

A Non-standard Geometric Quantization of the Harmonic Oscillator

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Abstract

In the standard geometric quantization of the harmonic oscillator in $(\mathbb{R}^2, dp \wedge dx)$, using standard holomorphic coordinate $z = x - ip$ induces a standard complex structure which Kähler polarizes the pre-quantum line bundle. Hence, the Quantum Hilbert space maybe identified with holomorphic functions which are L^2 integrable with respect to Gaussian measure, denoted by \mathbb{H} . The corresponding quantized Hamiltonian is then given by $\hbar z \frac{\partial}{\partial z}$. We propose using a different almost complex structure J (under some restrictions) compatible with $dp \wedge dx$ and construct a Quantum Hilbert space \mathbb{H}_J containing non-trivial L^2 sections and a quantized Hamiltonian Q_J . The main result is that (\mathbb{H}_J, Q_J) is unitarily equivalent to $(\mathbb{H}, \hbar z \frac{\partial}{\partial z})$.

1 Introduction

Let (M, w) be a symplectic manifold of dimension $2n$. Quantization of a symplectic manifold requires a construction of a Hermitian complex line bundle $B \rightarrow M$ with a metric compatible connection ∇ such that the curvature of ∇ is $\frac{1}{i\hbar}\omega$. Here, \hbar is Planck's constant divided by 2π . This line bundle B exists if and only if $\frac{1}{2\pi\hbar}w \in H^2(M, \mathbb{Z})$. If the manifold is simply connected, then up to equivalence, there is a unique line bundle (B) with a covariant derivative ∇ such that the curvature is equal to $\frac{1}{i\hbar}\omega$.

The pre-quantum Hilbert space is then the space of sections of B which are square integrable with respect to the Liouville volume form ω^n . We can then define the pre-quantum operator $Q(f)$ of a function $f : M \rightarrow \mathbb{R}$ as

$$Q(f) = -i\hbar\nabla_{X_f} + M_f \quad (1.1)$$

where X_f is the Hamiltonian vector field given by

$$df + \omega(X_f, \cdot) = 0 \quad (1.2)$$

and M_f is the multiplication operator.

It is generally agreed that the pre-quantum Hilbert space is too big. To 'trim' down the size of this Hilbert space, one chooses a polarization on the tangent space of $T_m M$. By complexification of the tangent bundle $TM^{\mathbb{C}}$, one considers a polarization of $TM^{\mathbb{C}}$ by picking out a subset of dimension n out of these $2n$ variables. Namely, we choose n directions such that the sections in $TM^{\mathbb{C}}$ are constant. This is Kähler polarization.

In more details, we define an almost complex structure J on TM compatible with ω such that the triple (M, ω, J) becomes a Kähler manifold. Then our Quantum Hilbert space, \mathbb{H} is defined to be all complex polarized L^2 sections in $TM^{\mathbb{C}}$ such that it is constant in the $T^{(0,1)}M$ directions. However, given such a section l , there is no reason for $Q(f)l$ to be back in the Hilbert space \mathbb{H} , where $Q(f)$ is as defined in Equation (1.1). Hence we need to impose restrictions on the choice of J . It turns out that J needs to be constant on the flow generated by the Hamiltonian vector field X_f in order for $Q(f)l$ to be back in \mathbb{H} . Given a Kähler potential ϕ , this quantum Hilbert space is unitarily equivalent to the space of holomorphic, square integrable with respect to $e^{-\phi/\hbar}\omega^n$ functions, denoted by $\mathcal{H}L^2(M, e^{-\phi/\hbar}\omega^n)$. This unitary map is given by $ke^{-\phi/2\hbar} \in \mathbb{H} \mapsto k \in \mathcal{H}L^2(M, e^{-\phi/\hbar}\omega^n)$. We then define an operator $\tilde{Q}(f)$ acting on $\mathcal{H}L^2(M, e^{-\phi/\hbar}\omega^n)$, unitarily equivalent to $Q(f)$ via this map.

In this article, we consider a harmonic oscillator in \mathbb{R}^2 with the Hamiltonian $H = \frac{1}{2}(x^2 + p^2)$ where p is the conjugate momentum. In the 'standard' geometric quantization, the choice of J_{std} is

$$J_{std} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This J_{std} is compatible with the canonical $\omega = dp \wedge dx$ and further more, is constant in the flow of X_H . We choose our holomorphic coordinates to be $z = x - ip$ and our Kähler potential to be H . Under this map $(x, p) \mapsto z = x - ip$, we identify our Hilbert space $\mathcal{HL}_{std}^2(\mathbb{R}^2, e^{-H/\hbar}\omega)$, with $\hat{\mathbb{H}}_{std}$, the space of holomorphic functions in \mathbb{C} , square integrable with respect to Gaussian measure. Under this map, the operator $\hat{Q}_{std}(H)$ corresponding to $\tilde{Q}_{std}(H)$, is given by $\hat{Q}_{std}(H) = \hbar z \frac{\partial}{\partial z}$. (See Section 3.)

However, there are many complex structures J one can define on TM which are compatible with $dp \wedge dx$ and constant on the flow of X_H , different from J_{std} . By choosing a J (under some restrictions) and Kähler polarize as described earlier, we obtained a 'new' Hilbert space $\mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega)$ with an operator $\tilde{Q}(H)$. We now would like to compare these 2 Hilbert spaces ($\hat{\mathbb{H}}_{std}, \hat{Q}_{std}(H)$) and $(\mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega), \tilde{Q}(H))$ and see if these 2 spaces are unitarily equivalent to each other. There is no reason why these 2 should be unitarily equivalent. However, it turns out that they are. This is the content of this article.

This article is organized as follows. We begin by describing Kähler polarization in detail in the next section. The reader familiar with the Kähler polarization of the harmonic oscillator may skip directly to Section 4.

After the theoretical setup, we will work out the example of the harmonic oscillator in \mathbb{R}^2 , using the standard almost complex structure $J_{std} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is a standard Kähler polarization of the harmonic oscillator in \mathbb{R}^2 .

Following the standard description of the harmonic oscillator, we will describe Kähler polarization using a different complex structure J . We will then show that this Quantum Hilbert space $(\mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega), \tilde{Q}(H))$ contains non-trivial complex polarized sections which are L^2 integrable and that it is unitarily equivalent to the Quantum Hilbert space $(\hat{\mathbb{H}}_{std}, \hat{Q}_{std}(H))$ obtained using J_{std} .

2 Kähler Polarization

We begin with the definition of a Kähler manifold.

Definition 2.1 A *Kähler manifold* is a triple, (M, ω, J) where M is a complex manifold with almost complex structure J associated to the complex structure on M , ω is a symplectic form on M such that

1. $J^*\omega = \omega$, (i.e. $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in T_m M$ and $m \in M$), and
2. $v \rightarrow \omega(Jv, v)$ is a positive definite quadratic form.

