and

$$\langle U\xi, U\eta\rangle = \theta(\alpha_1)\overline{\theta(\beta_1)} + \theta(\alpha_2)\overline{\theta(\beta_2)} = \theta(\alpha_1\bar{\beta}_1 + \alpha_2\bar{\beta}_2) = \theta(\langle\xi,\eta\rangle).$$

We have thus verified (D.2)–(D.5) with $\kappa = \theta$ for vectors in C' and hence for all vectors in C.

We have already seen that the map U was specified by

$$U(\varphi + \xi) = U\varphi + U\xi = \varphi' + U\xi \quad \text{with} \quad P_{U(\varphi+\xi)} = TP_{\varphi+\xi},$$

where $\xi \in C$. The most general vector in $\mathscr{H}$ can be written as

$$\alpha\varphi + \eta \quad \text{with} \quad n \in C.$$

If $\alpha \neq 0$, set $\xi = (\alpha^{-1}\eta)$ and

$$U(\alpha\xi + \eta) = U[\alpha(\varphi + \xi)] = \kappa(\alpha)(\varphi' + U\xi), \quad \text{where } k = \theta.$$

We now have defined U for all vectors in $\mathscr{H}$. Equations (D.2), (D.3) and (D.5) clearly hold, and (D.4) is easily verified. This completes the proof of Wigner's theorem.

Appendix E

COMPACT GROUPS, HAAR MEASURE, AND THE PETER–WEYL THEOREM

A *topological group* is a group, G, which is a topological space such that the maps $G \times G \to G$ sending $a, b \rightsquigarrow ab$ and $G \to G$ sending $a \rightsquigarrow a^{-1}$ are continuous. The group G is called *compact* if it is a topological group and the underlying topological structure of G is that of a compact space. Many of the theorems proved in Chapter 2 for finite groups are valid with only minor modifications for the case of compact groups. One of the main techniques, that of averaging over the group, carries over with practically no change at all. In order to average over the group we have to replace the sum by an integral, and the first main theorem that we shall prove is that there exists a unique notion of invariant integration on the group. That is, that there exists a unique rule assigning to each continuous function, f, its integral, which we denote by $\int_G f(a)\,da$, which satisfies the usual axioms for an integral, and in addition is left invariant, i.e.

$$\int_G f(b^{-1}a)\,da = \int_G f(a)\,da \quad \text{for all} \quad b \in G \tag{E.1}$$

and is normalized,

$$\int_G 1\,da = 1. \tag{E.2}$$

This integral is known as the Haar integral, and the corresponding measure is called Haar measure. Our first task will be to establish the existence and uniqueness of the Haar integral, and establish some of its properties. In particular, we shall prove that the Haar integral is also right invariant, i.e. satisfies

$$\int_G f(ab)\,da = \int_G f(a)\,da. \tag{E.3}$$

The most familiar example of a topological group is the group $Gl(n)$ of non-singular matrices (either the group of $n \times n$ real matrices, $Gl(n, \mathbb{R})$ or the group of $n \times n$ complex matrices, $Gl(n, \mathbb{C})$). The topology is just the usual topology obtained by regarding $Gl(n)$ as a subset of the space of all $n \times n$ matrices, which has the topology of $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$. Any subgroup of $Gl(n)$ which is closed and bounded as a subset of the set of all matrices will then be a compact group. Thus the orthogonal group, $O(n)$, is a compact group, as is the rotation group $SO(n)$, of all orthogonal matrices with determinant 1. Similarly, the group $U(n)$ of all $n \times n$ unitary matrices is a compact group as is the special unitary group, of all unitary matrices of determinant 1. These will be among the most important of our examples.

In what follows, we will not use the full fact that our groups are compact, but only that they are *totally bounded*: a group G is totally bounded if there exists a fundamental family of neighborhoods of the origin, U_i, and, for each i, a finite number of elements $g_1, \ldots, g_{j(i)}$ such that the sets $g_1 U_i, \ldots, g_{j(i)} U_i$ cover G for each fixed i. Thus, if G is compact it is certainly totally bounded. Also, suppose that G possesses a metric, d, which is invariant under right and left translation, and that the diameter of G is finite with respect to this metric. Then we can take U_i to be the ball of radius $1/i$ about e. This gives another example of a totally bounded group. For the rest of this section, we will assume without mentioning it any further that G is a totally bounded group.

The existence of the Haar integral will follow from the following theorem.

Theorem E.1 (The mean ergodic theorem)
Let T be a linear transformation of a normed vector space, V, such that

(i) There is some constant, c, such that

$$\| T^n v \| \leq c \| v \|$$

for all $v \in V$ and all positive integers, n, and

(ii) For some fixed $w \in V$ the sequence

$$S_n w = (1/n)(w + Tw + \cdots + T^{n-1} w)$$

possesses a subsequence which converges to some element $\bar{w}$ (or, in fact, only converges weakly to $\bar{w}$, i.e. $f(S_{n_j} w - \bar{w}) \to 0$ for any continuous linear function, f).

Then $T\bar{w} = w$ and the sequence $S_n w$ converges to $\bar{w}$.

Proof We first observe that the subspace $\overline{(I-T)V}$ consists of all $z \in V$ such that $S_n z \to 0$. Indeed, if $z = (I-T)v$, then

$$\| S_n z \| = \| (1/n)(T^n v - v) \| \leq (c/n) \| v \| + (1/n) \| v \| \to 0,$$

so that every vector in $(I-T)V$ satisfies $S_n z \to 0$. On the other hand, (i) implies that the space of z satisfying $S_n z \to 0$ is closed, so that every element of $\overline{(I-T)V}$ satisfies $S_n z \to 0$. Conversely, suppose that $S_n z \to 0$. Then for any positive ε we can find some n such that

$$\| z - (z - S_n z) \| < \varepsilon.$$

We claim that for any n, $z - S_n z \in (I-T)V$. Indeed,

$$\begin{aligned} z - S_n z &= (1/n)\{(I-T)z + (I-T^2)x + \cdots + (I-T^{n-1})z\} \\ &= (1/n)\{(I-T)z + (I-T)(I+T)z + \cdots + (I-T) \\ &\quad \cdot (I + T + \cdots + T^{n-2})z \in (I-T)V\}. \end{aligned}$$

Now suppose that

$$S_{n_j} w \to \bar{w}.$$

Then

$$TS_{n_j} w - S_{n_j} w = (1/n)(T^{n_j} w - w) \to 0$$

so

$$T\bar{w} = \bar{w}.$$

Now

$$T^n w = T^n \bar{w} + T^n(w - \bar{w}) = \bar{w} + T^n(w - \bar{w})$$

so

$$S_n w = \bar{w} + S_n(w - \bar{w})$$

By assumption,

$$w - S_{n_j} w \to w - \bar{w}$$

and

$$w - S_{n_j} w \in (I-T)V.$$

Hence $w - \bar{w} \in \overline{(I-T)V}$ so that

$$S_n(w - \bar{w}) \to 0,$$

which is what we want to prove. (If we only assume that $S_{n_j} w \to \bar{w}$ weakly, the argument proceeds in much the same fashion but uses the Hahn–Banach theorem. We conclude, as before, that $f(Tw - \bar{w}) = 0$ holds for any continuous linear function, and hence $Tw = \bar{w}$. It also follows that $w - \bar{w} \in \overline{(I-T)V}$. Indeed, if not, there would exist some linear function, f, vanishing on $\overline{(I-T)V}$ with $f(w - \bar{w}) \neq 0$. But $w - S_{n_j} w \in (I-T)V$ and $f(w - S_{n_j} w) \to f(w - \bar{w})$, yielding a contradiction.) Notice that if T is a unitary operator on a Hilbert space, then for any w we can find a weakly convergent subsequence of $S_{n_j} w$ since the unit sphere of a Hilbert space is weakly compact. Thus for any unitary operator in a Hilbert space our theorem guarantees that $S_n w$ converges to an invariant element for any w. This is the usual form of the mean ergodic theorem.

