A Geometric Transformation Theory for PDE

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A surface in $\mathbb{E}^3$ is \textbf{pseudospherical} if it has constant Gaussian curvature $K = -1$. 

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The graph of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defines a pseudospherical surface if and only if $f$ satisfies the PDE

$$
\left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 - \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} + \left( 1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \right)^2 = 0
$$
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Note that this equation fails to be even quasilinear — it can only be classified as “rather unpleasant”.
By looking for solutions with a rotational symmetry, we can derive a new equation for the profile curve of a rotationally symmetric pseudospherical surface. This leads to the classical pseudosphere:
Bianchi’s Theorem

**Theorem (Bianchi)**

Let \( f, \hat{f} \) parametrize two surfaces in \( \mathbb{R}^3 \), and let \( N, \hat{N} \) be the corresponding normal maps. Suppose further that the four relations

- \(|f - \hat{f}| = 1\)
- \(N \perp \hat{N}\)
- \(N \perp f - \hat{f}\)
- \(\hat{N} \perp f - \hat{f}\)

hold at each point. Then \( f \) and \( \hat{f} \) are both parameterizations of pseudospherical surfaces.
The Bianchi relations are *geometric* in the sense that if $f \sim \hat{f}$ and $g$ is any Euclidean motion, $g \cdot f \sim g \cdot \hat{f}$.

Two surfaces are Bianchi-related at $p, \hat{p}$ exactly when the tangent planes are in the geometric configuration depicted below:

**Figure:** Two planes related by a $90^\circ$ unit-distance screw motion.
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**Theorem (Lie)**

Let $f$ parameterize a pseudospherical surface. Then there exists a pseudospherical surface $\hat{f}$ which is Bianchi-related to $f$. The transformed surface $\hat{f}$ may be computed from $f$ by solving a series of ordinary differential equations.
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Translation: Once you have one solution to the $K = -1$ equations, it is “easy” to compute a new one!
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Let $f : M \rightarrow \mathbb{E}^3$ be a parameterized surface.

**Definition**

A *Euclidean frame* over $f$ is a map $F : M \rightarrow ASO(3)$ such that

$$F \cdot O = f$$

where $O$ is the origin of $\mathbb{E}^3$. 
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Using the standard representation of $ASO(3)$, $F$ must take the block form

$$F = \begin{bmatrix} 1 & 0 \\ f & R \end{bmatrix}, \quad R \in SO(3)$$
A Euclidean frame \( F = \begin{bmatrix} 1 & 0 \\ f & R \end{bmatrix} \) is adapted if

\[ \text{Re}_3 = N \]

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**Lemma**

A Euclidean frame \( F \) is adapted if and only if \( e^3 (F^{-1} dF) = 0 \).
Let $\beta_\theta \in ASO(3)$ be a $90^\circ$ unit-displacement screw motion in the $\cos(\theta)e_1 + \sin(\theta)e_2$ direction.

**Lemma**

*If $f, \hat{f} : M \rightarrow \mathbb{E}^3$ are Bianchi-related and $F$ is an adapted frame over $f$, then there is a unique function $\theta : M \rightarrow S^1$ such that*

$$\hat{F} = F \cdot \beta_\theta$$

*is an adapted frame over $\hat{f}$.***
To find the function $\theta$, we need to know when $F \cdot \beta_\theta$ is adapted. Let

$$F^{-1}dF = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\tau^1 & 0 & \lambda & \nu^1 \\
\tau^2 & -\lambda & 0 & \nu^2 \\
0 & -\nu^1 & -\nu^2 & 0
\end{bmatrix}$$

Then $F \cdot \beta_\theta$ is adapted if and only if

$$0 = e^3(\text{Ad}(\beta^{-1}_\theta)(F^{-1}dF) + \beta^{-1}_\theta d\beta_\theta)
= \lambda + \sin(\theta)\tau^1 - \cos(\theta)\tau^2 - d\theta$$
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Then $F \cdot \beta_\theta$ is adapted if and only if

$$ 0 = e^3(\text{Ad}(\beta_\theta^{-1})(F^{-1}dF) + \beta_\theta^{-1}d\beta_\theta) $$

$$ = \lambda + \sin(\theta)\tau^1 - \cos(\theta)\tau^2 - d\theta $$

...but why should this equation have any solutions?
We have no reason to expect that a function $\theta$ exists which also satisfies