Let (M, ω, J) be a Kähler manifold. The form ω and the almost complex structure J will be extended to $TM^{\mathbb{C}}$ by complex linearity.

Notation 2.2 We will denote the space of holomorphic functions on M by $\mathcal{H}(M)$.

Definition 2.3 We say a vector field Z is of *type $(1,0)$* if $Z = X - iJX$, for some vector field X . A section $k \in \Gamma(B)$ is said to be *complex polarized* if

$$\nabla_{\bar{Z}} k = 0$$

for all vector fields Z of type $(1,0)$, where B is a Hermitian complex line bundle.

Remark 2.4 Note that $f \in \mathcal{H}(M)$ iff $\bar{Z}f = 0$ for all type $(1,0)$ vector fields Z .

Definition 2.5 The *Quantum Hilbert space* is

$$\mathbb{H} = \{k \in \mathcal{K} : k \text{ is complex polarized}\}$$

where $\mathcal{K} = L^2(B, \omega^n)$.

We would like to make the prequantum operators $Q(f) = -i\hbar\nabla_{X_f} + M_f$ act on \mathbb{H} . There is no canonical way to do this since typically (ignoring domain questions) $Q(f)\mathbb{H} \not\subset \mathbb{H}$. The following theorem gives a sufficient condition on f so that $Q(f)\mathbb{H} \subset \mathbb{H}$.

Theorem 2.6 Suppose that (M, ω, J) is a Kähler manifold and B is a Hermitian complex line bundle over M with a metric - covariant derivative ∇ such that $\text{Curvature}(\nabla) = \frac{1}{i\hbar}\omega$. If $f \in C^\infty(M)$ is such that $L_{X_f}J = 0$ where L denotes Lie derivative, then $Q(f)\mathbb{H} \subset \mathbb{H}$.

Proof. If $k \in \mathbb{H}$ (i.e. $\nabla_{\bar{Z}}k = 0$ for all $Z \in T^{(1,0)}M$), we must show that $\nabla_{\bar{Z}}(Q(f)k) = 0$ for all type $(1, 0)$ vector fields Z . Now

$$\begin{aligned}
\nabla_{\bar{Z}}(Q(f)k) &= \nabla_{\bar{Z}}((-i\hbar\nabla_{X_f} + f)k) = -i\hbar\nabla_{\bar{Z}}\nabla_{X_f}k + \nabla_{\bar{Z}}(fk) \\
&= -i\hbar[\nabla_{\bar{Z}}, \nabla_{X_f}]k + (\bar{Z}f)k \\
&= -i\hbar\frac{\omega(\bar{Z}, X_f)}{i\hbar}k - i\hbar\nabla_{[\bar{Z}, X_f]}k + df(\bar{Z})k \\
&= -df(\bar{Z})k - i\hbar\nabla_{[\bar{Z}, X_f]}k + df(\bar{Z})k \\
&= -i\hbar\nabla_{[\bar{Z}, X_f]}k.
\end{aligned}$$

So to finish the proof, we need only realize that $[\bar{Z}, X_f]$ is of type $(0, 1)$. To verify this remark, write $\bar{Z} = W + iJW$ for some real vector field W , then

$$\begin{aligned}
-[\bar{Z}, X_f] &= L_{X_f}\bar{Z} = L_{X_f}(W + iJW) \\
&= L_{X_f}W + i(L_{X_f}J)W + iJL_{X_f}W \\
&= L_{X_f}W + iJL_{X_f}W \in T^{(0,1)}.
\end{aligned}$$

■

Because of this theorem, if we want to quantize the Hamiltonian system with Hamiltonian f , then we should choose J such that $L_{X_f}J = 0$. In this case, we will be able to quantize f in an unambiguous way. Since

$$L_{X_f}J = \left. \frac{d}{dt} \right|_0 e_*^{-tX_f} J_{e^{tX_f}} e_*^{tX_f},$$

where e^{tX_f} is the flow of X_f , it follows that

$$J_{e^{tX_f}(m)} = e_*^{tX_f} J_m e_*^{-tX_f} \in \text{End}\left(T_{e^{tX_f}(m)}M\right). \quad (2.1)$$

So Equation (2.1) shows that J is determined on each point of the orbits of e^{tX_f} once it is known at one point. Thus if we can find a slice (S) to the orbits X_f , then J is determined by its values on S .

Remark 2.7 *In general, Equation (2.1) does not extend to give us an unambiguously defined J on M . For example, if the orbits of e^{tX_f} are periodic, then there will be several values of t such that $e^{tX_f}(x_0) = x$, where x_0 is the point in S which is in the same orbit as x . However, $J_{e^{tX_f}(x_0)}$ may be different for these values of t . To get an unambiguously defined J , the value of J at x_0 must commute with $e_*^{TX_f}$, where T is the period of the orbit of $e^{tX_f}(x_0)$.*

Proposition 2.8 *Suppose that S is a slice to the orbits X_f and J is given on S such that $J^2 = -1$, $J^*\omega = \omega$ on S , and $\omega(v, Jv) > 0$ for all $v \in T_m M \setminus \{0\}$ and $m \in S$. Then J defined on each orbit of e^{tX_f} by Equation (2.1) has these properties for all $m \in S$.*

Proof. By Equation (2.1), for $m \in S$ and $v, w \in T_{e^{tX_f}(m)}M$,

$$J_{e^{tX_f}(m)}^2 = \left(e_*^{tX_f} J_m e_*^{-tX_f} \right)^2 = e_*^{tX_f} J_m^2 e_*^{-tX_f} = e_*^{tX_f} (-1) e_*^{-tX_f} = -1,$$

$$\begin{aligned} \omega(J_{e^{tX_f}(m)}v, J_{e^{tX_f}(m)}w) &= \omega(e_*^{tX_f} J_m e_*^{-tX_f} v, e_*^{tX_f} J_m e_*^{-tX_f} w) \\ &= \omega(J_m e_*^{-tX_f} v, J_m e_*^{-tX_f} w) = \omega(e_*^{-tX_f} v, e_*^{-tX_f} w) \\ &= \omega(v, w), \end{aligned}$$

and

$$\omega(v, J_{e^{tX_f}(m)}v) = \omega(v, e_*^{tX_f} J_m e_*^{-tX_f} v) = \omega(e_*^{-tX_f} v, J_m e_*^{-tX_f} v) > 0,$$

where we have made repeated use of the fact that $(e^{tX_f})^* \omega = \omega$ for all t . This is because

$$L_{X_f} \omega = d\omega(X_f, \cdot) + i_{X_f} d\omega = -ddf = 0$$

and hence

$$0 = L_{X_f} \omega = \frac{d}{dt} \Big|_0 (e^{tX_f})^* \omega. \quad \blacksquare$$

Definition 2.9 *Let (M, ω, J) be a Kähler manifold. A real valued function ϕ is called a Kähler potential if*

$$i\partial\bar{\partial}\phi = \omega.$$

Theorem 2.10 *Suppose that (M, ω, J) is a Kähler manifold and that there exists a globally defined Kähler potential ϕ . Then*

$$\theta = -\operatorname{Im} \bar{\partial}\phi = -\frac{(\bar{\partial}\phi - \partial\phi)}{2i} \quad (2.2)$$

is a symplectic potential. That is $\omega = d\theta$.