We now apply the theorem to the construction of an invariant integral on a totally bounded group. Let V be the space of uniformly continuous functions on G with the norm taken to be the sup norm, i.e.

$$\| f \| = \sup_{g \in G} | f(g) |.$$

(If G is compact every continuous function is uniformly continuous and V is simply the space of continuous functions.) If we take all the g_k's that entered into the definition of uniform boundedness and arrange them into a sequence, we clearly get some sequence $\{g_k\}$ which is dense in V. Define the operator T on V by the formula

$$(Tf)(g) = \sum_k \frac{1}{2^k} f(g_k^{-1} g)$$

If $| f(a) - f(b) | < \varepsilon$ whenever $a^{-1} b \in U$, where U is some neighborhood of e, then

$$| Tf(a) - Tf(b) | = \sum \frac{1}{2^k} | f(g_k^{-1} a) - f(g_k^{-1} b) | < \varepsilon$$

so that T does map V into itself, and it is clear that $\| T^n f \| \leq \| f \|$ for all n and f. Now

$S_n f(g) = \sum a_j f(h_j g)$ for some sequence of elements, h_j in G and some sequence of real numbers, a_j, with $\sum a_j = 1$, $a_j \geqslant 0$. Thus, by the same sort of estimate we conclude that sequence of functions, $S_n f$ is equicontinuous, i.e. that for any $\varepsilon > 0$ we can find some fixed neighborhood, U, of e such that

$$|S_n f(a) - S_n f(b)| < \varepsilon \text{ for all } n \text{ if } a^{-1}b \in U.$$

Since G possesses a dense sequence of elements, this implies (by the usual selection type argument) that we can choose a convergent subsequence of $S_n f$. By Theorem E.1, we conclude that for any $f \in V$, the sequence $S_n f$ converges to some element $\bar{f}$ satisfying $T\bar{f} = \bar{f}$.

We claim that $\bar{f}$ is a constant, and intend to show that the map, $f \rightsquigarrow \bar{f}$ has all the properties of an integral. To prove that $\bar{f}$ is a constant, we may assume, by taking real and imaginary parts, that f is real valued. Let $M = \sup \bar{f}$. If $\bar{f}$ is not constant, then there will be some $h \in G$ with $\bar{f}(h) < M - 2\varepsilon$ for some $\varepsilon > 0$. Since $\bar{f}$ is uniformly continuous, we will then have $\bar{f}(g) < M - \varepsilon$ for all g such that $gh^{-1} \in U$, where U is some suitable neighborhood of the origin. By construction, a finite number of the $g_i U$ will cover all of G. Thus for any $g \in G$ we will have

$$gh^{-1} \in g_i U \quad \text{for some} \quad i < N$$

where N is some sufficiently large number, and hence

$$g_i^{-1} g h^{-1} \in U \quad \text{so that} \quad \bar{f}(g_i^{-1} g) < M - \varepsilon.$$

But then

$$\bar{f}(g) = T\bar{f}(g) = \sum \frac{1}{2^j} \bar{f}(g_j^{-1} g) \leqslant \sum_{j \neq i} \frac{1}{2^j} M + \frac{M - \varepsilon}{2^N} \leqslant M - \frac{\varepsilon}{2^N}$$

contradicting the definition of M.

Thus $\bar{f}$ is a constant. We denote the number $\bar{f}$ by $\int_G f(g)\,dg$ and note that

$$\textit{the map } f \rightsquigarrow \int_G f(g)\,dg \textit{ is linear,} \tag{E.4}$$

$$\int_G 1\,dg = 1 \tag{E.2}$$

and

$$\textit{if } f(g) \geqslant 0 \quad \textit{for all} \quad g \in G \textit{ then} \int_G f(g)\,dg \geqslant 0. \tag{E.5}$$

Thus we do indeed get all the properties of an integral. Notice that if we set $f_h(g) = f(gh)$ for some $h \in G$, then $(Tf)_h = T(f_h)$ from which it follows that

$$\int_G f(gh)\,dg = \int_G f(g)\,dg. \tag{E.3}$$

It follows from the construction of $\int_G f(g)\,dg$ that for any fixed function, f, and any $\varepsilon > 0$, we can find a sequence of real numbers, $a_k \geqslant 0$, and a sequence of group elements,

h_k, such that

$$\sup_{g \in G} \left| \sum a_k f(h_k^{-1} g) - \int_G f(g)\,dg \right| < \varepsilon. \tag{E.6}$$

Now we could just as well have started our whole procedure with a map $\tilde{T}$ given by $\tilde{T}f(g) = \sum \frac{1}{2^j} f(gp_j)$ for a suitable dense sequence of elements, p_j. The arguments would have gone exactly as before, and we would end up with some integral, $\bar{\int}$, satisfying (E.4), (E.2), (E.5), and with (E.3) replaced by

$$\bar{\int}_G f(p^{-1}g)\,dg = \bar{\int}_G f(g)\,dg$$

and with (E.6) replaced by

$$\sup_{g \in G} \left| \sum b_k f(g q_k) - \bar{\int}_G f(g)\,dg \right| < \varepsilon. \tag{E.7}$$

That is, left multiplication is replaced everywhere by right multiplication. Now (E.6) and (E.7) imply that

$$\sup_{g \in G} \left| \sum a_j b_k f(g_j^{-1} g q_k) - \int_G f(g)\,dg \right| < \varepsilon$$

and

$$\sup_{g \in G} \left| \sum a_j b_k f(g_j^{-1} g q_k) - \bar{\int}_G f(g)\,dg \right| < \varepsilon$$

so that

$$\int_G f(g)\,dg = \bar{\int}_G f(g)\,dg,$$

and therefore $\int_G f(g)\,dg$ is also left invariant, i.e. (E.1) holds. Notice that (E.2), (E.3), (E.4) and (E.5) uniquely characterize the integral $\int$, as do (E.1), (E.2), (E.4) and (E.5). Indeed, if $\tilde{\int}$ is some other integral satisfying (E.2)–(E.5) we have

$$\int_G f(g)\,dg - \varepsilon \leqslant \sum b_j f(g q_j) \leqslant \int_G f(g)\,dg + \varepsilon$$

for any real valued function, f, and suitable b_j with $\sum b_j = 1$ and suitable group elements, q_j. Applying the integral $\bar{\int}$ to this inequality yields

$$\left| \int_G f(g)\,dg - \bar{\int}_G f(g)\,dg \right| \leqslant \varepsilon.$$

Since this holds for all f and for all ε we conclude that $\int = \bar{\int}$. A similar argument shows that (E.1) works as well as (E.3). We have thus established the existence and uniqueness of the Haar integral. It is also useful to remark that it follows from our construction that if f is a uniformly continuous function satisfying $f \geqslant 0$, and if $f(h) > 0$ for some h, then $\int_G f(g)\,dg > 0$.

Using the Haar integral, much of the theory of *finite*-dimensional representations goes through with little change. For a topological group, a representation will mean a continuous homomorphism of G into the group of continuous linear maps, End (V), of some topological vector space, V. Here End (V) is endowed with one of its suitable topologies (and the choice of topology might affect the notion of representation). For finite-dimensional vector spaces over $\mathbb{C}$ (or $\mathbb{R}$) there is only one topology – the one we mentioned above for $Gl(n)$. Thus there is no ambiguity in discussing finite-dimensional representations of topological groups, in particular in discussing compact groups. For two continuous functions, f_1 and f_2, on the compact group, G (or two uniformly continuous functions on a totally bounded group) we define their scalar product by

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}\,dg$$

(which coincides with the definition given in Chapter 2 if the group happens to be finite). We now briefly indicate the minor changes to be made in redoing the results of Section 2.3 for the case of compact groups: Schur's lemma needs no change. In Proposition 3.1 replace the definition of S by

$$S = \int_G r_a^2 S_0 (r_a^1)^{-1}\,da.$$

Replace (3.1), Section 2.3, by

$$\text{If } r^1 \nsim r^2 \quad \text{then} \quad \int_G r_{kl}^2(a) r_{ij}^1(a)\,da = 0 \tag{E.8}$$

and replace (3.2) by

$$\int_G r_{kl}(a) r_{ij}(a^{-1})\,da = \frac{1}{n}\delta_{li}\delta_{kj}. \tag{E.9}$$

The method of averaging over the group (with sum replaced by integral) still applies to show that any finite-dimensional representation is equivalent to a unitary representation, and hence that every finite-dimensional representation is completely reducible. For finite-dimensional unitary representations (3.3), (3.4), and all of Section 2.4, hold without change. On the other hand, the regular representation, which we must define with slightly more care, will be infinite dimensional if G is not finite. For the purposes of the present section, we will take the regular representation to be the unitary representation of G on $L^2(G)$. That is, we consider the action of G on the space of continuous functions on G, but complete this space (relative to the scalar product, (,)), to obtain the Hilbert space $L^2(G)$, and (E.1) implies that G acts as unitary transformations. Unless G is finite, the space $L^2(G)$ will be infinite dimensional, and thus the character cannot be defined in the usual way as a function on G. (For

instance, the value of the character at e would have to be infinite.) In Appendix G we will see that is possible to generalise the definition of trace, and hence of character, so that it makes sense for many of the interesting infinite-dimensional representations that arise in practice, but the character will then be a distribution rather than a function. For the moment, however, we have no analogs of the results of Sections 2.5 and 2.6 for general compact groups, but we will return to these formulas in Appendix G.