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But if there were such a $\theta$, differentiating the above equation and using the relation $d(F^{-1}dF) = -F^{-1}dF \wedge F^{-1}dF$ gives

$$0 = d\lambda + \cos(\theta)d\theta \wedge \tau^1 + \sin(\theta)d\tau^1$$
$$+ \sin(\theta)d\theta \wedge \tau^2 - \cos(\theta)d\tau^2$$

$$= \nu^1 \wedge \nu^2 + \tau^1 \wedge \tau^2$$
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$$= \nu^1 \wedge \nu^2 + \tau^1 \wedge \tau^2$$

$$= K |df|^2 + |df|^2$$

Note the miracle — all dependence on $\theta$ has vanished!
We can interpret the previous calculation as saying that the overdetermined system of PDEs

$$d\theta = \lambda + \sin(\theta)\tau^1 - \cos(\theta)\tau^2$$

is consistent if and only if $K = -1$ on the surface $f$. 
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**Lemma**

A consistent overdetermined system of first-order PDEs can always be solved by a sequence of ordinary integrations.

This proves Lie's theorem: if \( f \) is pseudospherical, then by solving a sequence of ODEs we may find a second Bianchi-related pseudospherical surface \( \hat{f} \).
Application: Transforming the Pseudosphere
Underlying Equations

Using the curvature-line adapted frame $F$ on the pseudosphere:

$$\frac{\partial \theta}{\partial x} = \sin \theta \tanh x, \quad \frac{\partial \theta}{\partial y} = (1 - \cos \theta) \text{sech} x$$
Application: Transforming the Pseudosphere

Underlying Equations

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Starting with $\theta(0, 0) = \pi$, the second equation may be integrated along the curve $x = 0$ to obtain

\[
\theta(0, y) = \pi + 2 \tan^{-1}(y)
\]

Then integrate along each curve $y' = 0$ to get

\[
\theta(x, y) = \pi + 2 \tan^{-1}(y \sech x)
\]
Results

Using the $\theta$ constructed above, $\hat{F} = F \cdot \beta_\theta$ is an adapted frame for a new pseudospherical surface $\hat{f}$. 
Application: Transforming the Pseudosphere

Results

Using the $\theta$ constructed above, $\hat{F} = F \cdot \beta_{\theta}$ is an adapted frame for a new pseudospherical surface $\hat{f}$. In this case we obtain Kuen’s Surface, a very nontrivial surface with $K = -1$. 
Homogeneous Spaces

**Definition (Homogeneous Space)**

A **homogeneous space** is a smooth manifold $M$ equipped with a smooth transitive left action of a Lie group $G$. We call $G$ the **structure group** of $M$. 
Examples of Homogeneous Spaces

1. The round $n$-sphere $S^n$ is a homogeneous space for the rotation group $SO(n + 1)$. 
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3. The conformal sphere $\mathbb{C}P^1$ is a homogeneous space for the Möbius group $PSL(2, \mathbb{C})$. Note that this is a distinct space from $S^2$, even though they are diffeomorphic!
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4. If $H$ is any Lie subgroup of $G$, then the coset space $G/H$ is a homogeneous space with structure group $G$. 
An invariant relation on a homogeneous space $M$ is a subset of $M \times M$ which is fixed under the diagonal action of $G$.

**Theorem**

*The double coset space $\mathcal{R}_M = H \backslash G / H$ is isomorphic to the space of atomic invariant relations on $M = G/H$.**
Invariant Relations

An **invariant relation** on a homogeneous space $M$ is a subset of $M \times M$ which is fixed under the diagonal action of $G$.

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**Corollary**

Given any point $x \in M$ and atomic invariant relation $r \in \mathcal{R}_M$, there is an $H$-family of related points $y \sim_r x$.

