objects and a group G of arrows with source at $1 \in H$, along with a target map $t_0 : G \longrightarrow H$ and an action $\alpha : H \longrightarrow Aut(G)$ which satisfies the intertwining property

$$\begin{tikzcd} G \arrow[r, "\alpha(h)"] \arrow[d, "t_0"'] & G \arrow[d, "t_0"] \\ H \arrow[r, "\mathrm{Ad}(h)"] & H \end{tikzcd}$$

and the Peiffer identity

$$\begin{tikzcd} G \arrow[d, "t_0"'] \arrow[dr, "\mathrm{Ad}"] & \\ H \arrow[r, "\alpha"] & Aut(G) \end{tikzcd}$$

Given such a 2-group G, H is the *group of objects* and $G \ltimes_\alpha H$ is the *group of arrows*. The source and target maps are given by

$$s(g,h) = h, \qquad t(g,h) = t_0(g) \cdot h$$

The identity arrow associated to the object h is

$$\mathrm{id}_h = (1,h)$$

and composition is given by

$$(g_2, t(g_1) \cdot h) \circ (g_1, h) = (g_2 \cdot g_1, h)$$

The intertwining property and the Peiffer identity ensure that G satisfies the axioms of a category.

The fact that $\circ$ is a homomorphism means that any 2-group satisfies the exchange identity, verified by a straightforward but messy computation using both the intertwining property and the Peiffer identity:

$$((g_3, t(g_1)h_1) \circ (g_1, h_1)) \cdot ((g_4, t(g_2)h_2) \circ (g_2, h_2)) =$$
$$= (g_3 g_1, h_1) \cdot (g_4 g_2, h_2) = (g_3 g_1 \cdot \alpha(h_1)(g_4 g_2), h_1 h_2)$$

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[1] In this paper, we only deal with *strict* 2-groups, also called *crossed modules.* A weaker notion of 2-group may be achieved by internalizing groups in Cat and demoting identities to isomorphisms. This is studied extensively in [2] and [1].

$$\begin{aligned}((g_3, t(g_1)h_1) \cdot (g_4, t(g_2)h_2)) \circ ((g_1, h_1) \cdot (g_2, h_2)) &= \\ (g_3\alpha(t(g_1)h_1)(g_4), t(g_1)h_1t(g_2)h_2) \circ (g_1\alpha(h_1)(g_2), h_1h_2) &= \\ (g_3g_1\alpha(h_1)(g_4)g_1^{-1}, t(g_1)h_1t(g_2)h_2) \circ (g_1\alpha(h_1)(g_2), h_1h_2) &= \\ (g_3g_1 \cdot \alpha(h_1)(g_4g_2), h_1h_2)&\end{aligned}$$

In other words, 2-groups are exactly the sort of object which can be consistently multiplied over a *surface*, just as (1-)groups are exactly the sort of object which can be consistently multiplied over a *path*.

Example 4.3. One of the most common 2-groups is the *automorphism 2-group* Aut_G of a group G. The object group is $Aut(G)$ and the arrow group is $G \ltimes_\alpha Aut(G)$, where the action α is evaluation. The target map is $t_0 = \mathrm{Ad} : G \longrightarrow Aut(G)$.

Closely related is the *adjoint 2-group* Ad_G of G. This is the subcategory of Aut_G connected to the identity — in other words, exactly like Aut_G but with $Inn(G)$ replacing $Aut(G)$.

Example 4.4. Given pointed topological spaces X, Y with $Y \subset X$, there is the long exact homotopy sequence

$$\ldots \pi_k(X) \xrightarrow{q} \pi_k(X;Y) \xrightarrow{\partial} \pi_{k-1}(Y) \xrightarrow{i} \pi_{k-1}(X) \longrightarrow \ldots$$

Associated to this sequence is the fundamental 2-group with object group $\pi_1(Y)$, arrow group $\pi_2(X;Y)$, target map $t_0 = \partial$ and action of $\pi_1(Y)$ on $\pi_2(X;Y)$ by basepoint change. Historically, this was the first known example of a 2-group and captures nonabelian homotopy information even though $\pi_2(Z)$ is always abelian (again by the Eckmann-Hilton argument!).

Example 4.5. Given any group G there are two natural ways to construct a 2-group. The first is to use the 1-element group as the group of objects and G as the group of arrows. On the other hand, there is a unique 2-group Sk_G called the *sketch* of G such that G is the group of objects and $Hom(g_1, g_2)$ is a single element for all g_1, g_2.

References

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