The assertions concerning induced representations need a certain amount of reformulation for the case of compact groups. The key tool is an analysis of the regular representation, i.e. the representation of G on $L^2(G)$. The essential facts are summarized in the celebrated Peter–Weyl theorem, to be stated below. Before stating the theorem, it is convenient to make a definition.

Definition E.1

A (uniformly) continuous function, f, is called a *representative function* if the space spanned by all its translates, gf is finite dimensional, where, as usual, $gf(a) = f(g^{-1}a)$, $g \in G$.

Suppose that f is a representative function, and let $g_1 f, \ldots, g_n f$ be a maximal set of linearly independent translates of f. Then for any $g \in G$

$$f(g^{-1}u) = gf(u) = \sum h_i(g)(g_i f)(u) = \sum h_i(g) f_i(u)$$

where the h_i are suitable continuous functions on G and where we have set $f_i = g_i f$. Conversely, if f is a function satisfying such an equation, it is clear that f is a representative function. Replacing g by g^{-1} in the above equation shows that the set of right translates, $f(\cdot u)$ also span a finite-dimensional space. In particular, if we set $\tilde{f}(a) = f(a^{-1})$ then $\tilde{f}$ is a representative function, and conversely. Let W be the space spanned by the translates of f, so that the f_i form a basis for W. By the above equation

$$gf_j = (gg_j)f = \sum h_i(gg_j) f_i$$

so that the values $h_i(gg_j)$ are precisely the matrix entries of the representation of G on W, relative to the basis f_i. But

$$\tilde{f}(g) = f(g^{-1}) = \sum h_i(gg_j) f_i(e).$$

Since the $f_i(e)$ are just constants in the above equation, we see that $\tilde{f}$ is a linear combination of the matrix elements of some finite-dimensional representation of G, and hence so is f. Conversely, if we start with some finite-dimensional representation of G, and let r_{ij} be its matrix elements relative to some basis, then it is clear that each of the r_{ij} is a representative function. Since every finite-dimensional representation can be written as a direct sum of irreducibles, we conclude that the space of representative functions consists precisely of linear combinations of matrix coefficients of finite-dimensional representations.

We are now in a position to state the Peter–Weyl theorem:

Theorem E.2

(1) The representation functions are dense in $L^2(G)$

(2) The space $L^2(G)$ decomposes into a Hilbert space direct sum of irreducible representations of G, each of which is finite dimensional.

(3) Every irreducible representation of G is finite dimensional.

(4) Each irreducible representation of G occurs in $L^2(G)$ with a multiplicity equal to its dimension.

(5) Any unitary representation of G on any Hilbert space decomposes into a Hilbert space direct sum of (finite-dimensional) irreducible representations.

We begin by proving a special case of (3):

Proposition E.1

Let W be a closed subspace of $L^2(G)$ which is irreducible in the sense that it possesses no proper closed subspaces which are invariant under G. Then W is finite dimensional.

Let f be a unit vector in $L^2(G)$ and consider the function k_f defined on $G \times G$ by the formula

$$k_f(a,b) = \int_G \overline{f(ga)} f(gb)\,dg.$$

Since $f \in L^2(G)$, it is easy to see that k_f is a continuous function on $G \times G$. On the other hand, let P_f denote projection onto the line spanned by f, so that $P_f v = (v, f)$ and define the operator K_f by

$$K_f = \int_G g^{-1} P_f g\,dg,$$

so that for any v, $w \in L^2(G)$, we have, by unitarity,

$$(K_f v, w) = \int_G (P_f g v, g w)\,dg$$

$$= \int_G (gv, f)(f, gw)\,dg.$$

Assume, for the moment, that v and w are continuous functions. Then this last expression can be written as

$$\int_G (gv,f)(f,gw)\,dg = \int_G \int_G v(g^{-1}a)\overline{f(a)}\,da \int_G f(b)\overline{w(g^{-1}b)}\,db\,dg$$

$$= \int_G \int_G v(a)\overline{f(ga)}\,da \int_G f(gb)\overline{w(b)}\,db\,dg$$

$$= \int_G \int_G k_f(a,b) v(a)\overline{w(b)}\,da\,db$$

$$= \left(\int_G k_f(a,\cdot) v(a)\,da, w \right)$$

so that

$$K_f v = \int_G k_f(a,\cdot) v(a)\,da.$$

Thus K_f is given as an integral operator with the continuous kernel k_f. In particular, K_f carries any bounded set in $L^2(G)$ into an equicontinuous set, and hence is a compact operator:

$$|K_f v(b_1) - K_f v(b_2)| \leqslant \|v\| \sup_{a \in G} |k_f(a,b_1) - k_f(a,b_2)|.$$

Finally, since K_f was obtained from P_f by averaging over the group, it follows in the usual fashion that K_f commutes with all elements of G. If f lies in some invariant subspace, W, then it is clear that $K_f(L^2(G)) \subset W$. The operator K_f is self-adjoint and

$$(K_f f, f) = \int_G (P_f g f, g f)\,dg > 0$$

since $(P_f g f, g f) \geqslant 0$ for all g and $(P_f f, f) = 1$. Thus $K_f \neq 0$. Therefore K_f, being a compact operator, has an eigenspace corresponding to some non-zero eigenvalue, and this eigenspace must be finite dimensional (since K_f is compact) and invariant under G (since K_f commutes with the elements of G). Thus, if W is a non-trivial invariant subspace of $L^2(G)$, we have produced a non-trivial invariant subspace of W which is finite dimensional (and hence closed). If W is irreducible, this subspace must coincide with W, proving the proposition.

Now let U_i be a fundamental sequence of neighborhoods of e, and let $f_i \geqslant 0$ be a sequence of continuous functions with supp $f_i \subset U_i$ and $f_i(e) > 0$. By multiplying f_i by a suitable constant we may further assume that $\int |f_i(g)|\,dg = 1$. We set

$$R_i = R_{f_i} = \int_G f(a) a\,da$$

in the regular representation, and notice that

$$R_i v(b) = \int_G f_i(a) v(a^{-1}b)\,da = \int_G f_i(ab^{-1}) v(a^{-1})\,da = \int_G f_i(ba^{-1}) v(a)\,da$$

so that R_i is an integral operator with continuous kernel $h_i(a,b) = f_i(ba^{-1})$ and hence is compact. On the other hand, the sequence $R_i v$ approaches v as $i \to \infty$. Indeed,

$$\|R_i v - v\| = \left\| \int_G f_i(a)(av - v)\,da \right\|$$

$$\leqslant \int_{U_i} |f_i(a)|\,\|av - v\|\,da$$

which approaches zero since $\int_G |f_i(a)|\,da = 1$ and $av \to v$ as $a \to e$. Let us write

$$L^2(G) = H_{0,i} + \sum_j H_{j,i} \quad \text{(Hilbert space direct sum)},$$

where $H_{0,i}$ is the zero eigenspace of R_i and the $H_{j,i}$ are non-zero eigenspaces, and hence the $H_{j,i}$ are finite dimensional for $j \neq 0$. We claim that

$$L^2(G) \text{ is the closure of the subspace } \sum_i \sum_{j \geqslant 1} H_{j,i}.$$

Indeed, if v is orthogonal to this subspace, then $v \in \cap H_{0,i}$ so that $R_i v = 0$ for all i, implying that $v = 0$.