For $U_T^3$, $SO(2) \backslash ASO(3) / SO(2)$ is coordinatized by the four relations which appear in Lie’s theorem.
Generalized Bianchi and Lie Theorems

Now we can seek a generalization of Bianchi’s relations and Lie’s theorem to other homogeneous spaces, along the lines of

**Theorem (Generalized Bianchi)**

Let \( f, \hat{f} : X \rightarrow M \) be two surfaces in the homogeneous space \( M \) and \( r \in \mathcal{R}_M \) a relation such that [??]. If \( f_p \sim_r \hat{f}_p \) for all \( p \in X \), then \( f \) and \( \hat{f} \) satisfy the differential equations \( \Delta, \hat{\Delta} \) respectively.

**Theorem (Generalized Lie)**

If \( f \) satisfies \( \Delta \) and \( r \in \mathcal{R}_M \) is as above, then there is a surface \( \hat{f} \) such that \( f \sim_r \hat{f} \) and \( \hat{f} \) satisfies \( \hat{\Delta} \). \( \hat{f} \) may be constructed from \( f \) by solving a sequence of ODEs.
Every smooth first-order ODE $\Delta$ for one function of one variable is of the form

$$F_\Delta(x, y, y') = 0$$

for some smooth function $F_\Delta : \mathbb{R}^3 \longrightarrow \mathbb{R}$. We may think of formal solutions to $\Delta$ as being curves $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^3$ such that $F_\Delta \circ \gamma = 0$. 
Every smooth first-order ODE $\Delta$ for one function of one variable is of the form

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for some smooth function $F_{\Delta} : \mathbb{R}^3 \rightarrow \mathbb{R}$. We may think of formal solutions to $\Delta$ as being curves $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$ such that $F_{\Delta} \circ \gamma = 0$.

Giving $\mathbb{R}^3$ the coordinates $x, y, p$, a formal solution is an actual solution to $\Delta$ exactly when

$$\gamma^*(dy - p \, dx) = 0$$
The same idea works for general differential equations, of any order and in any number of dependent and independent variables.
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**Lemma**

*To any PDE $\Delta$ there is a manifold $M_\Delta$ and a differential ideal $\Theta_\Delta \leq \Omega^\bullet_{M_\Delta}(\mathbb{R})$ such that solutions to $\Delta$ are in a natural correspondence with maps $f : U \rightarrow M_\Delta$ satisfying $f^*\Theta_\Delta = 0$.*

The pair $(M_\Delta, \Theta_\Delta)$ is called an **exterior differential system**, or EDS.
Exterior Differential Systems
Example of an EDS for a PDE

Let $\Delta$ be the Euler-Tricomi equation $\frac{\partial^2 u}{\partial x^2} = x \frac{\partial^2 u}{\partial y^2}$. Then $\Delta$ is equivalent to the EDS generated by the 1-forms

$$\alpha = du - p \, dx - q \, dy, \quad \beta = p \, dy + xq \, dx$$

on $\mathbb{R}^5$ with coordinates $x, y, u, p, q$. 
Exterior Differential Systems
Example of an EDS for a PDE

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on $\mathbb{R}^5$ with coordinates $x, y, u, p, q$.

If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^5$ satisfies $f^* \Theta = 0$ then from $f^* \alpha = 0$ we get

$$p(x, y) = \frac{\partial u}{\partial x}, \quad q(x, y) = \frac{\partial u}{\partial y}$$
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$$p(x, y) = \frac{\partial u}{\partial x}, \quad q(x, y) = \frac{\partial u}{\partial y}$$

Combined with $f^*d\beta = 0$,

$$0 = dp \wedge dy + x\,dq \wedge dx = \left( \frac{\partial^2 u}{\partial x^2} - x \frac{\partial^2 u}{\partial y^2} \right) \, dx \wedge dy$$
Let $\Theta$ be an EDS which is differentially generated by a set $I$ of 1-forms. We call $\Theta$ integrable (or Frobenius) if

$$dI = 0 \mod I$$

so that $\Theta$ is algebraically generated by $I$ as well. $\Theta$ is integrable if and only if $I^\perp$ is a Frobenius distribution.
Integrable Extensions of EDSs

Integrability

Let $\Theta$ be an EDS which is differentially generated by a set $I$ of 1-forms. We call $\Theta$ integrable (or Frobenius) if

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Let $\pi : Y \longrightarrow X$ be a submersion, $\Theta$ an EDS on $X$, and $\tilde{\Theta}$ an EDS on $Y$ such that $\pi^*\Theta \subseteq \tilde{\Theta}$.