Proof. The proof is the following computation:

$$\begin{aligned} d\theta &= (\partial + \bar{\partial})\theta = -\frac{(\partial + \bar{\partial})(\bar{\partial}\phi - \partial\phi)}{2i} \\ &= -\frac{(\partial\bar{\partial}\phi - \bar{\partial}\partial\phi)}{2i} = i\partial\bar{\partial}\phi = \omega. \end{aligned}$$

■

Theorem 2.11 *Suppose that (M, ω, J) is a Kähler manifold and that there exists a globally defined Kähler potential ϕ . Let θ be as in Equation (2.2) and $B = M \times \mathbb{C}$ be the Hermitian line bundle with covariant derivative $\nabla = d + \frac{1}{i\hbar}\theta$ where θ is described as above. Then the Quantum Hilbert space \mathbb{H} may be described as*

$$\mathbb{H} = \{ke^{-\phi/2\hbar} \in \mathcal{K} : k \in \mathcal{H}(M)\} \cong \mathcal{H}L^2(M, e^{-\phi/\hbar}\omega^n).$$

Let $f \in C^\infty(M)$, then for any $k \in \mathcal{H}(M)$,

$$\begin{aligned} \tilde{Q}(H)k &:= e^{\phi/2\hbar}Q(f)\left(ke^{-\phi/2\hbar}\right) = -i\hbar X_f k + (f + iZ_f\phi)k \\ &= -i\hbar Z_f k + (f + iZ_f\phi)k, \end{aligned} \tag{2.3}$$

where $Z_f = \frac{1}{2}(X_f - iJX_f)$. If $L_{X_f}J = 0$, then Z_f is a holomorphic vector field and $(f + iZ_f\phi)$ is a holomorphic function.

Proof. The main point is to show that $\nabla_{\bar{Z}}e^{-\phi/2\hbar} = 0$ for all $Z \in T^{(1,0)}M$. This is easily checked using

$$\begin{aligned} \nabla_{\bar{Z}}e^{-\phi/2\hbar} &= (\bar{Z} + \frac{1}{i\hbar}\theta(\bar{Z}))e^{-\phi/2\hbar}, \\ \bar{Z}e^{-\phi/2\hbar} &= -\frac{\bar{Z}\phi}{2\hbar}e^{-\phi/2\hbar}, \end{aligned}$$

and from Equation (2.2),

$$\frac{1}{i\hbar}\theta(\bar{Z}) = -\frac{1}{i\hbar}\text{Im}\bar{\partial}\phi(\bar{Z}) = -\frac{1}{i\hbar}\frac{(\bar{\partial}\phi - \partial\phi)(\bar{Z})}{2i} = \frac{1}{2\hbar}\bar{Z}\phi.$$

This proves the theorem because of the product rule for ∇ .

In more detail, if $k \in \mathcal{H}(M)$, then

$$\nabla_{\bar{Z}}\left(ke^{-\phi/2\hbar}\right) = \bar{Z}k \cdot e^{-\phi/2\hbar} + k\nabla_{\bar{Z}}e^{-\phi/2\hbar} = 0 + 0 = 0.$$

Conversely if $ke^{-\phi/2\hbar} \in \mathbb{H}$, then

$$0 = \nabla_{\bar{Z}} \left(ke^{-\phi/2\hbar} \right) = \bar{Z}k \cdot e^{-\phi/2\hbar} + k\nabla_{\bar{Z}} e^{-\phi/2\hbar} = \bar{Z}k \cdot e^{-\phi/2\hbar}$$

which implies that $\bar{Z}k = 0$ for all $Z \in T^{(1,0)}M$, i.e. that $k \in \mathcal{H}(M)$. We may write $X_f = Z_f + \bar{Z}_f$. Because $\nabla_{\bar{Z}_f} (ke^{-\phi/2\hbar}) = 0$ and

$$\theta(Z_f) = -\frac{(\bar{\partial}\phi - \partial\phi)(Z_f)}{2i} = \frac{Z_f\phi}{2i},$$

we find that

$$\begin{aligned} e^{\phi/2\hbar}Q(f) \left(ke^{-\phi/2\hbar} \right) &= e^{\phi/2\hbar} (-i\hbar\nabla_{X_f} + f) \left(ke^{-\phi/2\hbar} \right) \\ &= e^{\phi/2\hbar} (-i\hbar\nabla_{Z_f} + f) \left(ke^{-\phi/2\hbar} \right) \\ &= -i\hbar Z_f k + f k - i\hbar k e^{\phi/2\hbar} \nabla_{Z_f} e^{-\phi/2\hbar} \\ &= -i\hbar Z_f k + f k - i\hbar k e^{\phi/2\hbar} \left(Z_f + \frac{1}{i\hbar} \theta(Z_f) \right) e^{-\phi/2\hbar} \\ &= -i\hbar Z_f k + \left(f - \frac{Z_f\phi}{2i} + i\hbar \frac{Z_f\phi}{2\hbar} \right) k \\ &= -i\hbar Z_f k + (f + iZ_f\phi) k. \end{aligned}$$

Let $\{z_j\}$ be a set of local holomorphic coordinates on M . Now suppose that $L_{X_f}J = 0$ and write X for X_f , Z for Z_f , ∂_j for $\frac{\partial}{\partial z^j}$ and $Z = \sum_j c^j \partial_j$. It is well known that $L_X J = 0$ is equivalent to requiring the coefficients c^j to be holomorphic, i.e. that Z is a holomorphic vector field. (See Appendix.) To show that $(f + iZ_f\phi)$ is holomorphic it suffices to prove that $\bar{W}(f + iZ_f\phi) = 0$ for all type $T^{(1,0)}M$ vector fields W . Recall that $\omega = i\partial\bar{\partial}\phi$ and

$$\begin{aligned} \partial\bar{\partial}\phi(\bar{W}, Z) &= d\bar{\partial}\phi(\bar{W}, Z) = \bar{W}\bar{\partial}\phi(Z) - Z\bar{\partial}\phi(\bar{W}) - \bar{\partial}\phi([\bar{W}, Z]) \\ &= -Z\bar{W}\phi - [\bar{W}, Z]\phi = -\bar{W}Z\phi, \end{aligned}$$

wherein we have used $L_Z J = 0$ to conclude that $[\bar{W}, Z]$ is still type $T^{(0,1)}M$. (See the last paragraph in the proof of Theorem 2.6.) Using this result,

$$\begin{aligned} \bar{W}(f + iZ_f\phi) &= \bar{W}f + i\bar{W}Z_f\phi = \bar{W}f - i\partial\bar{\partial}\phi(\bar{W}, Z_f) \\ &= df(\bar{W}) - \omega(\bar{W}, Z_f) = df(\bar{W}) + \omega(Z_f, \bar{W}) = 0. \end{aligned}$$