For each j and i, let $W_{j,i}$ denote the intersection of all the closed invariant subspaces of $L^2(G)$ which contain $H_{j,i}$. It, itself, is clearly a closed invariant subspace containing $H_{j,i}$ and hence is the minimal such subspace. We claim that any closed invariant subspace of $W_{j,i}$ must have a non-zero intersection with $H_{j,i}$. Indeed, suppose that U is a closed invariant subspace of $W_{j,i}$ whose intersection with $H_{j,i}$ were zero. Then the orthogonal complement of U in $W_{j,i}$ would be a closed invariant subspace containing $H_{j,i}$ and hence must coincide with $W_{j,i}$, implying that $U = 0$. Consider all the intersections $U \cap H_{j,i}$, as U ranges over the closed invariant subspaces. The spaces $U \cap H_{j,i}$ are all finite dimensional and non-zero. Let us pick such a subspace, $H^1_{j,i}$, which has minimum dimension, and let us set

$$W^1_{j,i} = \cap U,$$

where U ranges over closed invariant subspaces satisfying $U \subset W_{i,j}$ and $U \cap H_{j,i} = H^1_{j,i}$. Thus $W^1_{j,i}$ is the smallest closed invariant subspace of $W_{j,i}$ whose intersection with $H_{j,i}$ is $H^1_{j,i}$. But this implies that $W^1_{j,i}$ is irreducible. Indeed, any proper closed subspace of $W^1_{j,i}$, would have to intersect $H_{j,i}$ in a proper subspace of $H^1_{j,i}$ contradicting our choice of $H^1_{j,i}$. Let us now replace $H_{j,i}$ by $H_{j,i} \cap (H^1_{j,i})^\perp$ and $W_{j,i}$ *by* $W_{j,i} \cap (W^1_{j,i})^\perp$. Proceeding as before, we will find a collection, $H^1_{j,i}, H^2_{j,i}, \ldots$ of mutually orthogonal Subspaces of $H_{j,i}$, and a collection $W^1_{j,i}$, $W^2_{j,i}, \ldots$ of irreducible mutually orthogonal subspaces with $W^k_{j,i} \cap H_{j,i} = H^k_{j,i}$. Since $H_{j,i}$ is finite dimensional, there will be only finitely many such subspaces, $H^k_{j,i}$ and $W^k_{j,i}$.

Suppose that $(W^k_{j,i}, W^l_{r,s}) \neq \{0\}$. Then the orthogonal projection onto $W^k_{j,i}$ is non-trivial, when restricted to $W^l_{r,s}$, and hence, by Schur's lemma, these two irreducible subspaces give equivalent representations. It follows from the expresson for R_i that it has the same eigenvalue on these two subspaces, so that $W^l_{r,s}$ has a non-trivial intersection with $H_{j,i}$ and hence $W^l_{r,s} \subset W_{j,i}$.

Let us relabel the $W_{j,i}$ as W_1, W_2, etc. and set

$$U_1 = W_1, \quad U_2 = W_2 \cap U_1^\perp, \ldots,$$

$$U_{j+1} = W_j \cap (U_1 \oplus \cdots \oplus U_j)^\perp, \ldots$$

so that

$$W_1 + \cdots + W_j = U_1 \oplus \cdots \oplus U_j \text{ (orthogonal direct sum).}$$

By the above remarks, each U_j is the direct sum of finitely many irreducibles (which we know to be finite dimensional) and, since $L^2(G) = \overline{\sum H_{ij}}$ we conclude that $L^2(G)$ is the Hilbert space direct sum of the U_j's. This proves assertion (2) of the Peter–Weyl theorem.

We have actually proved a little more, namely that there are only finitely many irreducible subspaces in the direct sum decomposition of a given type. Indeed, on equivalent irreducible representations, the operators R_i must have the same eigenvalues, and hence all such irreducibles correspond to the same $H_{j,i}$, and thus there cannot be an infinite number of mutually orthogonal irreducible subspaces of the same type.

Before proceeding with the rest of the proof of the Peter–Weyl theorem, it will be of some use to us to isolate the preceding argument and state the conclusions in a broader context which will have other applications for us. Let G be a continuous group with left invariant measure $\int dg$, and let r be a unitary representation of G. If f is any continuous function with compact support defined on G, we can form the operator R_f. We say that the representation is *completely continuous* if each of the operators R_f is a compact operator. An examination of the above argument shows that we have proved the following proposition.

Proposition E.2

Let r be a completely continuous representation of the topological group G on the Hilbert space, H. Then we can decompose H into a Hilbert space direct sum $H = \oplus W_i$ of irreducible subspaces, W_i where there are only finitely many irreducible subspaces of any given equivalence class of irreducible representations.

Notice that if W is a closed invariant subspace of a completely continuous representation, then the restriction of the representation to W is again completely continuous.

We now return to the Peter–Weyl theorem. If G is any compact group, we can map $L^2(G)$ into $L^2(G \times G)$ by sending f into θf, where $\theta f(a,b) = f(ab^{-1})$. (The image consists of those functions in $L^2(G \times G)$ which are invariant under right translation by elements of the diagonal subgroup, i.e. the functions h on $G \times G$ which satisfy $h(ag, bg) = h(a,b)$ for all $g \in G$.) Notice that

$$\|\theta f\|^2 = \int_G \int_G |f(ab^{-1})|^2 \, da \, db = \int_G |f(a)|^2 \, da = \|f\|^2$$

so that the map θ is unitary. If we consider G as the subgroup $G \times \{e\}$ of $G \times G$ then θ is a G morphism. Finally, since left translation commutes with right translation, the image of θ is an invariant subspace of $L^2(G \times G)$, for the regular representation of $G \times G$. Thus, by restriction, we can regard $G \times G$ as acting on $L^2(G)$, and, by the representation of $G \times G$ on $L^2(G)$ is completely continuous. By Proposition E.2 we know that $L^2(G)$ decomposes under $G \times G$ into a direct sum of irreducibles with each type occurring a finite number of times, and by Proposition E.1 (applied to $G \times G$, where $L^2(G)$ is a subspace of $L^2(G \times G)$), we know that each irreducible is finite dimensional. We claim that every finite-dimensional irreducible representation of $G \times G$ occurs exactly once in this decomposition. Indeed let W_1 and W_2 be two irreducible subspaces of $L^2(G)$ under $G \times G$ which define equivalent representations. We claim that $W_1 = W_2$. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be bases of W_1 and W_2 such that $G \times G$ has the same matrix representation relative to these bases. Let $F(a,b) =$

$\sum u_i(a)v_i(b)$. Then, letting (r_{ij}) denote the matrix of the irreducible representation in question, we have

$$\begin{aligned} F(gah^{-1}, gbh^{-1}) &= \sum u_j(gah^{-1})\overline{v_j(gbh^{-1})} \\ &= \left(\sum_j r_{ij}(g,h)\overline{r_{ij}(g,h)}\right)\sum u_i(a)\overline{v_i(b)} \\ &= F(a,b) \end{aligned}$$

since the matrix (r_{ij}) is unitary. In particular,

$$\begin{aligned} \sum u_i(g)\overline{v_i(e)} &= F(g,e) = F(e,g^{-1}) \\ &= \sum u_i(e)v_i\overline{(g^{-1})} = \sum u_i(e)\overline{\tilde{v}_i(g)}. \end{aligned}$$

If we set $v = \sum u_i(e)v_i \in W_2$, the above equation asserts that $\bar{\tilde{v}} \in W_1$. Now the space of all functions of the form $\bar{\tilde{v}}$, $v \in W_2$ is clearly an invariant subspace for $G \times G$, and hence must coincide with W_1. If we take $W_1 = W_2$ in the above argument, we see that $\bar{\tilde{u}} \in W_1$ if and only if $u \in W_1$, so that any two equivalent irreducibles in the decomposition of $L^2(G)$ under $G \times G$ coincide.

