Revisiting the 'unreasonable effectiveness' of mathematics

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Although the phrase 'unreasonable effectiveness of mathematics' is widely used, it is not clear what it means. To understand this phrase critically, we first need to understand the meaning of mathematics and what it means to use it in the sciences. This paper begins by considering the different views on the nature of mathematics, the diversity of which points to the difficulty in understanding what mathematics really is, a difficulty which adds to the mysteriousness of the applicability of mathematics. It is also not clear as to what is applied when we apply mathematics. What is clear however is that mathematics cannot be applied to the world but only to some descriptions of the world. This description occurs through the medium of language and models, thus leading us to consider the role of mathematics as language. The use of a language like English to describe the world is itself 'unreasonably effective' and the puzzle with mathematics is just one reflection of this larger mystery of the relation between language and the world. The concluding parts of this paper argue how the view of mathematics as language can help us understand the mechanisms for its effective applicability.

SOME words and phrases are destined to capture the imagination and in so doing get widely used. As a consequence, they are also open to serious misunderstanding. 'Unreasonable effectiveness of mathematics' is one such phrase which is often invoked but little analysed or understood. This phrase was made famous by Eugene Wigner in the Richard Courant Lecture in Mathematical Sciences at New York University in 1959, which was subsequently published in the *Communications in Pure and Applied Mathematics* in 1960. I will begin by summarizing Wigner's arguments in order to understand exactly what he meant when he used the phrase 'unreasonable effectiveness', after which I will analyse what mathematics and mathematization means, and then conclude with one explanation for the effectiveness of mathematics.

Wigner begins with a story of two friends, one of whom, a statistician, was working on population distribution[1]. When the statistician explained the symbol ð occurring in a particular distribution, the friend, who presumably was not a mathematician, thought it was a joke and said, 'surely the population has nothing to do with the circumference of a circle'. Wigner learns a lesson or two from this story. He first notes that mathematical concepts turn up unexpectedly thereby providing close descriptions of some phenomena. Secondly, he believes that because we do not know the reason why mathematics is so unexpectedly useful we will not be able to say with certainty whether a theory we hold true is uniquely appropriate to a phenomenon or not. With this as his starting point he analyses the usefulness of mathematics in the natural sciences and comments that this usefulness is mysterious and has 'no rational explanation' for it.

The significant use of mathematics in the sciences owes a great debt to the belief that the laws of nature are written in the language of mathematics, a statement attributed to Galileo and one which has been echoed for centuries after by figures such as Newton, Einstein and Feynman. Wigner too joins this chorus and begins by correctly noting that only some mathematical concepts are used in the formulation of laws of nature and these concepts are not chosen arbitrarily. One of the elements contributing to the mystery of mathematics lies in the physicist stumbling upon a mathematical concept that best describes a phenomenon only to find that the mathematician has already developed that concept independently. As examples, Wigner cites complex numbers and functions, the appropriateness of which is especially manifested in the formulation of the complex Hilbert space which is so essential to quantum mechanics. The surprising (to the common sense) and necessary role of complex numbers and functions along with the idea of analytic functions is one example of the 'miracle' of mathematization.

The important argument here is that mathematical concepts are not accidentally useful but are *necessary* in the sense that they are the 'correct language' of nature. Wigner offers three examples to illustrate this necessary relation. The first is that of Newton's law. Not only was this law based on 'scanty observations', it also contained the physically non-intuitive idea of the second derivative and yet exhibited an extremely high sense of accuracy. The second example is the matrix formulation of quantum mechanics. The miracle in this case, according to Wigner, lay in the fact that one could apply these matrix methods even in cases where Heisenberg's initial rules did not apply, as illustrated in the calculation of the lowest energy level of helium.

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The third example is that of quantum electrodynamics, particularly the theory of Lamb shift, a theory which again showed extremely high accuracy with experiment. From this, Wigner concludes that mathematical concepts, 'chosen for their manipulability', are not only appropriate but are also accurate formulation of the laws of nature. For him, these laws together with the laws of invariance are the foundation of the mathematical method in sciences. Finally, he considers the uniqueness of theories in physics and asks whether mathematics alone can help adjudicate which theories are essentially right. The problem here is that some theories which are known to be false also give 'amazingly accurate results'. The examples he gives of these 'false' theories are Bohr's early model of the atom, Ptolemy's epicycles and the free-electron theory.

Wigner concludes by saying that the 'miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve'. I have no comments on whether we *deserve* this 'gift' or not but as for understanding it, we can at least make an honest try – and this many philosophers have done.

But what is mathematics?

Much of what Wigner says must perforce depend on what he means by mathematics. Wigner says little about what mathematics is but what he says is suggestive. Wigner writes, 'mathematics is the science of skillful operations with concepts and rules invented just for this purpose. The principal emphasis is on the invention of concepts'. This ability to create concepts takes the mathematician into unchartered realms to the point of being imaginatively 'reckless'. Further, there is a notion of generality, simplicity and beauty inherent in this creation.