Definition

If $\tilde{\Theta}$ is generated by $\pi^*\Theta$ and $J \subseteq \Omega^1_Y(\mathbb{R})$ with $J$ a basis of $(\ker d\pi)^*$, then $\tilde{\Theta}$ is called an extension of $\Theta$. If additionally $dJ = 0 \mod J$, $\pi^*\Theta$ then the extension is called integrable.
If $\tilde{\Theta}$ is an extension of $\Theta$ then there are solutions to $\tilde{\Theta}$ lifting any solution to $\Theta$. If the extension is integrable, then finding a solution to $\tilde{\Theta}$ over a solution of $\Theta$ only involves solving a Frobenius system.

**Theorem (Estabrook-Wahlquist)**

Let $\tilde{\Theta}$ be an integrable extension of $\Theta$, $f : U \to X$ a solution to $\Theta$, and $q \in \pi^{-1}f(p)$ for some $p \in U$. Then there is a unique lift $\tilde{f} : U \to Y$ of $f$ through $q$, and $\tilde{f}$ may be constructed by solving a sequence of ordinary differential equations.
Geometric Exterior Differential Systems

Definition

Not every PDE on a manifold $M$ is interesting. In order to narrow our focus, we will define a class of PDEs which only contains equations involving geometric quantities.
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**Definition**

Let $M$ be a homogeneous space with structure group $G$. A **geometric exterior differential system (gEDS)** on $M$ is a differential ideal $\Theta \leq \Omega^\bullet_M(\mathbb{R})$ such that $\Theta$ is invariant under the action of $G$.

A gEDS is only general enough to encode “geometrically meaningful” differential equations.
Let $U\mathbb{TE}^3$ be the unit tangent bundle of Euclidean 3-space, so $U\mathbb{TE}^3 \cong \mathbb{E}^3 \times S^2$. An element $(T, R)$ of the Euclidean group $ASO(3)$ acts on $(p, n) \in \mathbb{E}^3 \times S^2$ by

$$(T, R) \cdot (p, n) = (T + R \cdot p, R \cdot n)$$

This action is clearly transitive, so $U\mathbb{TE}^3$ is a homogeneous space for the Euclidean group.
Geometric Exterior Differential Systems
Example: $U\mathbb{T}\mathbb{E}^3$

Let $U\mathbb{T}\mathbb{E}^3$ be the unit tangent bundle of Euclidean 3-space, so $U\mathbb{T}\mathbb{E}^3 \cong \mathbb{E}^3 \times S^2$. An element $(T, R)$ of the Euclidean group $ASO(3)$ acts on $(p, n) \in \mathbb{E}^3 \times S^2$ by

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*Claim:* The 1-form $\langle n, dp \rangle$ is invariant.
Let $U_T E^3$ be the unit tangent bundle of Euclidean 3-space, so $U_T E^3 \cong \mathbb{E}^3 \times S^2$. An element $(T, R)$ of the Euclidean group $ASO(3)$ acts on $(p, n) \in \mathbb{E}^3 \times S^2$ by

$$(T, R) \cdot (p, n) = (T + R \cdot p, R \cdot n)$$

This action is clearly transitive, so $U_T E^3$ is a homogeneous space for the Euclidean group.

Claim: The 1-form $\langle n, dp \rangle$ is invariant.

$$L^*_{(T, R)} \langle n, dp \rangle = \langle R \cdot n, d(T + R \cdot p) \rangle = \langle R \cdot n, R \cdot dp \rangle = \langle n, dp \rangle$$
Let $\Theta$ be the differential ideal on $U \mathbb{E}^3$ generated by $\langle n, dp \rangle$. Since $\langle n, dp \rangle$ is $G$-invariant, so is $\Theta$. Therefore, $\Theta$ is a gEDS.
Geometric Exterior Differential Systems

Example: $U\mathbb{E}^3$ continued

Let $\Theta$ be the differential ideal on $U\mathbb{E}^3$ generated by $\langle n, dp \rangle$. Since $\langle n, dp \rangle$ is $G$-invariant, so is $\Theta$. Therefore, $\Theta$ is a gEDS.