We now propose to show that these irreducibles are isomorphic under $G \times G$ to the irreducible representations of $G \times G$ on $\mathrm{Hom}(V,V) = V \otimes V^*$, as V ranges over the finite-dimensional irreducible representations of G. Indeed, given $u \otimes v^* \in V \otimes V^*$ we obtain a function $f_{u\otimes v^*}$ on G defined by $f_{u\otimes v^*}(g) = v^*(gu)$, and it is clear that this extends to a map of $V \otimes V^* \to L^2(G)$. The group $G \times G$ acts on $V \otimes V^*$ and

$$f_{(g,h)(u\otimes v^*)}(a) = f_{gu\otimes h^{*-1}v^*}(a) = v^*(h^{-1}agu) = f_{u\otimes v^*}(h^{-1}ag)$$

so that the map is equivariant for the action of $G \times G$. Since G acts irreducibly on V, and hence also on V^*, we can conclude that $G \times G$ acts irreducibly on $V \otimes V^*$. Indeed, if we examine the proof of the assertion on p. 66, we see that the proof works without change for the case of compact groups. Thus each $V \otimes V^*$ does occur exactly once in the decomposition of $L^2(G)$ under $G \times G$. If we knew that every irreducible finite-dimensional representation of $G \times G$ must be a tensor product of irreducibles, we could conclude that every irreducible subspace in $L^2(G)$ must be equivalent to a representation of $G \times G$ on $\mathrm{Hom}(V', V) = V \otimes V'^*$. By Schur's lemma, we could conclude that $V \sim V'$ if we found a non-zero element in this space which was invariant under all $(g,g) \in G \times G$. Let us establish the existence of such a non-zero G-invariant vector in any $G \times G$ irreducible subspace, W, of $L^2(G)$. Let $w_1, \ldots, w_n$ be an orthonormal basis of W. Let $B(a,b) = \overline{w_1(ab^{-1})}w_1 + \cdots + \overline{w_n(ab^{-1})}w_n$, so that $B: G \times G \to W$, and

$$\begin{aligned} B(ha, gb) &= \overline{w_1(hab^{-1}g^{-1})}w_1 + \cdots + w_n\overline{(hab^{-1}g^{-1})}w_n \\ &= (h,h)B(a,b). \end{aligned}$$

Since B does not map all of $G \times G$ into zero, it follows that $B(e,e) = 0$. But it also follows directly from the above equation and the definition of B that $B(e,e)$ is invariant

under all (g,g). We still must prove that irreducibles of $G \times G$ are tensor products of irreducibles of G. This follows from proposition E.3.

Proposition E.3.
Let r be an irreducible unitary representation of the topological group $G_1 \times G_2$ on a finite-dimensional vector space W, where G_1 and G_2 are topological groups and $G_1 \times G_2$ is given the product topology. Then $r \sim r_1 \otimes r_2$, where r_i is an irreducible unitary representation of G_i, $i = 1, 2$.

Proof Consider the restriction of r to the group G_1, considered as the subgroup $G_1 \times \{e\}$ of $G_1 \times G_2$. The space W decomposes into a direct sum of irreducibles, $W = V_1 \oplus \cdots \oplus V_k$ under G_1. We claim that all the V_i are equivalent as representation spaces of G_1. Indeed, let W_1 be the sum of G_1-invariant subspaces, U, such that $\mathrm{Hom}_G(V_1, U) = \{0\}$, so that W_1 is the maximal such subspace. It is clear that W_1 is G_1 invariant, and we claim that it is also G_2 invariant. Indeed, $r_{(e,h)}$ commutes with all $r_{(g,e)}$ and so $r_{(e,h)}T \in \mathrm{Hom}_{G_1}(V_1, r_{(e,h)}U)$ if and only if $T \in \mathrm{Hom}_{G_1}(V_1, U)$. Thus W_1 is $G_1 \times G_2$ invariant, and since $V_1 \cap W_1 = \{0\}$, we conclude that $W_1 = \{0\}$. Thus all the V_i are equivalent. Let $U_2 = \mathrm{Hom}_{G_1}(V_1, W)$. By Schur's lemma we know that $\dim U_2 = k$, and by the above considerations we know that G_2 acts on U_2 by sending $T \sim\to r_{(e,h)}T$. We have the obvious evaluation map of $V_1 \otimes U_2 \to W$ sending $v \otimes T \sim\to Tv$, $v \in V_1$ and $T \in U_2$. The group G_1 acts on V_1 and the group G_2 acts on U_2 so that $G_1 \times G_2$ acts on $V_1 \otimes U_2$. It is easy to check that the evaluation map is equivariant with respect to the action of $G_1 \times G_2$ on both sides, and is non-trivial. It is therefore surjective, and the representation of G_2 on U_2 is irreducible, proving the proposition.

Let us now see where we stand in the proof of the Peter–Weyl theorem. It is easy to check that the functions $f_{u\otimes v^*}$ are precisely the matrix elements for the representation of G on $V \otimes V^*$, when we take u and v to be basis elements of V. Thus the irreducible subspaces of $L^2(G)$ consist precisely of the representative functions for the various irreducible representations, and thus we have proved (1). We have also proved (2). We have proved that each finite-dimensional irreducible representation occurs in $L^2(G)$ with a multiplicity equal to its dimension, which is the assertion or (4), provided that we know (3), i.e. that each irreducible representation is finite dimensional. Since (3) is a consequence of (5), it suffices for us to prove (5).

Proof of (5) For any pair of continuous functions, f_1 and f_2, on G we define their convolution, $f_1 * f_2$ by the formula

$$(f_1 * f_2)(a) = \int_G f(ag^{-1})f_2(g)\,\mathrm{d}g.$$

If r is any unitary representation of G and we set

$$R_f = \int_G f(a)r_a\,\mathrm{d}a$$

then

$$R_{f_1}R_{f_2}=\int_G\int_G f_1(a)f_2(b)r_{ab}\,\mathrm{d}a\,\mathrm{d}b=\int_G\int_G f_1(ab^{-1})f_2(b)\,\mathrm{d}b r_a\,\mathrm{d}a$$

and so

$$R_{f_1}R_{f_2}=R_{f_1*f_2}. \tag{E.10}$$

If r^1 and r^2 are inequivalent irreducible representations we have for the matrix elements

$$r^1_{ij}*r^2_{kl}(g)=\Sigma r^1_{ij}(g)\int_G r^1_{ij}(b^{-1})r^2_{kl}(b)\,\mathrm{d}b=0$$

so that

$r^1_{ij}*r^2_{kl}=0$ *if* r^1 *and* r^2 *are inequivalent irreducible finite-dimensional representations,* (E.11)

and a similar argument shows that

$r_{ij}*r_{kl}=(1/n)\delta_{jk}r_{il}$ *if* r *is an irreducible representation of degree* n. (E.12)

Taking the trace of (E.11) and (E.13) gives, for irreducible characters

$$\chi^i*\chi^j=\begin{matrix}0 & \text{if} & \chi^i\neq\chi^j\\ (1/\chi^i(e))\chi^i & \text{if} & \chi^i=\chi^j\end{matrix} \tag{E.13}$$

where χ^j and χ^j are irreducible characters.