The last line follows from Equation (1.2). ■

The previous theorem tells that our Quantum Hilbert space \mathbb{H} (the space of L^2 complex polarized sections) may be identified with the space of holomorphic functions, L^2 integrable with respect to the form $e^{-\phi/\hbar}\omega^n$, denoted by $\mathcal{HL}^2(M, e^{-\phi/\hbar}\omega^n)$. The corresponding Hamiltonian operator $\tilde{Q}(H)$ is given by Equation (2.3). Note that this operator $\tilde{Q}(f)$ is unitarily equivalent to $Q(f)$.

3 Standard Kähler Polarization of the Harmonic Oscillator in \mathbb{R}^2

We can now apply the theory developed in the previous section to the simple harmonic oscillator. Let $M = T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ with the usual symplectic form $\omega = dp \wedge dx$. Note that a 2-dimensional symplectic vector space with any compatible almost complex structure (satisfying (1) and (2) in Definition 2.1) is always a Kähler manifold. This is because on \mathbb{R}^2 , any almost complex structure J is integrable, and thus by Newlander-Nirenberg Theorem, (\mathbb{R}^2, J) is a complex manifold. (See [Newlander and Nirenberg(1957)].) Let $H \in C^\infty(\mathbb{R}^2)$ be the Hamiltonian of the harmonic oscillator, H given by $H(x, p) = \frac{1}{2}p^2 + \frac{1}{2}x^2$.

The Hamiltonian vector field of H is

$$X_H = p\partial_x - x\partial_p,$$

where $\partial_x := \frac{\partial}{\partial x}$ and $\partial_p := \frac{\partial}{\partial p}$. Hamilton's equations of motion are

$$\begin{aligned} \dot{x}(t) &= H_p(x, p) = p(t), \\ \dot{p}(t) &= -H_x(x, p) = -x(t) \end{aligned}$$

or equivalently

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \Omega \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$$

where

$$\Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

Newton's equations of motion are

$$\ddot{x}(t) + x(t) = 0$$

from which we easily derive

$$e^{t\Omega} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \quad (3.2)$$

After identifying $T_m M$ with M , Equation (2.1) becomes

$$J_{e^{t\Omega}m} = e^{t\Omega} J_m e^{-t\Omega}. \quad (3.3)$$

Hence we seek to find a J_m on the slice of orbits of e^{tX_H} which has the properties in Proposition 2.8.

Proposition 3.1 *Suppose ω is the standard symplectic form on \mathbb{R}^2 and J be a 2×2 matrix such that $J^2 = -I$ and $J^*\omega = \omega^1$. Then J is of the form*

$$J = \begin{pmatrix} a & b \\ -(1+a^2)b^{-1} & -a \end{pmatrix} \quad (3.4)$$

and J is positive, i.e. and $\omega(Jv, v) > 0$ for all $v \neq 0$, iff $b > 0$. If we replace b by $(1+a^2)^{1/2}b$ in this last expression we arrive at an alternate form for J , namely

$$J = \begin{pmatrix} a & (1+a^2)^{1/2}b \\ -(1+a^2)^{1/2}b^{-1} & -a \end{pmatrix}.$$

Proof. Let $J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $\omega(v, w) := v \cdot J_0 w$. The condition $J^*\omega = \omega$ implies

$$Jv \cdot J_0 Jw = v \cdot J_0 w \text{ for all } v, w \in \mathbb{R}^2$$

which implies that $J^{\text{tr}} J_0 J = J_0$ where

$$J^{\text{tr}} J_0 J = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad - bc \\ -ad + bc & 0 \end{pmatrix}.$$

¹Notice that this condition implies that $\omega(Jv, w)$ is a symmetric form. Indeed,

$$\omega(Jv, w) = -\omega(Jv, JJw) = -\omega(v, Jw) = \omega(Jw, v).$$

So this condition is equivalent to $\det J = 1$ which is also clear from the exterior algebra construction of the determinant.

The condition that $J^2 = -1$ becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} bc + a^2 & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and hence that

$$b(a + d) = 0 = c(a + d) \text{ and } bc + a^2 = -1 = bc + d^2.$$

It follows from these equations that $a^2 = d^2$ or $a = \epsilon d$ with $\epsilon \in \{\pm 1\}$. If $\epsilon = 1$ we then have

$$ba = 0 = ca$$

in which case either $a = 0$ and hence $c = -1/b$ and hence

$$J = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

or $a \neq 0$ and hence $b = c = 0$ and in which case $bc + a^2 = -1$ does not have a solution. If $\epsilon = -1$, then we must still satisfy $bc + a^2 = -1$ or $bc = -1 - a^2$ or $c = -\frac{1+a^2}{b}$, i.e.

$$J = \begin{pmatrix} a & b \\ -(1 + a^2)b^{-1} & -a \end{pmatrix}.$$

Finally we have

$$\omega(Jv, v) = Jv \cdot J_0v = -J_0Jv \cdot v$$

and

$$\begin{aligned} -J_0J &= - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ -(1 + a^2)b^{-1} & -a \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{b}(a^2 + 1) & a \\ a & b \end{pmatrix} \end{aligned}$$

which is positive if $b > 0$ since $\det(-J_0J) = 1$. ■

Let us consider the case when $a = 0, b = 1$. Then Equation (3.3) reduces to the standard almost complex structure, $J_{std} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence $\xi(x, p) = x - ip = z$ is a global

holomorphic chart $\xi : \mathbb{R}^2 \rightarrow \mathbb{C}$ such that $\partial_z = \frac{1}{2}(\partial_x + i\partial_p)$ is a $T^{(1,0)}\mathbb{R}^2$ vector field. Let the Kähler potential $\phi = H = \frac{1}{2}(p^2 + x^2) = \frac{1}{2}|z|^2$, which follows from

$$\begin{aligned} i\partial\bar{\partial}\frac{1}{2}(x^2 + p^2) &= \frac{i}{8}(\partial_x^2 + \partial_p^2)(x^2 + p^2)dz \wedge d\bar{z} \\ &= \frac{i}{2}(dx - idp) \wedge (dx + idp) \\ &= dp \wedge dx. \end{aligned}$$

Note that the Kähler potential is never unique. We can always add the real part of a holomorphic function to it. Similarly we can also add the real part of an anti-holomorphic function. By Theorem 2.10, $\omega = d\theta$, where $\theta = \frac{1}{2}(pdx - xdp)$. In this case we may take $B = M \times \mathbb{C}$ and $\nabla = d + \frac{1}{i\hbar}\theta$. Then the covariant derivative ∇ is metric compatible and $\text{Curvature}(\nabla) = \frac{1}{i\hbar}d\theta$. Since \mathbb{R}^2 is simply connected, this Hermitian line bundle with this covariant derivative (B, ∇) is unique.