Integral manifolds of $\Theta$ are maps $f : U \rightarrow \mathbb{E}^3$, $n : U \rightarrow S^2$ such that $n$ is the normal map of $f$. In this sense, $\Theta$ is a geometric version of a contact ideal.
Let $M$ be a homogeneous space with structure group $G$, and let $H \leq G$ be the stabilizer of some point $p \in M$.

**Theorem (N–, 2008)**

*There is a one-to-one correspondence between gEDSs on $M$ and $\text{ad}^*(\mathfrak{h})$-submodules of $\mathfrak{h}^\perp \subseteq \mathfrak{g}^*$.*

*Usage:* This theorem allows us to replace gEDS calculations with much simpler computations on $\mathfrak{g}^*$.
The Exterior Algebra on a Homogeneous Space
The Exterior Derivative

A local frame on $U \subseteq M$ relative to $q$ is a map $\sigma : U \rightarrow G$ such that $\sigma(p) \cdot q = p$. 
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**Lemma**

Let $\delta : \mathfrak{g}^* \rightarrow \bigwedge^2 \mathfrak{g}^*$ be the negative dual of the Lie bracket,

$$(\delta \phi)(x, y) = -\phi([x, y])$$

and $\sigma$ a local frame. Then if $\Theta$ is a gEDS we have

$$\mu_{\sigma^{-1}}^*(\delta \Theta) = d(\mu_{\sigma^{-1}}^* \Theta)$$

where $\mu_g(p) = g \cdot p$ is the action of $G$ on $M$.

*Moral:* $\delta$ replaces $d$ as the exterior derivative for a gEDS.
Since we have a differential on $g^*$ and a pairing $\lrcorner$ of $g^*$ with $g$, we can define for any $\xi \in g$ the **Lie derivative** $L_\xi : \bigwedge^k g^* \longrightarrow \bigwedge^k g^*$ by

$$L_\xi \omega = \xi \lrcorner (\delta \omega) + \delta (\xi \lrcorner \omega)$$
Since we have a differential on $g^*$ and a pairing $\downarrow$ of $g^*$ with $g$, we can define for any $\xi \in g$ the Li derivative $L_\xi : \bigwedge^k g^* \to \bigwedge^k g^*$ by

$$L_\xi \omega = \xi \downarrow (\delta \omega) + \delta (\xi \downarrow \omega)$$

**Lemma**

Let $\mathfrak{h}$ be a subalgebra of $g$, and $\Theta \leq \bigwedge \mathfrak{h}^\perp$ an ideal closed under $\delta$. Then $\Theta$ is ad$^*(\mathfrak{h})$-invariant (and therefore a $g$EDS on $G/H$) if and only if

$$L_\xi \Theta = 0 \mod \Theta$$

for all $\xi \in \mathfrak{h}$. 
For each $\xi \in \mathfrak{h}$ there is a special operator $\nabla_\xi : g^* \longrightarrow \bigwedge^2 g^*$, useful in determining which relations lead to Bäcklund transformations. This operator is defined by the equation

$$\nabla_\xi \omega = \xi \lrcorner (\omega \wedge \delta \omega)$$

$\nabla$ is used to single out a lift of $f$ which realizes the relation to $\hat{f}$. 
The Main Theorem

Theorem (N--, 2009)

Let $\Theta \subseteq \mathfrak{h} \perp$ generate a gEDS on the homogeneous space $M = G/H$, and let $[\beta] \in H \backslash G/H$ be a geometric relation. Define $\Delta, \hat{\Delta} \in \text{Hom}(\mathfrak{h} \times \Theta, \bigwedge^2 g^*)$ by

$\Delta(\xi, \vartheta) = \nabla_\xi \text{Ad}^*(\beta^{-1})\vartheta$,

$\hat{\Delta}(\xi, \vartheta) = \nabla_\xi \text{Ad}^*(\beta)\vartheta$.