Wigner, like many scientists, blissfully ignores some of the seminal contributions from philosophy to the understanding of mathematics. His view of mathematics, emphasizing the importance of rules and the human creative element in creating concepts and rules, runs counter to some dominant views on mathematics. Although Wigner does not explicitly push this point further, it is clear that his understanding of mathematics as being rule-driven makes the effectiveness of it a much greater mystery. Namely, how is an activity of humans, driven as it is by rules we create and with human-centred ideas such as beauty, so well matched with the natural world? Wigner was right in his characterization of mathematics even though his analysis of the question is incomplete.

It will be useful to briefly discuss the dominant views of mathematics before we consider the question of applicability. I will summarize five different views on the nature of mathematics. The divergence of these positions clearly suggests that the mysteriousness of applicability has its origins in the 'mysteriousness' of mathematics itself.

Platonism

Let me first consider the realist view of mathematics. Realists about mathematics believe that mathematical entities exist independently of humans just as trees and tables do. Platonism about mathematical entities is the dominant realist tradition. Platonists believe that mathematical entities have an existence independent of human minds. These entities inhabit a special world, the Platonic world. Platonism thus believes not only in the independent existence of mathematical objects and relations but also believes that the 'reality' of that world explains the universal nature of mathematical truth. However, Platonism, although popular among mathematicians and scientists, runs into serious problems when confronted with the applicability of mathematics. In this case, the basic problem is to understand how these Platonic entities, which do not have spatial or temporal characteristics, can get in touch with our physical world, which is defined by spatio-temporal extension. In other words, how do we as humans access these Platonic objects? And how do these objects link up with our real world?

Logicism

One dominant view of mathematics relates it intrinsically to logic. Logic elucidates the structure and validity of arguments. Reduction of mathematics to logic, in particular deductive logic, meant that the complete domain of mathematical activity was a logical one. Echoing this, the influential logician and philosopher Frege argued that 'mathematics was nothing but the systematic construction of complex deductive arguments', a view which has been dubbed the logicist view of mathematics[2]. Russell attempted to show that all mathematical concepts could be redefined in terms of purely logical concepts. The reduction of mathematics to logic would then imply, for Russell, that all of mathematics, including its axioms and postulates, could be derived entirely from logical laws. However, as Dummett notes, there are various problems in this reduction of mathematics to logic, including Zermelo's axiom of choice and the axiom of infinity[3]. Moreover, there was a serious problem even with a fundamental mathematical entity, the set. If logicism is right, then a set should be a logical concept. However, it was clear that a set was not a logical concept – one reason being that there are many incompatible axiomatizations of set theory.

Formalism

Another view of mathematics, influential in its own way, is called formalism[4]. This school was largely associated with the German school of mathematics and most notably with the illustrious mathematician David Hilbert. The basic idea in the formalist view of mathematics is that mathematics is nothing but a set of rules and formal manipulations of mathematical symbols and terms according to these rules. For formalists there are no meanings attached to mathematical objects, equations or operations over and beyond these meaningless formal manipulations, whether in proof or applications. An analogy that has often been made is that mathematics is like a chess game, which has its objects such as pawns, queen, king and so on, and rules of movement for each of these pieces. The formalist view of mathematics argues that there is no meaning to mathematics over and beyond the game which is played with these mathematical objects according to some given rules. Not only was Hilbert a strong proponent of this view but so was G. H. Hardy who believed that mathematics was just like chess. Moreover, Hardy describes even formal mathematical proof in terms of the structure of chess: 'The *axioms* correspond to the given position of the pieces; the *process of proof* to the rules for moving them; and the *demonstrable formulae* to all possible positions which can occur in the game'[5]. The basic problem with formalism is that it seems difficult to accept mathematics as just a game; in particular its applicability to the sciences then seems totally arbitrary and forces us to ask, why is not chess applicable to the world like mathematics is? In fact, Frege believed that it is the applicability of mathematics alone that makes mathematics more than just a game. On the other hand, for Hardy, the very idea of applying mathematics was distasteful and he writes that mathematics which has practical uses is 'on the whole, rather dull' and has 'least aesthetic va [6].