By Theorem 2.11, Z_H is a holomorphic vector field. With $X_H = p\partial_x - x\partial_p$ and $z = x - ip$,

$$Z_H = \frac{1}{2}(X_H - iJX_H) = \frac{1}{2}(p\partial_x - x\partial_p + i(x\partial_x + p\partial_p)) = i(x - ip)\frac{1}{2}(\partial_x + i\partial_p) = iz\partial_z.$$

Our Kähler potential ϕ thus satisfies

$$H + iZ_H\phi = \frac{1}{2}z\bar{z} - z\frac{1}{2}\partial_z z\bar{z} = 0.$$

Therefore using Theorem 2.11, we may identify our Quantum Hilbert space, via the map $\xi : (x, p) \rightarrow z = x - ip$, with $\hat{\mathbb{H}}_{std} = \mathcal{H}L^2(\mathbb{C}, e^{-(x^2+p^2)/\hbar}dp \wedge dx)$. Under this 'ξ' identification, our quantized Hamiltonian $\hat{Q}_{std}(H)$ is thus given by

$$\hat{Q}_{std}(H) = -i\hbar \cdot iz\partial_z = \hbar z\partial_z.$$

It is then a straight forward calculation to show that $\{z^n\}_{n=0}^\infty$ are the eigenfunctions of $\hat{Q}_{std}(H)$. In fact, one can check that they are orthogonal and span $\hat{\mathbb{H}}_{std}$. Let us summarize these facts in a theorem.

Theorem 3.2 *Let $(\mathbb{R}^2, \omega, J_{std})$ be our Kähler manifold with $\omega = dp \wedge dx$ and*

$$J_{std} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $H = \frac{1}{2}(x^2 + p^2)$ be the Hamiltonian. Under the map $\xi : (x, p) \rightarrow z = x - ip$, our Quantum Hilbert space is identified with

$$\hat{\mathbb{H}}_{std} = \mathcal{H}L^2(\mathbb{C}, e^{-(x^2+p^2)/\hbar} dp \wedge dx).$$

The corresponding quantized Hamiltonian $\hat{Q}_{std}(H)$ is given by

$$\hat{Q}_{std}(H) = \hbar z \frac{\partial}{\partial z}$$

and $\{z^n : n = 0, 1, 2, \dots\}$ are the eigenfunctions of $\hat{Q}_{std}(H) = \hbar z \frac{\partial}{\partial z}$. Furthermore, they form an orthogonal basis for $\hat{\mathbb{H}}_{std}$.

4 A Non-standard Kähler Polarization

We now want to consider the quantization of the harmonic oscillator, using a different almost complex structure J , given in Proposition 3.1. Let $r^2 = x^2 + p^2$. For $r \geq 0$, let $a(r)$ and $b(r)$ be given smooth functions with some restrictions near 0. That is, $a(r) \rightarrow 0$, $b(r) \rightarrow 1$ as $r \rightarrow 0$.

Now the orbits of $e^{t\Omega}$ are circles, from Equation (3.2). Choose the slice of orbits of $e^{t\Omega}$ to be the positive real line. By Proposition 3.1, we should let

$$J_{(r,0)} := J_{(x,p)} \Big|_{x=r,p=0} = \begin{pmatrix} a(r) & b(r) \\ -\frac{1}{b(r)}(1+a^2(r)) & -a(r) \end{pmatrix}. \quad (4.1)$$

Then use Equation (3.3) to define J on each orbit of $e^{t\Omega}$. By Proposition 2.8, J defined in this way is compatible with the standard symplectic form ω .

It is clear from Equation (3.2) that 't' takes values $[0, 2\pi]$. The period of each orbit of $e^{t\Omega}$ is 2π and $e^{2\pi\Omega}$ is just the identity map from \mathbb{R}^2 to \mathbb{R}^2 . Hence we can use Equation (3.3) and define J unambiguously on all of \mathbb{R}^2 . Thus Equation (3.3) defines a J on all of \mathbb{R}^2 which is compatible with the standard symplectic form. (See also Remark 2.7.) The assumptions on a and b will guarantee that J is continuous on \mathbb{R}^2 , but it may not be smooth at the origin. We make an additional assumption that J is smooth at the origin.

Let

$$V = \begin{pmatrix} b & , & i - a \end{pmatrix} e^{-t\Omega} \begin{pmatrix} \partial_x \\ \partial_p \end{pmatrix}$$

and

$$\bar{V} = \begin{pmatrix} b & , & -i - a \end{pmatrix} e^{-t\Omega} \begin{pmatrix} \partial_x \\ \partial_p \end{pmatrix}.$$

Note that here, 't' is not the usual angular variable, in the sense that $e^{t\Omega}$ is rotation in the clockwise direction, as opposed to the usual angular variable in polar coordinates, where positive means going in the anti-clockwise direction. It is easily checked that $JV = iV$ and $J\bar{V} = -i\bar{V}$.

As remarked earlier, $(\mathbb{R}^2, \omega, J)$ with a compatible almost complex structure J is always a Kähler manifold. By the Uniformization Theorem (See [Farkas and Kra(1992)]), since \mathbb{R}^2 is simply connected and non-compact, $(\mathbb{R}^2, \omega, J)$ is conformally equivalent to either \mathbb{C} or the complex unit disc \mathbb{D} . Let ζ be such a map $\zeta : \mathbb{R}^2 \rightarrow \mathbb{C}$ or \mathbb{D} . We will write down a formula for ζ in section 4.1. But for the time being, we don't need the explicit formula. We will assume that $\zeta(0) = 0$. Then there exists a non-zero function ψ such that

$$\begin{aligned} \zeta_*^{-1} \frac{\partial}{\partial w} &= \psi V, \\ \zeta_*^{-1} \frac{\partial}{\partial \bar{w}} &= \bar{\psi} \bar{V} \end{aligned}$$

where w is a linear holomorphic coordinate in \mathbb{C} or \mathbb{D} .