If $\beta$ is such that for all $\xi \in \mathfrak{h}$ we have $L_\xi \Delta = L_\xi \hat{\Delta} = 0 \mod \Theta$ then:

1. (Generalized Bianchi) If $f$ and $\hat{f}$ are $\Theta$-adapted and $f \sim_{[\beta]} \hat{f}$, then $f, \hat{f}$ satisfy the differential equations $\Delta, \hat{\Delta}$ resp.

2. (Generalized Lie) If $f$ is $\Theta$-adapted and satisfies $\Delta$, then we can construct a $\hat{f}$ which is $[\beta]$-related to $f$, is $\Theta$-adapted, and which satisfies $\hat{\Delta}$. Furthermore, $\hat{f}$ may be constructed by integrating a series of ODEs.
The Main Theorem

Proof Outline

Define $\Theta_1 = \Theta \cup \Delta$, $\Theta_2 = \Theta \cup \hat{\Delta}$, where $\Theta$ is a gEDS generated by 1-forms. The goal is to construct a gEDS which is an integrable extension of both $\Theta_1$ and $\Theta_2$. 
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Equip $g \times g$ with the projections

$$
\begin{array}{ccc}
g \times g & \xrightarrow{\pi_1} & g \\
& \text{Ad}(\beta^{-1}) \circ \pi_2 & \xrightarrow{} & g
\end{array}
$$

Denote these projections by $\pi$, $\hat{\pi}$ for short.
The Main Theorem
Proof Outline

1. Define a gEDS $\Omega$ on $g \times g$ differentially generated by the 2-forms $\pi^* \Delta$, $\hat{\pi}^* \hat{\Delta}$, the 1-forms $\pi^* \Theta$, and

   \[ \left\{ \pi^* \phi - \hat{\pi}^* \phi \mid \phi \in g^* \right\} \]

   A map $(F, \hat{F}) : X \longrightarrow G \times G$ is an integral manifold of $\Omega$ iff there is an element $g \in G$ such that $g \cdot F \sim_{[\beta]} \hat{F}$. 
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The Main Theorem

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3. $\Omega$ is a gEDS for the diagonal action of $\mathfrak{h}$ only when $L_{\mathfrak{h}} \Omega = 0$. This is automatic, except for $L_{\mathfrak{h}} \Delta = 0$. 


As an extended example, let us use the main theorem to find the geometric Bäcklund transformations for surfaces in Euclidean 3-space. A generic relation $[\beta]$ on $UTE^3$ has a representation of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ X & 1 & 0 & 0 \\ Y & 0 & \cos \varphi & -\sin \varphi \\ Z & 0 & \sin \varphi & \cos \varphi \end{bmatrix} \implies \begin{cases} |\hat{f} - f|^2 &= X^2 + Y^2 + Z^2 \\ \langle n, \hat{n} \rangle &= \cos \varphi \\ \langle n, \hat{f} - f \rangle &= Z \\ \langle \hat{n}, \hat{f} - f \rangle &= Z \cos \varphi - Y \sin \varphi \end{cases}$$
Applying the Main Theorem
Canonical Form of Euclidean Relations

As an extended example, let us use the main theorem to find the geometric Bäcklund transformations for surfaces in Euclidean 3-space. A generic relation $[\beta]$ on $UT^{\mathbb{R}}_3$ has a representation of the form

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\begin{bmatrix}
1 & 0 & 0 & 0 \\
X & 1 & 0 & 0 \\
Y & 0 & \cos \varphi & -\sin \varphi \\
Z & 0 & \sin \varphi & \cos \varphi
\end{bmatrix}
\Rightarrow
\begin{cases}
|\hat{f} - f|^2 = X^2 + Y^2 + Z^2 \\
\langle n, \hat{n} \rangle = \cos \varphi \\
\langle n, \hat{f} - f \rangle = Z \\
\langle \hat{n}, \hat{f} - f \rangle = Z \cos \varphi - Y \sin \varphi
\end{cases}
$$