Intuitionism

In contrast to formalism is intuitionism[7]. The contrast is also illustrated in the nationalities associated with these two views. Intuitionism was predominantly influenced by the French while formalism was developed by German mathematicians. The father of intuitionism was the French mathematician Brouwer who (ironically?) drew upon the German philosopher Immanuel Kant's ideas of intuition and *a priori* truth of mathematics. Intuitionism accepts the 'obviousness' of mathematical entities and places them on par with objects such as chairs and tables. It is in this sense that Godel says that we can perceive mathematical objects like sets in a manner similar to our perception of objects in our world. Godel suggests that 'we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true'[8]. This is an intriguing way of understanding perception; namely, perception of something is not the reason for it being true but recognizing the truth of something actually suggests its perceptability. However, mathematical intuitionism seems counter-intuitive, at least with regard to our common understanding of perception. The intuitionists find the idea of infinity problematic and Brouwer argues that the formalists' approach to infinity and transfinite set theory is 'meaningless' since these are beyond the limits of mathematical intuitions. For the intuitionists, mathematics is something to be created and not discovered, and the role of a creator is best exhibited when the mathematician has to exhibit proof for all existential mathematical assertions.

Mathematics as language

Finally, let me consider the view that mathematics is a product of human imagination, is grounded in our experience with the world and functions like a language. First is the obvious fact that mathematics is a product of humans and is created through our interaction with the world. This implies that the world catalyses mathematical ideas, including the kinds of mathematical entities such as numbers, sets, functions and so on. For example, the mathematical principle of linearity illustrates the physical principle of superposition[9]. If we think of mathematics as beginning with numbers along with some operations like addition we can find an immediate link between human experience (including the activity of counting and aggregating), the structure of the world around us and mathematics. While this does not mean that every mathematical entity or operation is somehow connected to our activity in this world it suggests that the distance between mathematics and our world is not that far removed in the first place. And this relation between the world, humans and mathematics can be analysed in different ways. Steiner argues that anthropocentrism pervades applied mathematics[10]. Rotman argues that mathematics is not divorced from the world and mathematical concepts and objects arise first from the world[11]. Sarukkai has shown how various discursive strategies actually help to create mathematics as we know it now[12]. This view of mathematics offers a canonical answer to the puzzle of unreasonable effectiveness of mathematics. Part of the puzzle lies in the mysteriousness of the relation between two different kinds of worlds – the physical and the mathematical. But if we question the proposition that these are different worlds and argue that mathematics actually 'arises' from the world then the unnatural connection is no longer there, thereby diluting the puzzle as far as this relation is concerned. I will discuss this view of mathematics in much more detail towards the end.

So we find that there is no simple answer to what mathematics really is. This ambiguity about the scope and depth of mathematics gets transferred to the question of applicability. The mysteriousness that is enshrined in the phrase 'unreasonable effectiveness' of mathematics reflects as much a confusion about the mysteriousness of what mathematics is as much as its applicability. Moreover, there are many different types of applicability and different meanings to applicability. I will briefly discuss this issue in a later section.

Mathematization in modern science: lessons from the early days

How exactly did mathematization of the sciences begin? Historians trace the origin of the modern sensibility of mathematization from Galileo onwards although he was not the first person to use mathematics to describe the world. The

Greeks placed mathematics on the highest pedestal; as is well known, the golden section was one of the most privileged concepts in Greek art, architecture, ethics and science. Indian astronomy made extensive use of mathematics, as did Ptolemy. But what was special to Galileo was that he combined mathematics with experimentation, thereby justifying his being called the father of modern science.

Although Galileo radically changed some fundamental presuppositions, his effort nevertheless was built on work by others. For example, there were mathematical-philosophers predating Galileo who, among other things, had analysed the idea of motion in great detail. What Galileo did was to relook at the phenomenon of motion in order to describe it as faithfully as possible with the help of mathematics, corresponding to his belief that physical events were *describable* correctly by mathematics. Again what differentiated Galileo from other natural philosophers who used mathematics was his insistence that experiments were necessary to test and verify the mathematical description. This 'harmony between the world of experience and the mathematical form of knowledge, to be attained through experiment and critical observation'[13] was the unique contribution of Galileo.

Let me analyse one particular component of Galileo's method to illustrate why the method of mathematization seems to work so effectively. Galileo's mathematics was not calculus but number sequences. He discovered by his experiment on motion that the distance of free fall of an object is proportional to the square of the interval of time. How does mathematics manifest itself in this case? Let us assume that we have the necessary apparatus to do this experiment. We drop a ball and find the distance it travels after one second, two seconds, three seconds and so on. Just by noting the distance travelled, we can see a pattern, which is that the distance fallen is in multiples of 4, 9 and so on. Without needing to know any physical laws or calculus we can conjecture that distance varies as the square of the distance[14].

The basic point is this: a pattern about free fall motion is discernible by a particular kind of observation that measures some parameter, in this case distance. Neither the act of measurement nor the use of numbers constitutes mathematization of this problem. But what they do is to illustrate a pattern about motion which is not otherwise discernable. That the distance varies as time squared is of profound importance – this observation plays an important role in helping Newton postulate the gravitational force law as an inverse square law.

