Beginning with the simple-sounding question of how many faces a convex polytope can have, we discuss the enumeration theory for flags in Eulerian partially ordered sets. We emphasize the two main examples, face posets of convex polytopes and regular CW-spheres, and Bruhat intervals in Coxeter groups. We review the two algebraic approaches to flag enumeration and their relation via duality of Hopf algebras. One result of this is a direct expression for an important invariant of a Bruhat interval, its Kazhdan-Lusztig polynomial, in terms of a new combinatorial invariant, the complete cd-index. Finally, we summarize the theory of combinatorial Hopf algebras, developed by Aguiar, which gives a unifying framework for the quasisymmetric generating functions developed here.

This talk is aimed for a general mathematical audience. In particular, most of it should be accessible to graduate students.