$[\beta]$ is a symmetric relation ($[\beta] = [\beta^{-1}]$) when

$$Y \sin \varphi = Z(1 + \cos \varphi)$$
The contact system $\Theta$ on $UT\mathbb{E}^3$ is generated by the single element $e^3 \in \text{asso}(3)^\ast$. Since $\mathfrak{h}$ is generated by $e_1^2$, we get the lone constraint

$$L_{e_1^2} \nabla_{e_1^2} \text{Ad}^\ast (\beta^{-1}) e^3 = 0 \mod \Theta$$
Applying the Main Theorem
Using the $\nabla$ Operator

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The Differential Equation

The corresponding differential operator $\Delta$ is

$$(\sin^2 \varphi) e^1 \wedge e^2 + (Y \sin \varphi) (e_1^3 \wedge e^2 - e_2^3 \wedge e^1) + (X^2 + Y^2) e_1^3 \wedge e_2^3$$
Applying the Main Theorem
Geometric Bäcklund Transformations in $\text{UTE}^3$

Theorem (Bäcklund Transformations in $\text{UTE}^3$)

Let $[\beta] \in \mathcal{R}_{\text{UTE}^3}$ be a geometric relation of the form

\[
|\hat{f} - f|^2 = X^2 + Z^2 \csc^2(\varphi/2)
\]

\[
\langle n, \hat{n} \rangle = \cos \varphi
\]

\[
\langle n, \hat{f} - f \rangle = Z
\]

\[
\langle \hat{n}, \hat{f} - f \rangle = -Z
\]

Then $[\beta]$ induces a geometric Bäcklund transformation between surfaces satisfying the affine Weingarten equation

\[
sin^2 \varphi + 2H((1 + \cos \varphi)Z) + K(X^2 + Z^2 \cot^2 \frac{\varphi}{2}) = 0
\]
Specialize to surfaces satisfying the affine Weingarten equation

\[ 2K + 2H + 1 = 0. \]

The unit-radius cylinder is a simple solution to the Weingarten equation \( 2K + 2H + 1 = 0 \). Let us parameterize the cylinder by \( f(u, v) = (\cos u, \sin u, v)^T \).
The simplest transform of the cylinder is a surface of revolution satisfying $2K + 2H + 1 = 0$ with profile curve

$$\gamma(t) = \frac{1}{4\pi^2 e^{2t} + 1} \left( \begin{array}{c} 4\pi^2 e^{2t} - 4\pi e^t - 1 \\ 4\pi^2 (t - 1)e^{2t} - 4\pi e^t + t + 1 \end{array} \right)$$
There is actually a 1-parameter family of transformations acting on solutions to $2K + 2H + 1 = 0$. The case just analyzed corresponds to the parameter $-\pi/2$. Here is a solution where the parameter is nearly $-\pi$: 
More generally, the cylinder transforms to a solution of \(2K + 2H + 1 = 0\) parameterized by

\[
\hat{f}(u, v) = \frac{1}{e^{2u \cot \varphi} + e^{2v \csc \varphi}}.
\]

\[
\begin{pmatrix}
(2e^u \cot \varphi - v \csc \varphi + e^u \cot \varphi) \sin \varphi \sin u + e^u \cot \varphi \cos \varphi \cos u + \cos(u + \varphi) e^{2v \csc \varphi} \\
(2e^u \cot \varphi - v \csc \varphi + e^u \cot \varphi) \sin \varphi \cos u + e^u \cot \varphi \cos \varphi \sin u + \sin(u + \varphi) e^{2v \csc \varphi} \\
(v - \sin \varphi) e^{2v \csc \varphi} + ve^u \cot \varphi + (2e^u \cot \varphi - v \csc \varphi + e^u \cot \varphi) \sin \varphi
\end{pmatrix}
\]

where \(\varphi\) is the parameter mentioned previously.
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There are many interesting Bäcklund transformations which are “infinitesimally geometric”, in the sense that the relevant relation is a relation on $g \times \hat{g}$ rather than $G \times \hat{G}$. Let $f$ be a surface and