Suppose somebody claimed that we could as well have described the fall of the object in English instead of mathematics. So when asked to describe this free fall, this person could say that the object falls fast, faster and … Note that in using English we do not have the capacity to specify the relation between fast and faster. Mathematics, as a language, has this capacity to tell us something about relations. It can tell us that the distance fallen after two seconds is not only greater than the distance fallen after one second but that the distance is four times more. So the use of numbers gives us more information about the distances compared to the use of phrases such as 'greater than'.

But this still does not explain the mystery of mathematization. Suppose we had numbers but did not have multiplication or the concept of proportion. Then we can conceivably give values for distance fallen but we will find no proportional relation between them. Say an object falls 16 ft after the first second and 64 ft after the second second. Let us suppose (however improbable it may seem!) that our mathematics has no concept of multiplication and division but only addition. Then looking at these numbers we cannot find the law that distance varies as time squared. So just having numbers is not enough but we also need an appropriate set of operations. The question therefore is: if we discover new operations and new kinds of numbers would we be able to have a 'better' description of nature? But how do we know what operations are needed? Can nature tell us that? Or does mathematics first offer us this? Also, note that there are already prior physical concepts in use even in this simple problem. Even in a simple mathematical description there are many physical concepts which makes possible the mathematization. For example, before Newton's law can be written down in a mathematical form the physical ideas of force, mass and acceleration need to be present. Describing acceleration as a second-derivative comes after the physical intuition of acceleration as a 'property' of the moving object. Thus, the miracle is not in the use of second derivative as Wigner has it but in the discovery of acceleration as an essential physical concept. Even in the case of Newton's equations, Newton himself notes in his *Principia* that Galileo had known the first two laws of motion – this without the use of the second derivative!

Descartes, one of the most influential mathematicians and philosophers of all time, believed that physics is a branch of mathematics as well exemplified by his statement that 'no other principles are required in physics than are used in Geometry or Abstract Mathematics, nor should any be desired, for all natural phenomena are explained by them'[15]. However, his view on mass is an instructive example about the pitfalls of ignoring the differences in the ideas of the physical and the mathematical. Consider two ways of characterizing mass: mass as extensional and as point-like. In one sense, mass as extensional reflects a brute facticity whereas mass as point-like seems to be counter-intuitive to the common sense. Descartes, for all his belief that physics is a branch of mathematics, conceptualized mass as being extensional. Newton, on the other hand, believed that the essence of mass was to be point-like, a move which allowed him to formulate his physics. Although Descartes had formulated the principle of inertia which was 'formally equivalent' to that of Newton, he did not discover Newtonian physics partly because of his belief in the essence of matter as being extensional. Descartes' belief that physics was a branch of mathematics came in the way of his acknowledging the importance of experiments in physics. Although he had formulated his rule of inertial motion and set of rules of impact, they were incorrect because he did not consider the vector nature of momentum. Cohen notes that Descartes could have easily discovered his mistake by simple experiments[16].

As a final example, I will briefly consider a seminal contribution of Newton to the process of mathematization. This particular method of mathematization which he initiated continues to influence the way mathematics is used in modern science. Cohen isolates one aspect of Newton's use of mathematics, what he calls 'Newton style', as illustrated in Newton's derivation of Kepler's law. First, Newton considers a purely mathematical system, nothing to do with how the world is but dictated by the concerns of pure mathematics. Here a 'single mass-point moves about a centre of force'[17]. Mathematically, if the centre of force is stationary and if the force is always directed towards the centre then Kepler's law of areas can be derived. This is a mathematical problem and treated as such. From this model he goes on to derive the other laws of Kepler, under appropriate conditions. After doing this, Newton compares this imaginary world with the real one. This immediately necessitates him to deal with two-particle motion since the centre of force is also a massive object. Then he develops the mathematics of this system. Next, when he compares his model with the real world, he finds that he has to take into account a much more complex world which has more than two bodies in the solar system. This dynamic interplay between mathematical ideas and comparison with the physical world made Newton realize that laws are absolutely correct only as mathematical laws but in physics they are only approximations (what he called 'hypothes
of mathematization is a continuation of this method (that is, the creation of ideal models) along with the concomitant realization of the important role of the notion of approximation[18].

What exactly is being mathematized and applied?

Let us begin with a catalogue of the furniture of mathematics – there are objects such as numbers, sets, functions and matrices; operators such as the ones used in arithmetic and calculus; rules of operation which make possible calculation; the equality sign (and associated with it appropriate inequalities) and a host of concepts such as continuous, analytical, differentiable and so on. So the first point to note is that applying mathematics could mean applying any or all of the above elements that belong to mathematics.