Combining (E.10) with (E.13) shows once again that

$$P_i=\chi^i(e)R_{\bar{\chi}^i}$$

is a projection operator for any unitary representation, r. We first propose to show that the sum of all these projections, as χ^i ranges over all the finite-dimensional irreducible characters, is the identity operator. The following amounts to the same thing: we want to show that if v is a vector in the representation space such that $P_iv=0$ for all i, then $v=0$. Indeed, let us define the function f by

$$f(g)=\langle v, r_gv\rangle.$$

Then

$$\overline{f*\chi^i}(a)=\int_G\langle v,\chi^i(b)r_{ab^{-1}}v\rangle\,\mathrm{d}b=(1/\chi^i(e))\langle v,r_aP_iv\rangle=0.$$

Now the function f is continuous, and hence has an expansion in terms of the (orthogonal system of functions given by) the matrix coefficients, r^k_{ij}, as r^k ranges over the irreducible representations of G: if we set

$$c^k_{ij}(f)=(f,r^k_{ij}) \tag{E.14}$$

then

$$f=\sum n_k^{\frac{1}{2}}c^k_{ij}(f)r^k_{ij},\quad \|f\|^2=\sum_k n_k\Big(\sum_{i,j}|c^k_{ij}(f)|^2\Big). \tag{E.15}$$

This is true for any $f\in L^2(G)$ by virtue of that portion of the Peter–Weyl theorem that

we have already proved. Now

$$\operatorname{tr}R^k_fR^{k*}_f=\sum_{i,j}\int_G f(a)r^k_{ij}(a)\,\mathrm{d}a\int\bar{f}(b)\bar{r}_{ij}(b)\,\mathrm{d}b$$
$$=\sum|c^k_{ij}(\bar{f})|^2$$

and thus, since $\|f\|^2=\|\bar{f}\|^2$ we can rewrite the second equation in (E.15) as

$$\|f\|^2=\sum n_k\operatorname{tr}R^k_fR^{k*}_f. \tag{E.16}$$

Now for any representation and any continuous function, f, we have

$$R_{f^*}=R_{\bar{\bar{f}}},\ \text{where}\ \bar{\bar{f}}(g)=\overline{f(g^{-1})} \tag{E.17}$$

as can be easily verified. We can thus rewrite (E.16) as

$$\|f\|^2=\sum n_k\operatorname{tr}R^k_{\bar{\bar{f}}*f}. \tag{E.18}$$

Equations (E.14)–(E.18) are valid for all functions f. Let us now apply (E.18) to the function f given by $f(g)=\langle v,r_gv\rangle$. We know that

$$f*\chi^{-k}=0$$

and therefore

$$\bar{\bar{f}}*f*\chi^k=0.$$

But

$$(\bar{\bar{f}}*f*\chi^k)(e)=\operatorname{tr}R^k_{\bar{\bar{f}}*f},$$

and we conclude from (E.18) that $f=0$. Since the function f is continuous, we conclude that $f(e)=0$, which is just the assertion $v=0$.

We have thus proved that no v is orthogonal to all the spaces P_iH, where H is the Hilbert space of the representation and P_i ranges over all the projections associated with the finite-dimensional irreducible representations. To complete the proof of the Peter–Weyl theorem, it suffices to show that if $P_iv=v$, then v lies in a finite-dimensional invariant subspace. Let W be the space spanned by all the vectors r_av, $a\in G$. We will show that W is finite dimensional, and, in fact, has dimensional at most n_i^2. For this it suffices to show that any collection of more than n_i^2 vectors of the form $r_{a_j}v$ must be linearly dependent, or that the matrix whose entries are $\langle r_{a_i}v,r_{a_j}v\rangle$ is singular. This will certainly be the case if we can show that functions f_j given by

$$f_j(g)=\langle v,r_{ga_j}v\rangle=f(ga_j),\quad f(b)=\langle v,r_bv\rangle$$

are linearly dependent. But,

$$n_i(f*\bar{\chi}^i)(b)=\langle v,r_bP_iv\rangle=f(b)$$

and so

$$f(ga)=n_i\int_G f(gab^{-1})\chi_i(b)\,\mathrm{d}b$$
$$=n_i\int_G f(ba^{-1}g^{-1})\chi_i(b^{-1})\,\mathrm{d}b$$
$$=n_i\int_G f(b)\chi_i(b^{-1}ga)\,\mathrm{d}b$$

is a superposition of the matrix function $r^i_{jl}(g)$, and there are only n_i^2 linearly independent such functions. This completes the proof of the Peter–Weyl theorem.

There are a number of immediate corollaries and minor improvements of the Peter–Weyl theorem which are worth recording. We recall that a function, f, is called a central function if it satisfies $f(ab)=f(ba)$, i.e. if it is constant on conjugacy classes. For a central function, f, the operator $C_f=\int_G f(a)r_{a^{-1}}\,da$ satisfies

$$C_f r_b=\int_G f(a)r_{a^{-1}b}\,da=\int_G f(ba)r_{a^{-1}}\,da=\int_G f(ab)r_{a^{-1}}\,da=r_bC_f.$$

By Schur's lemma, if r is irreducible then C_f is a scalar operator and

$$\operatorname{tr} C_f=(f,\chi),\ \text{so}\ C_f=(1/\chi(e))(f,\chi)I.$$

Writing this out as an equation for the matrix entries with $\chi=\chi^k$ gives

$$(f,r^k_{ij})=\frac{1}{n_k}\delta_{ij}(f,\chi^k).$$

If we now apply (E.14) and (E.15) we get

$$\text{if } f \text{ is a central function then } f=\sum_k (f,\chi^k)\chi^k. \tag{E.19}$$

The equality in (E.15) and (E.19) are in the sense of $L^2(G)$, the series converge in the norm of the Hilbert space. If f is a continuous function, it need not be true that the series converge in the sup norm. Nevertheless, the representative functions are dense in the sup norm: This is the content of the next proposition, which asserts that any continuous function can be uniformly approximated by representative functions. But, as we mentioned, the specific 'Fourier series' given by the right-hand side in (E.15) or (E.19) need not converge in the uniform (sup) norm.

Proposition E.4

Given any continuous function, f, on the compact group, G, and given any $\varepsilon>0$, there exists a representative function, q, such that

$$\sup_{g\in G}|f(g)-q(g)|<\varepsilon.$$

Proof If h is any continuous function on G and $p\in V\otimes V^*$ is a representative function, then, letting r denote the regular representation of G on $L^2(G)$, we have

$$R_hp\in V\otimes V^*.$$

On the other hand, if v denotes any element of $L^2(G)$, we have

$$(R_hv)(a)=\int_G h(g)v(g^{-1}a)\,dg=\int_G h(ag^{-1})v(g)\,dg=h*v.$$

Suppose we choose some neighborhood, U, of e such that

$$|f(x)-f(y)|<\varepsilon \quad \text{for } xy^{-1}\in U.$$

Let us choose h to be a non-negative continuous function with support in U, and with $\int_G h(g)\,dg=1$. Then it is easy to see that

$$|(R_hf)(a)-f(a)|=|(h*f)(a)-f(a)|<\frac{\varepsilon}{2}.$$

If v is continuous we have

$$|h*v(a)|\leqslant\|h\|\ \|v\|.$$

Now choose a representative function, p, such that

$$\|f-p\|\leqslant\frac{\varepsilon}{2\|h\|}.$$

Then

$$|f(a)-h*p(a)|\leqslant|f(a)-h*f(a)|+|h*f(a)-h*p(a)|\leqslant\frac{\varepsilon}{2}+\frac{\varepsilon}{2},$$

proving the proposition with $q=h*p$.

As an immediate corollary we obtain

Proposition E.5

A compact group is commutative if and only if all its irreducible representations are one dimensional.

Proof If G is compact, then every irreducible representation must be finite dimensional. If G is not commutative, we can find a and b in G with $ab\neq ba$, and hence a continuous function f, with $f(ab)\neq f(ba)$. If all irreducible representations were one dimensional, all representative functions would be central, so that f could not be approximated by representative functions.

Let us give some examples of non-finite compact groups. The most familiar example is the group $C_\infty=O^+(2)$ of rotations in the plane. Its elements consist of rotations through angle θ, where θ is determined mod 2π. In other words, the group is $\mathbb{R}/2\pi\mathbb{Z}$. If we use θ (mod 2π) to parametrize the group then it is clear that the Haar measure is $(1/2\pi)\,d\theta$. The group is commutative so that all its irreducible representations are one dimensional. Any one-dimensional represention is given by a $\mathbb{C}$-valued continuous function, φ, satisfying $\varphi(\theta_1+\theta_2)=\varphi(\theta_1)\varphi(\theta_2)$ and $\varphi(\theta+2\pi)=\varphi(\theta)$. The only such functions are the exponential functions, χ^n:

$$\chi^n(\theta)=\exp in\theta.$$

The orthogonality relations between the characters reduce to the usual orthogonality relations between exponentials. The coefficients in (E.14) are the usual Fourier coefficients. The right-hand side of the first equation in (E.15) is the standard Fourier series expansion of f, and (E.15) asserts that the Fourier series converges in L^2, with the second assertion in (E.15) being the Plancharel formula for Fourier series. Proposition E.4 is then (a version of) the Weirstrass approximation theorem. The equation $\chi^m\cdot\chi^n=\chi^{m+n}$ tells us all there is to know about the tensor product of two representations.