$$F^{-1}dF = \omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tau^1 & 0 & \lambda & \nu^1 \\ \tau^2 & -\lambda & 0 & \nu^2 \\ 0 & -\nu^1 & -\nu^2 & 0 \end{bmatrix}$$
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- A surface $f$ is minimal if and only if

$$\hat{\omega} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\tau^2 & 0 & \lambda & \nu^1 \\ \tau^1 & -\lambda & 0 & \nu^2 \\ 0 & -\nu^1 & -\nu^2 & 0 \end{bmatrix}$$

is integrable.
A surface $f$ has constant mean curvature 1 (CMC) if and only if

$$\hat{\omega} = \begin{bmatrix} 0 & \tau^2 & -\tau^1 & 0 \\ -\tau^2 & 0 & \lambda & \nu^1 \\ \tau^1 & -\lambda & 0 & \nu^2 \\ 0 & -\nu^1 & -\nu^2 & 0 \end{bmatrix}$$

is integrable in $\mathfrak{so}(4)$.

The surface corresponding to $\hat{\omega}$ is minimal in $S^3$. Combined with the Dorfmeister-Pedit-Wu method, we get a new Weierstrass representation for CMC surfaces (N–, 2006).
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(N–, 2008) There are similar transformations taking Nambu-Goto strings in $\mathcal{L}^{2,1}$ to themselves, and taking Nambu-Goto strings in $(2, 1)$-dimensional deSitter space to strings subject to an additional field in $\mathcal{L}^{2,1}$. 
In the classical case, it happens that $\theta$ is a solution to the sine-Gordon equation

$$\frac{\partial^2 \theta}{\partial u^2} - \frac{\partial^2 \theta}{\partial v^2} = \frac{1}{2} \sin(2\theta)$$

The geometric Bäcklund transformations also induce Bäcklund transformations of the sine-Gordon equation.
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The geometric Bäcklund transformations also induce Bäcklund transformations of the sine-Gordon equation.

**Conjecture**

By comparing the special frames appearing in the transformations to canonical frames, we should be able to systematically associate a totally integrable PDE for $H$-valued functions to each geometric transformation on $G/H$. 
In the classical Bäcklund transform, the relations $|\hat{f} - f| = 1$ and $\langle n, \hat{n} \rangle = 0$ can be replaced with $|\hat{f} - f| = \sin \alpha$, $\langle n, \hat{n} \rangle = \cos \alpha$ to get a Bäcklund transform $\beta_\alpha$.

**Theorem (Bianchi’s Permutability Theorem)**

*Let $f$ be a pseudospherical surface, and $\beta_\alpha f$ its transform. Then*

$$\beta_\alpha \beta_\alpha' f = \beta_\alpha' \beta_\alpha f$$

*Furthermore, $\beta_\alpha \beta_\alpha' f$ is an algebraic function of $f$, $\beta_\alpha f$, and $\beta_\alpha' f$.*

Leads to a *nonlinear superposition principle* for these surfaces. Can this be generalized?
The techniques described here only pick out “elementary” equations, so we get things like $aK + bH + c = 0$ but not $H^2 - K = 0$, despite the fact that $H^2 - K = (\kappa_1 - \kappa_2)^2$ is a Euclidean invariant.

Is there a natural analog of prolongation which gives access to these invariants?
Thank you for your time!
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Integrating an Overdetermined System

To integrate an overdetermined PDE $df = \Phi(\vec{x}, f)$, pick a complete flag $X_1 \subset X_2 \subset \cdots \subset X$ on the independent variables and solve for $f$ inductively. Moving up each step in the flag only involves integrating an ODE.
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Kuen’s Surface
Kuen’s Surface

\[ \hat{f}(u, v) = \left( \begin{array}{c}
\frac{2}{1+v^2 \text{sech}^2 u} (\text{sech} u \cos v + v \text{sech} u \sin v) \\
\frac{2}{1+v^2 \text{sech}^2 u} (\text{sech} u \sin v - v \text{sech} u \cos v) \\
u - \frac{2}{1+v^2 \text{sech}^2 u} \tanh u 
\end{array} \right) \]