What is mathematics being applied *to*? Even in the simple example of applying a number we notice an interesting facet of application. For example, let us say that we first start with a statement (in English): 'there are some apples on the table', then apply the concept of number to this and get, say, the statement 'there are ten apples on the table'. This is a proto application of mathematics. But what is getting applied to what? Here, the idea of a number is being 'applied' to a sentence in English. The concept of number is not being applied to the real apples in the world but to a particular description of the world which is first expressed in English with the help of the word 'some'. The lesson from this simple example is one that is central to the process of mathematization: mathematics is first and foremost applied not to phenomena in themselves but to descriptions of phenomena. The two common modes of describing phenomena are through language and through idealized models (it can be argued that models themselves are one kind of linguistic description). Here I would like to focus attention on language and in particular to considering the possibility that the first defining characteristic of mathematical application is not the application of mathematics to the world as such but to other language(s). One way of understanding this is as follows: models and languages mediate between mathematics and the physical world.

Consider the example given by Wigner. Wigner was surprised that a second-derivative, which stands for acceleration, was integral to the mathematical formulation yet had no common-sensical correlate. But did Newton really write his equation this way? It is well known that in *Principia* Newton states his law as follows: 'The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed'[19]. What exactly has been mathematized in this case? Where is the mathematics in Newton's law expressed in the modern form which says force is equal to mass times acceleration? In writing force as F, there has really been no mathematics done. In the usual form of the equation $F = ma$, we only have a *symbolic shorthand* for a longer sentence and this simple strategy is an important element of mathematization. Moreover, force, mass and acceleration are *not* 'mathematical' concepts. They are physical ideas and the genius of Newton lay in formulating the *appropriate* physical concepts first. (For Galileo, appropriate for science implies that they can be measured.) The mystery for Wigner would lie in the fact that the physical idea of acceleration can actually be correctly described by a second derivative.

Mathematics is therefore applied not to the world but to language and this application can even be at the level of creating appropriate symbolizations. It may also be argued that mathematization is actually an application of mathematics to (idealized) models. In the case of planetary motion, for example, one applies mathematics to idealized pictures and models of the planets. The use of mathematics in order to create new descriptions of pictures and models is also closely related to the mechanism of applying mathematics to language. In the next section I will discuss a little more about this particular process of application of mathematics.

Finally, the common belief that there is a miraculous correspondence between mathematical entities and physical concepts might suggest that recognition of this correspondence is instantaneous. But this is hardly the case.

Physical concepts like mass or force get refined over centuries. During this process, they come to be associated with various physical and mathematical meanings till they settle down to some stable mode of description. The real mystery might occur when mathematics itself begins to supply physical concepts[20].

The problem is compounded when we consider the following: the space of mathematics is much larger than that which is applied or perhaps even applicable. There is a surplus of mathematics and only a part of it finds use in the sciences. And more problematically, the same mathematics can be used to model and describe worlds which are not only very different but also contradictory to our world. That is, as far as the truths of our world are concerned, mathematics is quite indifferent to them. And if we believe that science correctly describes our world then this indifference of mathematics to the 'truths' of our world is a potential embarrassment for science if we want to claim that mathematics is essential to it.

Explaining the obvious: the unreasonable effectiveness of language[21]

Here is one possible way of analysing the usefulness of mathematics in the sciences. First, mathematics constitutes a particular kind of description of the world. Description is an activity of language. Languages describe the world around us. Different languages offer descriptions that are unique to that language. The unique elements of a language include the kinds of concepts the language possesses, its grammatical structure and its larger vocabulary and meaning. A same phenomenon can in principle be described in different ways by using different languages.

The capacity of language to 'correctly' describe our world is already mysterious. The problem is simple. Assume that the world is given to us. The world, distinct from language, is nothing but a collection of objects and events. Language arises in learning to talk about the objects and events of the world. Language not only seems to give us a 'proper' description of the world but also allows us to negotiate and intervene with the world in various ways. For simplicity, in what follows let me consider English as an example of a language. Describing the world with the help of English seems to capture some important facets of the world. Consider this simple example. Say we are seeing two objects in front of us and we describe our perception by saying 'one object is to the left of the other'. The capacity of English to create a word called 'left', which describes not an object in itself but a relation, is itself surprising but what is more amazing is that the linguistic statement 'one object is to the left of the other' seems to correctly match with our perception. But we somehow seem to take it for granted that there is no mysteriousness in the capacity of English to describe the world. We do not think that the use of English suggests an 'unreasonable effectiveness' just as the use of mathematics does. What could possibly be the reason for this lack of surprise at the role of language?

One possible reason is this: a natural language like English seems to largely arise out of our interaction with the world. The word 'tree' denotes an object tree – suggesting that we create a word in our language to say something about an object that we already have in front of us. We can name by pointing to things and children often learn the association of a word to thing through the act of pointing. In this naï ve sense, words in a language seem to be derivative to the real world around us and arise in response to the given world. Objects and events surround us and we use language to talk about them leading us to the commonly held view that the world comes first and language follows the dictates of the world. This in a way reduces the mysteriousness in the act of using language to talk about the world, because it is *expected* that a natural language like English, since it arises from the wellspring of the world, should well describe the world.