We will continue to label points on \mathbb{R}^2 with (x, p) . The uniformization map ζ maps $(x, p) \in \mathbb{R}^2 \mapsto w = \zeta(x, p) \in \mathbb{C}$ or \mathbb{D} .

Notation 4.1 *We will in future write $\frac{\partial}{\partial w}$ as ∂_w . Similarly write $\frac{\partial}{\partial \bar{w}}$ as $\bar{\partial}_w$. In linear coordinates, we will write $w = \alpha - i\beta$. This is to be consistent with our $z = x - ip$. Therefore,*

$$\partial_w = \frac{1}{2}(\partial_\alpha + i\partial_\beta), \quad \bar{\partial}_w = \frac{1}{2}(\partial_\alpha - i\partial_\beta),$$

where we write

$$\partial_\alpha := \frac{\partial}{\partial \alpha}, \quad \partial_\beta := \frac{\partial}{\partial \beta}.$$

4.1 Uniformization Map

By definition,

$$J_{(r,t)} := J_{e^{t\Omega}(r,0)} = e^{t\Omega} J_{(r,0)} e^{-t\Omega}.$$

It is convenient to use this chart (r, t) , where $x = r \cos t$, $p = -r \sin t$. Hence we can write ∂_x and ∂_p as

$$\partial_x = \cos t \partial_r - \frac{\sin t}{r} \partial_t, \quad \partial_p = -\sin t \partial_r - \frac{\cos t}{r} \partial_t$$

where $\partial_r := \frac{\partial}{\partial r}$ and $\partial_t := \frac{\partial}{\partial t}$. As a reminder, this 't' is rotation in the clockwise direction.

Thus in this chart (r, t) ,

$$\begin{aligned} \bar{V} &= e^{t\Omega} \begin{pmatrix} b \\ -i-a \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_p \end{pmatrix} \\ &= \begin{pmatrix} b & -i-a \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t & -\frac{\sin t}{r} \\ -\sin t & -\frac{\cos t}{r} \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_t \end{pmatrix} \\ &= b\partial_r + \frac{i+a}{r}\partial_t. \end{aligned}$$

Now, we seek holomorphic functions $f(r, t)$ such that

$$\bar{V}f = 0$$

or

$$\left(b\partial_r + \frac{i+a}{r}\partial_t \right) f(r, t) = 0. \quad (4.2)$$

Lemma 4.2 *Let $iX_H = i(p\partial_x - x\partial_p)$. Suppose $f \in C^\infty(\mathbb{R}^2, \mathbb{C})$ such that $-iX_H f = \lambda f$, $\lambda \in \mathbb{R} \setminus \{0\}$. Then*

$$f(r, t) = f(r, 0)e^{i\lambda t}.$$

Proof. Note that

$$\begin{aligned} -iX_H f(r, t) &= -i \frac{\partial}{\partial s} \Big|_{s=0} f(r, t+s) \\ &= \lambda f(r, t). \end{aligned}$$

This is the same as

$$-i \frac{\partial}{\partial t} f(r, t) = \lambda f(r, t),$$

or

$$\frac{\partial}{\partial t} f(r, t) = i\lambda f(r, t).$$

So

$$f(r, t) = f(r, 0)e^{i\lambda t}.$$

■

With this lemma, we can choose f to be a product of a radial function and a function dependent only on the angular variable. Let us now consider a function of the form $f(r)e^{it}$, which is an eigenfunction of $-iX_H$ with eigenvalue 1. Note that f is only dependent on the radial distance r and that f might take complex values. If $f(r)e^{it}$ is holomorphic, it must satisfy the Cauchy-Riemann Equations. The Cauchy-Riemann Equations (4.2) are

$$b(r)f'(r)e^{it} + i\frac{i+a(r)}{r}f(r)e^{it} = 0$$

or

$$f'(r) = \frac{1-ia(r)}{rb(r)}f(r). \quad (4.3)$$

This differential equation is of separable type and the solution is given by

$$f(r) = f(1)e^{c(r)}, \quad r \in (0, \infty), \quad (4.4)$$

where $c(r) = \int_1^r \frac{1-ia(r)}{rb(r)} dr$. Since we are looking for non-trivial solutions, there is no loss of generality in assuming $f(1) = 1$. However, since a, b are smooth and $a(r) \rightarrow 0, b(r) \rightarrow 1$ as $r \rightarrow 0, f(r) \rightarrow 0$ as $r \rightarrow 0$. Hence, we can extend the solution of Equation (4.3) to $[0, \infty)$, given by

$$f(r) = \begin{cases} e^{c(r)} & r \in (0, \infty); \\ 0, & r = 0. \end{cases} \quad (4.5)$$

On closer examination of the second part of the formula, the magnitude of $f, |f|$ is given by $\exp(\int_1^r 1/rb(r) dr)$ and the 'angular' part is given by $\exp(-\int_1^r a(r)/(rb(r)) dr)$. Thus one might think of b as controlling the rate at which $|f|$ is increasing and a is controlling the 'angular' part. Also, the smaller b is, the greater the increase in the $|f|$.

It is not difficult to see that f maps the positive x -axis in \mathbb{R}^2 to a line in \mathbb{C} which does not cross itself. Since $b > 0, |f|$ is always increasing and continuous and thus $0 \leq r \mapsto |f(r)| = \rho$ is a one-to-one and onto map on $[0, \infty)$. In other words, the function $f(\cdot)$ maps r to one

and only one point on the circle with radius $\rho = |f(r)|$. Therefore, the map $r \mapsto f(r)$ traces a path l in the complex plane. When we apply e^{it} to f , we are in fact rotating this path l in the counterclockwise direction and after a revolution of 2π , we return back to the original path l . Thus $f(r)e^{it}$ is a one to one and onto map from \mathbb{R}^2 to \mathbb{C} and furthermore, is holomorphic, i.e. it satisfies the Cauchy-Riemann Equation (4.2). Henceforth, we can choose our uniformization map $\zeta : \mathbb{R}^2 \rightarrow \mathbb{C}$ to be

$$\zeta(r, t) = f(r)e^{it}$$

where $f(r)$ is defined in Equation (4.5).