As our second group, consider the orthogonal group in two dimensions, $O(2) = D_\infty$. It contains C_∞ as a normal commutative subgroup of index 2, and $O(2)$ is the semidirect product of C_∞ with $\mathbb{Z}_2$: let τ denote any reflection in $O(2)$, and let us denote rotation through angle θ by ρ_θ. Then every element of the group is either of the form ρ_θ (if it is a rotation) or of the form $\tau\rho_\theta$ (if it is a reflection), and we have the relations

$$\tau^2 = 1, \quad \tau\rho_\theta\tau = \rho_{-\theta}.$$

If f is any continuous function on $O(2)$ it is easy to check that

$$\int_G f(g)\,\mathrm{d}g = \frac{1}{4\pi}\int_0^{2\pi} f(\rho_\theta)\,\mathrm{d}\theta + \frac{1}{4\pi}\int_0^{2\pi} f(\tau\rho_\theta)\,\mathrm{d}\theta$$

is invariant under left (and right) translations and hence gives the formula for the invariant integral. Let us construct irreducible representations of $O(2)$ by analogy with our construction of the irreducible representations of the finite dihedral groups. We get two one-dimensional representations coming from the representations of $\mathbb{Z}_2$, regarded as the quotient group $O(2)/O^+(2)$. These have characters

$$\varphi^1 : \varphi^1(g) = 1$$

and

$$\varphi^2 : \varphi(\rho_\theta) = 1, \quad \varphi(\tau\rho_\theta) = -1.$$

We also have the two-dimensional irreducible representations, r^k, $k = 1, 2, \ldots$, given by

$$r^k_{\rho_\theta} = \begin{pmatrix} \mathrm{e}^{ik\theta} & 0 \\ 0 & \mathrm{e}^{-ik\theta} \end{pmatrix}$$

$$r^k_{\tau\rho_\theta} = \begin{pmatrix} 0 & \mathrm{e}^{-ik\theta} \\ \mathrm{e}^{ik\theta} & 0 \end{pmatrix}$$

with character, χ^k, given by

$$\chi^k(\rho_\theta) = 2\cos k\theta \quad \chi^k(\tau\rho_\theta) = 0.$$

We claim that these are all the irreducible representations. To prove this it suffices to show that the characters listed above form an orthonormal basis for the space of central function in $L^2(G)$. Now if a function, f, is central, then $f(\rho_{-\theta}\tau\rho_\theta) = f(\tau\rho_{2\theta}) = f(\tau)$, so that f is a constant on elements of the form $\tau\rho_\theta$, while $f(\tau\rho_\theta\tau) = f(\rho_{-\theta})$ so that f is an even function of θ. By a linear combination of φ^1 and φ^2 we can obtain all the constant functions of $\tau\rho_\theta$ and the cosines form an orthonormal basis for the even functions, proving that we have indeed found all the irreducible representations of D_∞. We have

$$\varphi_1 \cdot \chi = \chi \quad \text{for any character, } \chi$$

$$\varphi_2 \cdot \varphi_2 = \varphi_1, \quad \varphi_2 \cdot \chi^k = \chi^k$$

and

$$\chi^k \cdot \chi^l = \chi^{k+l} + \chi^{k-l}\ (k \neq l), \quad \chi^k \cdot \chi^k = \chi^{2k} + \varphi_1 + \varphi_2,$$

since $4\cos k\theta\cos l\theta = 2\cos(k+l)\theta + 2\cos(k-l)\theta$. This gives us the decomposition of the tensor product of two representations.

An entirely different kind of compact, or rather totally bounded, group is provided by the theory of almost periodic functions, which plays an important role in celestial mechanics. Let G be any group (not necessarily carrying a topology). A function, f, on G is called *almost periodic* if the family of functions, f_g, on $G \times G$, where

$$f_g(a, b) = f(agb)$$

is totally bounded relative to the uniform norm on $G \times G$. The set of all almost periodic functions forms a vector space, and each almost periodic function induces a notion of distance on G by setting

$$d_f(a, b) = \sup_{g,h\in G} |f(gah) - f(gbh)|.$$

It is clear that

$$d_f(a, b) = d_f(aa', ba') = d_f(a'a, a'b) \quad \text{for any } a' \in G,$$

i.e. d_f is both right and left invariant. Also

$$d_f(a_1a_2, b_1b_2) \leqslant d_f(a_1a_2, a_1b_2) + d_f(a_1b_2, b_1b_2) = d_f(a_2, b_2) + d_f(a_1, b_1),$$

$$d_f(a^{-1}, b^{-1}) = d_f(aa^{-1}b, ab^{-1}b) = d_f(b, a) = d_f(a, b)$$

and

$$|f(a) - f(b)| \leqslant d_f(a, b).$$

If $F = \{f_1, \ldots, f_n\}$ is any finite collection of almost periodic functions, and we set

$$d_F(a, b) = \max_i d_f(a, b),$$

then it is clear that d_F also satisfies the above relations, i.e.

$$d_F(a, b) = d_F(a'a, a'b) = d_F(aa', ba') \tag{E.20}$$

$$d_F(a_1a_2, b_1b_2) \leqslant d_F(a_1, b_1) + d_F(a_2, b_2) \tag{E.21}$$

$$d_F(a^{-1}, b^{-1}) = d_F(a, b) \tag{E.22}$$

and

$$|f(a) - f(b)| \leqslant d_F(a, b) \quad \text{if} \quad f \in F. \tag{E.23}$$

If we let $H_F = \{a \mid d_F(a, e) = 0\}$ then it follows from (E.20) and (E.21) that H_F is a normal subgroup of G. Let G_F denote the quotient group, $G_F = G/H_F$. It follows from (E.20)–(E.22) that d_F induces a right and left invariant metric on $G_{F'}$ which we shall continue to denote by d_F. It follows from (E.23) that any $f \in F$ induces a continuous function on G_F which we shall continue to denote by f. It is clear that if F and F' are two finite sets of almost periodic functions with $F \subset F'$ then

$$d_F(a, b) \leqslant d_{F'}(a, b)$$

and therefore $H_{F'} \subset H_F$ so that we get an induced map of $G_F \to G_{F'}$ which is continuous. If f is any almost periodic function, then the assertion that f_a is uniformly bounded means that for any $\varepsilon > 0$ we can find a finite set, $\{a_1, \ldots, a_k\}$ of elements if G such that

$$\min_i d_f(a, a_i) < \varepsilon \quad a \in G.$$

It follows that the same is true if we replace d_f by d_F, and thus the group G_F is uniformly bounded. (We could, if we like, complete it relative to the metric d_F and obtain a compact group, $\bar{G}_F$). We can now apply our theorem concerning the existence and uniqueness of Haar measure to conclude the following result due to von-Neumann.

Proposition E.6

Let A denote the space of almost periodic functions on the group G. There is a unique linear functional, μ, on A which satisfies

$$\mu(1) = 1$$

$$\mu(f) \geqslant 0 \quad \text{if } f \geqslant 0$$

and

$$\mu(f_g) = \mu(f), \quad \text{where } f_g(a) = f(g^{-1}a).$$

We can use the functional μ to introduce a scalar product, $(f_1, f_2) = \mu(f_1 \bar{f}_2)$ on the space of almost periodic functions, A, and complete it to get a Hilbert space, talk about almost periodic representative functions, apply the Peter–Weyl theorem and so on. We refer the reader to any standard modern text on almost periodic functions for the details. As an illustration of the scope of the results proved so far, we examine the classical case of almost periodic functions on the (commutative) group $\mathbb{R}$. For any real number, λ, the exponential function $\chi_\lambda(x) = \exp i\lambda x$ is clearly almost periodic, and satisfies

$$(\chi_\lambda)_y(x) = (\exp - i\lambda y)\chi_\lambda(x)$$

Thus $\mu(\chi_\lambda) = (\exp i\lambda y)\mu(x_\lambda)$ and therefore

$$\mu(\chi_\lambda) = \begin{matrix} 0 & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0 \end{matrix}$$

Notice that this same set of equalities holds for

$$\nu(\varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(t)\, dt$$

with $\phi = \chi_\lambda$.