And this is exactly where mathematics is seen to differ from English. Mathematical objects are not seen as those that belong to the natural world. Many mathematicians and scientists in fact believe quite the opposite – namely, mathematical entities belong to a Platonic world. However, mathematics functions in a way similar to natural language in the sense that the mathematical language is also a language, one which describes the mathematical world. For example, it gives names (such as 'sets') to mathematical objects (namely, sets), presumably existing in the world of mathematics. Therefore, the surprise is all the more exaggerated when it is found that mathematical objects, which presumably exist independently of our physical world, are very apt in describing our physical world. The surprise arises in finding that mathematics is doing a work which it supposedly should not be doing. And ironically, it seems to be doing it 'better' than natural language. In what sense is it doing a better job?

That mathematics does a better job than natural languages is perhaps most forcefully explained by the predictive success of the sciences based on mathematics. It is the predictive success of the sciences, based on mathematics, which gives the most important validation of mathematics. The mysteriousness of the effectiveness of mathematics is enhanced when a scientist stumbles upon a mathematical term which is then found to be the best fit to a particular physical description, like in the case of groups and symmetry or gauge theory and fibre bundles. Echoing this sentiment, Weinberg says that it is 'positively spooky how the physicist finds the mathematician has been there before him or her'[22].

However, both these descriptions of the character of English and mathematics are only partly right. English, although arising from a response to our natural world, also has the capacity to generate words which stand for physically non-existent objects. Abstract nouns, for example, refer to an abstract entity. Even the very act of having a word 'number', referring to a mathematical entity number, shows the capacity of natural language to refer to things which are beyond our physical world. Further, English generates a large amount of words which have nothing to do with physical objects. And the flip side of this is also that mathematics is not to be understood as being totally concerned with a Platonic world. So, both English and mathematics share some important, common features of languages, including the capacity to use both of them in different kinds of predictions. Mathematical description seems to be far more suited to certain types of description, typically quantitative, whereas a description in English may have superior qualitative expressions.

There are also some important differences between mathematics and English that we need to note[23]. In the context of applicability, I believe that the most important distinction which we need to focus upon is the observation that mathematics is not 'one' language like English. It is actually a collection of sub-languages each of which has some common links. Geometry, algebra, topology, etc. are sub-languages of a larger entity called mathematics. These are sub-languages in the sense that they function like a separate language in terms of the concepts they possess, the methodologies they use, their aesthetics and so on, yet share a common world with each other. Each discipline of mathematics is actually like a sub-language and in talking about mathematics as a language, as something homogenous, we overlook this important diversity and difference of its many sub-languages.

This diverse character of mathematics is very important and actually offers an explanation of why mathematics is so unreasonably effective. The different sub-languages that constitute mathematics make the descriptive enterprise of mathematics very interesting. Languages, when they are used to describe, explain, define, argue and so on, have specific narrative structures. Languages create narratives. A description is one kind of narration. The nature and the effectiveness of the description depend on the narrative structure of a language. In mathematics, the narrative structure is composed of the different elements of its different sub-languages, thereby expanding the scope of its narrative capability. Therefore, description in mathematics consists of much larger and more complex narratives than description restricted to only English. Let me give a simple illustration of how this is done[24].

Consider light reflecting from a mirror. How can we invoke mathematics here? What kinds of descriptions can we develop about this event with the use of mathematics? First, as is commonly done, we can give a pictorial representation of this process. The mirror is represented by a straight line and the incoming ray and the outgoing ray by two straight lines. Drawing the normal, we have the angle between the rays and the normal. This pictorial representation is very useful for science in that it allows us to do what we want with an idealized system. Mathematics comes into play on this idealized picture when we 'name' angles and use properties of terms such as momentum. So from a picture of the process we move into geometry (a sub-language of mathematics) of the system. This allows us to define and describe components of the momenta, forces and so on. At this stage we begin to do geometry – in the particular case of the reflection of light we do geometry on a plane. The results of these calculations will depend on some results that belong to the domain of this geometry and not the domain of the real phenomenon.

So typically this is what happens in the process of mathematization. The event in the world is first represented pictorially, for example, which can then be expressed in another sub-language, say geometry, and then in algebra and so on. Each one of these steps takes the real world event into different narrative domains. For example, light bouncing off a mirror has no velocity component in the real world but a mathematical description talks as if the components of momentum are real. So the shift into pictures and other sub-languages succeeds in adding new descriptions of the original event. It is important to note that these descriptions are unique to the different sub-languages. Description in the pictorial form is very different when compared to the ones derived from the geometrical narrative, which is itself very different from the one derived from using algebra. For example, once we enter the descriptive space of algebra we have a new vocabulary that is available to us to describe the process: continuity, rate of change, equations of motion and so on. This vocabulary, which was not present in the earlier sub-languages of pictorial representation or geometry, succeeds in expanding the narrative possibilities of this process. In the realm of algebra, the vocabulary allows us to talk of motion in higher dimensions, the possibility of transformations of co-ordinates, even the physically non-intuitive idea of transforming momenta into co-ordinates and so on. The important sub-language of calculus along with algebra allows us to develop extremely rich narratives about a simple process such as a ray of light bouncing off a mirror.