4.2 Formula for Kähler potential

By definition of Kähler potential, we want to find a function ϕ such that

$$i\partial_w\bar{\partial}_w(\phi \circ \zeta^{-1} \circ \xi)dw \wedge d\bar{w} = (\zeta^{-1} \circ \xi)^*\omega.$$

Note that $w = \alpha - i\beta = \xi(\alpha, \beta)$ in linear coordinates. Now,

$$\partial_w\bar{\partial}_w = \frac{1}{2}(\partial_\alpha + i\partial_\beta) \cdot \frac{1}{2}(\partial_\alpha - i\partial_\beta) = \frac{1}{4}\Delta$$

where $\Delta = \partial_\alpha^2 + \partial_\beta^2$. Define $\hat{\phi} := \phi \circ \zeta^{-1} \circ \xi$. Hence

$$i(\partial_w\bar{\partial}_w\hat{\phi}) dw \wedge d\bar{w} = i\frac{\Delta\hat{\phi}}{4} dw \wedge d\bar{w} = (\zeta^{-1} \circ \xi)^*(dp \wedge dx) = \begin{vmatrix} \hat{p}_\beta & \hat{x}_\beta \\ \hat{p}_\alpha & \hat{x}_\alpha \end{vmatrix} d\beta \wedge d\alpha,$$

where $\hat{x} := x \circ \zeta^{-1} \circ \xi$ and $\hat{p} := p \circ \zeta^{-1} \circ \xi$. One notices that the right hand side is just the Jacobian matrix

$$J_{\beta,\alpha}(\hat{p}, \hat{x}) = \begin{vmatrix} \hat{p}_\beta & \hat{x}_\beta \\ \hat{p}_\alpha & \hat{x}_\alpha \end{vmatrix}$$

of $\zeta^{-1} \circ \xi$. But

$$idw \wedge d\bar{w} = 2d\beta \wedge d\alpha.$$

Thus

$$\frac{\Delta\hat{\phi}}{4} \cdot 2d\beta \wedge d\alpha = J_{\beta,\alpha}(\hat{p}, \hat{x})d\beta \wedge d\alpha.$$

Hence $\hat{\phi}$ must satisfy the following partial differential equation

$$\frac{1}{2}\Delta\hat{\phi} = J_{\beta,\alpha}(\hat{p}, \hat{x}).$$

Let us change to coordinates (r, t) and (ρ, θ) in \mathbb{R}^2 . To be consistent, ρ is the usual radial coordinate and θ is measuring rotation in the clockwise direction. Now under the uniformization map $\zeta : (r, t) \rightarrow \chi(r)e^{i\eta(r)+it}$, with

$$\chi(r) = |\zeta(r)| = |f(r)|$$

and

$$\eta(r) = \int_1^r \frac{-ia(r)}{rb(r)} dr.$$

Note that χ is an increasing positive valued function, dependent on r only. Using ξ^{-1} to identify \mathbb{C} with \mathbb{R}^2 , as a map from \mathbb{R}^2 to \mathbb{R}^2 , $(r, t) \mapsto (\chi(r), \eta(r) + t)$. Let $\mu(\rho, \theta) := \theta - \eta \circ \chi^{-1}(\rho)$. Inverting this map, $\zeta^{-1} \circ \xi : (\rho, \theta) \rightarrow (\chi^{-1}(\rho), \mu(\rho, \theta))$. So,

$$\begin{aligned}\hat{x}(\rho, \theta) &= \chi^{-1}(\rho) \cos \mu(\rho, \theta), \\ \hat{p}(\rho, \theta) &= -\chi^{-1}(\rho) \sin \mu(\rho, \theta).\end{aligned}$$

In coordinates (ρ, θ) ,

$$\partial_\alpha = \cos \theta \partial_\rho - \frac{\sin \theta}{\rho} \partial_\theta, \quad \partial_\beta = -\sin \theta \partial_\rho - \frac{\cos \theta}{\rho} \partial_\theta,$$

where $\partial_\rho := \frac{\partial}{\partial \rho}$ and $\partial_\theta := \frac{\partial}{\partial \theta}$. By a direct computation, we get

$$\begin{aligned}\hat{p}_\beta &= \sin \theta \sin \mu \frac{d\chi^{-1}}{d\rho} - \chi^{-1} \sin \theta \cos \mu (\eta \circ \chi^{-1})_\rho + \cos \theta \frac{\cos \mu}{\rho} \chi^{-1}, \\ \hat{x}_\alpha &= \cos \theta \cos \mu \frac{d\chi^{-1}}{d\rho} + \chi^{-1} \cos \theta \sin \mu (\eta \circ \chi^{-1})_\rho + \sin \theta \frac{\sin \mu}{\rho} \chi^{-1}, \\ \hat{p}_\alpha &= -\cos \theta \sin \mu \frac{d\chi^{-1}}{d\rho} + \chi^{-1} \cos \theta \cos \mu (\eta \circ \chi^{-1})_\rho + \sin \theta \frac{\cos \mu}{\rho} \chi^{-1}, \\ \hat{x}_\beta &= -\sin \theta \cos \mu \frac{d\chi^{-1}}{d\rho} - \chi^{-1} \sin \theta \sin \mu (\eta \circ \chi^{-1})_\rho + \cos \theta \frac{\sin \mu}{\rho} \chi^{-1}.\end{aligned}$$

Finally, after some manipulation, we get

$$\begin{vmatrix} \hat{p}_\beta & \hat{x}_\beta \\ \hat{p}_\alpha & \hat{x}_\alpha \end{vmatrix} = \frac{\chi^{-1}}{\rho} \frac{d\chi^{-1}}{d\rho}.$$

Converting the Laplacian in terms of coordinates (ρ, θ) , we now have to solve

$$\frac{1}{2} \Delta \hat{\phi} = \frac{1}{2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \right) \hat{\phi} = \frac{\chi^{-1}}{\rho} \frac{d\chi^{-1}}{d\rho}. \quad (4.6)$$

To motivate the solution of this partial differential equation, let us look at $iZ_H \phi + H$. By Theorem 2.11, $iZ_H \phi + H$ is holomorphic. Since we can add a real part of any holomorphic

function to the Kähler potential, we will assume that $iZ_H\phi + H$ is 0. Now for the moment, let us further assume that ϕ is a radial function, independent of the angular coordinate. It is a reasonable assumption since the RHS of Equation (4.6) is a radial function.