Now let f be any almost periodic function on $\mathbb{R}$, and let $F = \{f\}$. It is then clear that the $d_f(a, b) = 0$ if and only if $f_a = f_b$, where $f_a(x) = f_b(x)$ and so we can identify the group G_F with the set of all f_a and the map of $\mathbb{R} \to \mathbb{R}_F$ sends a into f_a. The group $\mathbb{R}_F$ is commutative, and each (uniformly) continuous character on $\mathbb{R}_F$ induces a continuous character on $\mathbb{R}$ which must then be of the form χ_λ. Now suppose that the function f, in addition to being almost periodic, is also continuous as a function on $\mathbb{R}$. Then the function on $\mathbb{R}_F$ sending f_a into $f_a(0) = f(a)$ is a continuous function on $\mathbb{R}_F$. Hence, by Proposition E.4, we can find, for any $\varepsilon > 0$, characters $\chi_{\lambda_1}, \ldots, \chi_{\lambda_k}$ and constants

$c_1, \ldots, c_k$ so that

$$\sup_{a \in \mathbb{R}} |f(a) - \sum c_j \exp i\lambda_j a| < \varepsilon.$$

This is the content of the celebrated 'Bohr approximation theorem' which asserts that any continuous almost periodic function on $\mathbb{R}$ can be uniformly approximated by a linear combination of exponentials. Notice, among other things, that this implies that $\nu(f)$ converges for any continuous almost periodic function and that, since $\nu = \mu$ for exponentials,

$$\mu(f) = \lim \frac{1}{T} \int_0^T f(t)\, dt$$

for any continuous almost periodic function. For any λ we can form the scalar product

$$a_f(\lambda) = (f, \chi_\lambda) = \lim \frac{1}{T} \int_0^T f(t)(\exp - i\lambda t)\, dt$$

and, by the second equation in (E.15), we know that

$$\lim 1/T \int_0^T |f(t)|^2\, dt = \sum_\lambda |a_f(\lambda)|^2.$$

In particular, there are only countably many values of λ for which the 'Fourier coefficients' $a_f(\lambda)$ do not vanish, and these λ's are called the 'frequencies' of the almost periodic function, f. The series $\sum a_f(\lambda_j)\chi_{\lambda_j}$ converges to f in the L^2 sense (relative to the mean value, μ) by (E.15). It is not difficult to show that in the Bohr approximation theorem we need only use those λ which are frequencies of f, but, of course, the actual 'Fourier series' need not converge. We leave the details to the reader.

We now return to the general theory of compact groups and close this section with a brief discussion of homogeneous vector bundles and induced representations.

Let G be a topological group and M a topological space. When we speak of an action of G on M, we now demand the additional requirement that the map $G \times M \to M$ be continuous. In the definition of a vector bundle, E over a topological space, M, we require, in addition to the previous requirements, that E be a topological space and that the projection $\pi: E \to M$ be continuous. We can let $C(E)$ denote the space of continuous sections of E and let $C_0(E)$ denote the space of continuous sections of compact support. It is clear that any $g \in G$ acts as a linear transformation on $C(E)$ and $C_0(E)$, sending the section s, into the section gs, where, as usual, $(gs)(x) = g(s(x))$. It is also clear that $(g_1 g_2)s = g_1(g_2 s)$. In the main we will be interested in the situation where the fibers, E_x, are finite-dimensional spaces, although much of what we shall say is valid in greater generality.

Let H be a closed subgroup of G. Then G/H is a topological space on which G acts transitively. If r is a representation of H on some finite-dimensional vector space, V, then we can form the topological space $E = G \times_H V$. As before, E consists of equivalence classes of pairs (g, v), where $(gh, v) = (g, r_h v)$. The topology on E is the quotient topology inherited from the product topology on $G \times V$. A section $s: G/H \to E$ gives a function,

f_s, from G to V by the equation $s(gH)=[(g, f_s(g))]$, and f_s satisfies the identity

$$f(gh)=r_h f(g) \quad \text{for any } h\in H. \tag{E.24}$$

Conversely, any function from G to V which satisfies (E.24) clearly determines a section of E over G/H. It is clear that s is continuous if and only if f_s is continuous.

Let E be a continuous vector bundle over a topological space, M, and suppose that each of the fibers, E_x, is a Hilbert space under a scalar product. $\langle\ ,\ \rangle_x$, with the property that for any two continuous sections, s_1 and s_2, the function on M given by $x \rightsquigarrow \langle s_1(x), s_2(x)\rangle_x$ is continuous. We then say that E is a Hermitian vector bundle. If a group G acts on E so that the map $g: E_x \to E_{gx}$ is unitary for all g and x, then we say that E is a homogeneous Hermitian vector bundle for G. Suppose that E is a Hermitian vector bundle over M and that we have a notion of integration over M, i.e. we are given a measure on M. Then we can introduce a scalar product on $C_0(E)$ by setting

$$(s_1, s_2)=\int_M \langle s_1(x), s_2(x)\rangle_x \,\mathrm{d}x$$

The completion of the space $C_0(E)$ relative to this scalar product is called $L^2(E)$. If E is a homogeneous Hermitian vector bundle for G and G preserves the measure on M then we clearly get a unitary representation of G on $L^2(E)$. An analysis of the representation of G on $L^2(E)$ is somewhat complicated, although most of the theorems concerning the Mackey decomposition can be reformulated so as to hold in this context. In fact, they were originally proved by Mackey in even greater generality. We shall not go into this point here. Even if the group G is compact, the situation is a bit complicated, because the action of G on $L^2(E)$ need not be completely continuous. However, if G acts transitively on M, i.e. if $M = G/H$, where H is a closed subgroup of the compact group, G, and E is the vector bundle induced from some unitary representation of H on a Hilbert space V, the situation is quite simple. First of all, there is a well defined integral on M which is invariant under G. Indeed, any function, f, on M gives rise to a function, $\hat{f}$, on G defined by

$$\hat{f}(g)=f(gH),$$

and we set

$$\int_M f(x)\,\mathrm{d}x=\int_G \hat{f}(g)\,\mathrm{d}g.$$

If s_1 and s_2 are continuous sections of E and we take

$$f(x)=\langle s_1(x), s_2(x)\rangle_x,$$

then it is easy to see that

$$\hat{f}(g)=\langle f_{s_1}(g), f_{s_2}(g)\rangle$$

where the scalar product on the right is the scalar product on V, and where, as above, f_s denotes the V-valued function on G corresponding to the section s. We can thus regard $L^2(E)$ as a subspace of $L^2(G, V)$, where $L^2(G, V)$ is the L^2 completion of the space of V-

valued functions on G, namely $L^2(E)$ is the subspace consisting of those $\varphi\in L^2(G)$ which satisfy

$$\varphi(gh)=r_h\varphi(g) \quad \text{for } h\in H.$$

It then follows directly that the representation of G on $L^2(E)$ is completely reducible, so that Proposition E.2 applies. We conclude that the induced representation, i.e. the representation of G on $L^2(E)$, decomposes into a direct sum of finite-dimensional irreducible subspaces, each occurring with finite multiplicity. In Appendix G we shall show how to introduce the notion of a character for the representation of G on $L^2(E)$, and we will be able to conclude that the Frobenius reciprocity theorem holds for induced representations of compact groups. (Actually, in Appendix G, we will be discussing a more restrictive class of groups – the Lie groups, to be defined below. However, an examination of our argument will show it to be valid in the more general case considered here.) A direct proof of the Frobenius reciprocity theorem is also quite easy, but we will not present it here.