Thus, we see that the process of mathematization using the many sub-languages of mathematics enlarges the possible descriptions one can have of a process. There are literally no conceivable limits to what sub-languages we can use for this description. If, for example, someone finds the vocabulary and grammar of topology useful in the description of a bouncing ball then it becomes part of the larger mathematical description of this process.

So, first and foremost, using mathematics to describe the physical world is a means of finding ways to create multiple descriptions of a physical object or event. We can see that a language like English will only create limited narratives about a phenomenon because it does not have the rich sub-languages that mathematics has. When we use mathematics as a language to describe a process we first of all create a rich storehouse of possible narratives. What among them will fit the world is an issue that mathematics is unconcerned about. The job of mathematics in sciences is essentially to proliferate narratives and the more number of narrative descriptions are possible the better *probability* that there will be a fit somewhere, sometime.

Compounding the problem – the pictorial role of mathematics

A great deal of creative mathematics, both pure and applied, depends on analogy. I want to illustrate a few cases of analogies that have to do with formal patterns of the mathematical symbols. It is remarkable that mimicking the patterns of written mathematical terms yields profound new ideas in physics. The history of mathematics and science is replete with this strategy. The following elementary example is primarily to illustrate a seemingly arbitrary method used in mathematization. A detailed analysis of the role of form in mathematical discourse is extremely illuminating but here I will only briefly touch upon this issue[25].

The basic strategy is this: by looking at the way in which mathematical expressions are written and arise in the course of calculation we are able to identify some new information. A simple example is that of a term which looks like $1/2ab^2$. When we see such an expression in the context of some calculation it seems natural to identify a with a mass term and b with a velocity term since this expression *looks* like a kinetic energy term. Identifying and discovering such terms can be very important steps in theoretical research in science. Consider the following example from Landau and Lifshitz's *Mechanics*. Consider two particles with masses m_1, m_2 and velocities v_1 and v_2 in an interactive potential field. The total kinetic energy of the system is the sum of the kinetic energies of the two particles. In this system, we can rewrite the total kinetic energy as one term which looks like $1/2ab^2$. Now, looking at this we interpret a as the mass term (the reduced mass) and b as the velocity term. Further, the authors claim that because the expression of two-particle kinetic energy terms reduces to that of one kinetic energy term the two-particle motion is *equivalent* to the motion of one particle[26].

This strategy of 'discovering' mass is one that is practiced right across the many disciplines of science. In physics, it is extensively used in areas ranging from classical physics to particle physics. In fact, the identification of mass terms in quantum field theory follows similar 'pattern recognition' of symbolic terms. The importance of mathematical form should not be underestimated. Mathematical form is not about doing mathematics alone; it is also about writing mathematics in some specific ways, the underlying belief being that physical terms are expressed by unique mathematical forms. Thus, classical kinetic energy will be of the form $1/2mv^2$ or in terms of momentum as $p^2/2m$. When we move from classical to quantum physics, the identification of 'kinetic energy' continues to be $p^2/2m$ at the formal level although the physical meaning of kinetic energy for a wave is very different from that of a particle. In quantum theory, we replace p by an operator but the form of the term remains the same, as seen in the Schrödinger's equation.

Interestingly, Steiner points out that there is another formal analogy in the Schrödinger equation, which is that this *formally* identical to the equation for a *monochromatic* light wave in a *nonhomogeneous* medium'[27]. Similarly, the various meanings ascribed to the mass term in classical, electromagnetic, relativistic and quantum mechanical theories were significantly dependent on formal symbolic identification[28]. There are innumerable examples of the importance of symbolic manipulation based on formal similarity. In fact, I would go to the extent of saying that the effectiveness of mathematization significantly depends on the power of symbols to act like *pictures* of ideas, concepts and events. The role of mathematics in the sciences seems to be essentially dependent on the possibility of using mathematical symbols as 'pictures'. For example, we could look upon $1/2ab^2$ as the 'picture' of kinetic energy. So even in contexts that are very different we can still recognize that picture and identify it with the kinetic energy of that object or system. Similarly, the generation of 'alphabets' in mathematics is itself a very creative process and these alphabets many times visually suggest the kinds of things that can be done with them[29].

The above discussion indicates the complexity involved in the process of mathematization of the world. The great challenge to science will lie not only in the creation of new mathematics but also in the possibility of creating new modes of expressions and new languages in the unending scientific search for mapping the universe.

Notes and references

1. Wigner, E. P., The unreasonable effectiveness of mathematics in the natural sciences. *Commun. Pure Appl. Math.*, 1960, **13**.
2. For example, see *Philosophy of Mathematics* (ed. Jacquette, D.), Blackwell, Oxford, 2002.
3. Dummett, M., What is mathematics about? In *Philosophy of Mathematics: An Anthology* (ed. Jacquette, D.), Blackwell, Oxford, 2002, pp. 19–29.
4. See Brouwer, L. E. J., Intuitionism and formalism. In *Philosophy of Mathematics: An Anthology* (ed. Jacquette, D.), Blackwell, Oxford, 2002, pp. 269–276.
5. Hardy, G. H., Mathematical proof. In *Philosophy of Mathematics: An Anthology* (ed. Jacquette, D.), Blackwell, Oxford, 2002, p. 180.
6. Kanigel, R., *The Man Who Knew Infinity*, Scribners, London, 1991, p. 147.
7. See Brouwer, L. E. J., Intuitionism and formalism. In *Philosophy of Mathematics: An Anthology* (ed. Jacquette, D.), Blackwell, Oxford, 2002, pp. 269–276.
8. Quoted in Parsons, C., Mathematical intuition. In *Philosophy of Mathematics: An Anthology* (ed. Jacquette, D.), Blackwell, Oxford, 2002, p. 277.
9. Steiner, M., *The Applicability of Mathematics as a Philosophical Problem*, Harvard University Press, Cambridge, 1998, p. 30.
10. Steiner, M., *The Applicability of Mathematics as a Philosophical Problem*, Harvard University Press, Cambridge, 1998.
11. Rotman, B., *Ad Infinitum: The Ghost in Turing's Machine*, Stanford University Press, Stanford, 1993.
12. Sarukkai, S., *Translating the World: Science and Language*, University Press of America, Lanham, 2002.
13. Cohen, I. B., *Revolutions in Science*, Harvard University Press, Belknap, 1985, p. 139.
14. For Galileo it was much more difficult since he had to extrapolate in a complicated way from observations on an inclined plane.

15. Cohen, B., *Revolutions in Science*, Harvard University Press, Belknap, p. 156.
16. Cohen, B., *Revolutions in Science*, Harvard University Press, Belknap, p. 155.
17. Cohen, B., *Revolutions in Science*, Harvard University Press, Belknap, p. 166.
18. For a discussion on approximation in this context, see Sarukkai, S., *Translating the World: Science and Language*, University Press of America, London, pp. 114–118.
19. Chandrasekhar, S., *Newton's Principia for the Common Reader*, Clarendon Press, Oxford, 1995, p. 23.
20. See Sunil Mukhi's contribution in this collection.
21. For a more detailed discussion on some of these aspects of language and the unreasonable effectiveness of English, see Sarukkai, S., Applying mathematics: The paradoxical relation between mathematics, language and reality. *Econ. Pol. Weekly*, 2003, **XXXVIII**, **35**, 3662–3670.
22. Weinberg, S., Lecture on the applicability of mathematics, *Not. Am. Math. Soc.*, 33:5, 1986. However, we should also note that stumbling upon the right mathematical terms is quite similar to a musician stumbling upon the right notes or a poet stumbling upon the right word or phrase. What special value can be added to a 'right term' just because one stumbles upon it?
23. I do not subscribe to the opinion that the degree of precision and non-ambiguity distinguishes natural language and mathematics. The association of precise meanings with mathematical terms and ambiguity with English words does not represent the true picture. For more on semantic plurality and metaphorical use of mathematical terms, see Sarukkai, S., *Translating the World*. A related issue is whether mathematics can be totally divorced from its connection with natural language and whether mathematics needs natural language in an essential sense. For more on this see, Sarukkai, S., Mathematics, language and translation. *META*, 2001, **46**, 664–674. (The full text of the article is available at http://www.erudit.org/revue/meta/2001/v46/n4/.)
24. For more detailed discussion, see Sarukkai, S., *Translating the World: Science and Language*, University Press of America, London, 2002, pp. 60–70.
25. For more details, see Sarukkai, S., *Translating the World*, University Press of America, London, 2002, Part One.
26. Landau, L. D. and Lifshitz, L. M., *Mechanics*, Pergamon Press, Oxford, 1976, p. 29.
27. Steiner, M., *The Applicability of Mathematics as a Philosophical Problem*, Harvard University Press, Cambridge, 1998, p. 79.
28. See Jammer, M., *Concepts of Mass*, Harvard University Press, Cambridge, 1961.
29. For examples of alphabetisation in mathematics and in general the role of discursive strategies in mathematics, see Sarukkai, S., *Translating the World*, University Press of America, Lanham, 2002.

ACKNOWLEDGEMENTS. S. K. Rangarajan was instrumental in clarifying many ideas on mathematics and I am indebted to him for his patient attempt to understand my views even though it might have grated his sensibilities about mathematics! I also thank Dhanu Nayak for comments on various points about language.