Now $-i\zeta_*Z_H$ is a vector field on \mathbb{C} . Therefore,

$$-i\zeta_*Z_Hw = -iZ_H(w \circ \zeta) = -iX_H\zeta = \zeta.$$

The second equality follows from $X_H = Z_H + \overline{Z_H}$ and ζ is holomorphic. Hence, the push forward of $-iZ_H$ under ζ is $w\partial_w$. Since $i\zeta_*Z_H(\hat{\phi} \circ \xi^{-1}) + H \circ \zeta^{-1} = 0$, thus,

$$\begin{aligned} -w\partial_w(\hat{\phi} \circ \xi^{-1}) + \frac{1}{2}|\chi^{-1}|^2 \circ \xi^{-1} &= 0, \\ w\frac{\partial}{\partial w}(\hat{\phi} \circ \xi^{-1}) &= \frac{1}{2}|\chi^{-1}|^2 \circ \xi^{-1}. \end{aligned}$$

Now in cartesian coordinates, $w = \alpha - i\beta$ and $\partial_w = (\partial_\alpha + i\partial_\beta)/2$. Note that $\alpha = \rho \cos \theta$, $\beta = -\rho \sin \theta$, hence $\alpha - i\beta = \rho e^{i\theta}$. Therefore,

$$\begin{aligned} (\alpha - i\beta)\frac{1}{2}(\partial_\alpha + i\partial_\beta)\hat{\phi} &= \frac{1}{2}|\chi^{-1}|^2, \\ \frac{1}{2}\rho e^{i\theta} \left[\cos \theta \partial_\rho - \frac{\sin \theta}{\rho} \partial_\theta + i \left(-\sin \theta \partial_\rho - \frac{\cos \theta}{\rho} \right) \partial_\theta \right] \hat{\phi} &= \frac{1}{2}|\chi^{-1}(\rho)|^2, \\ \rho e^{i\theta} e^{-i\theta} \frac{d}{d\rho} \hat{\phi} &= |\chi^{-1}|^2, \\ \rho \frac{d\hat{\phi}}{d\rho} &= |\chi^{-1}|^2. \end{aligned}$$

Solving for this differential equation, we get the following formula for $\hat{\phi}$,

$$\hat{\phi}(\rho) = \int_0^\rho \frac{|\chi^{-1}|^2(\rho)}{\rho} d\rho. \quad (4.7)$$

So now we have a likely candidate for $\hat{\phi}$, under the assumption that it is a radial function. We now have to verify that it indeed is a Kähler potential, using Equation (4.6). Using the formula as in Equation (4.7),

$$\begin{aligned} \frac{\partial^2}{\partial \rho^2} \hat{\phi} &= \frac{\partial}{\partial \rho} \left(\frac{|\chi^{-1}|^2}{\rho} \right) \\ &= \frac{2}{\rho} |\chi^{-1}| \frac{d|\chi^{-1}|}{d\rho} - \frac{1}{\rho^2} |\chi^{-1}|^2, \end{aligned} \quad (4.8)$$

and

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \hat{\phi} = \frac{|\chi^{-1}|^2}{\rho^2}. \quad (4.9)$$

Adding up Equations (4.8) and (4.9),

$$\frac{1}{2}\Delta\hat{\phi} = \frac{1}{2}\left[\frac{\partial^2}{\partial\rho^2}\hat{\phi} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\hat{\phi}\right] = \frac{\chi^{-1}}{\rho}\frac{d\chi^{-1}}{d\rho}.$$

This verifies that $\hat{\phi}$ defined by Equation (4.7) is indeed a Kähler potential. Since $r \mapsto \chi(r) = \rho$, we have the following theorem.

Theorem 4.3 *The Kähler potential ϕ is a radial function on \mathbb{R}^2 and is given by*

$$\phi(r) = \int_0^{\chi(r)} \frac{|\chi^{-1}|^2(\rho)}{\rho} d\rho. \quad (4.10)$$

With this choice, $iZ_H\phi + H = 0$.

4.3 On the Unitary Equivalence

Define $Q(H)$ as in Equation (1.1) with (B, ∇) and $\tilde{Q}(H)$ as described in Theorem 2.11. We will show that $(\mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar\omega}), \tilde{Q}(f))$ is unitarily equivalent to $(\hat{\mathbb{H}}_{std}, \hat{Q}_{std}(H))$.

Proposition 4.4 *The holomorphic eigenfunctions of $-iX_H$ are given by $f(r)^n e^{int}$ for $n = 0, 1, 2, \dots$, where f is a radial function that solves*

$$f'(r) = \frac{(1 - ia(r))}{b(r)r} f(r).$$

Proof. By Lemma 4.2, we know that all eigenfunctions of $-iX_H$ are in the form $g(r)e^{i\lambda t}$ for some constant λ and radial function g . However, since t is the angular coordinate with values running from 0 to 2π and we are looking at holomorphic functions, λ has to be an integer. The Cauchy-Riemann equation for an eigenfunction f_n with integer eigenvalue n is given by

$$f'_n(r) = \frac{n(1 - ia(r))}{b(r)r} f_n(r).$$

The only solution is given by f^n , where $f = f_1$. Now $f(r) \rightarrow 0$ as $r \rightarrow 0$, and thus for negative values of n , $f^n(r)$ has a singularity point at the origin. Hence, n has to be non-negative and the only possible holomorphic eigenfunctions are $(f(r)e^{it})^n$ for $n = 0, 1, 2, \dots$

■

Corollary 4.5 *The only holomorphic eigenfunctions of $\tilde{Q}(f)$ are given by $\{\zeta^n : n = 0, 1, 2, \dots\}$.*

Proof. By definition,

$$Z_H = \frac{1}{2}(X_H - iJX_H)$$

and hence

$$X_H = Z_H + \overline{Z_H}.$$

Thus, for any holomorphic function $k \in \mathcal{H}(\mathbb{C})$,

$$X_H(k \circ \zeta) = Z_H(k \circ \zeta).$$

Hence the only holomorphic eigenfunctions of $-iZ_H$ are given by $\{\zeta^n : n = 0, 1, 2, \dots\}$.

With the choice of ϕ , $iZ_H\phi + H = 0$ and therefore,

$$\tilde{Q}(H)\zeta^n = -iZ_H\zeta^n = -iX_H\zeta^n = \hbar n\zeta^n.$$

■

Theorem 4.6 *Suppose for constants $m, M, 0 < m \leq b \leq M$. Then $(\mathcal{H}L^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega), \tilde{Q}(H))$ is unitarily equivalent to $(\hat{\mathbb{H}}_{std}, \hat{Q}_{std}(H))$.*

Proof. We will first show that ζ^n are L^2 integrable with respect to $e^{-\phi/\hbar}\omega$ for $n = 0, 1, 2, \dots$. Recall $\chi(r) = |\zeta(r)| = f(r)$. Note that

$$\frac{d\chi(r)}{dr} = \frac{1}{rb(r)} e^{\int_1^r \frac{1}{rb(\tau)} d\tau}.$$

By the substitution rule, $\rho = \chi(r)$ and thus $d\rho = \chi' dr$ or $dr = d\rho/\chi'$. For $n \geq 0$, we want to compute

$$\begin{aligned} \int_{\mathbb{R}^2} |\zeta|^{2n} e^{-\phi/\hbar} \omega &= 2\pi \int_{\mathbb{R}^+} \chi^{2n}(r) e^{-\phi(r)/\hbar} r dr \\ &= 2\pi \int_{\mathbb{R}^+} \rho^{2n} (\chi^{-1}(\rho))^2 e^{-\hat{\phi}(\rho)/\hbar} b(\chi^{-1}(\rho)) e^{-\int_1^{\chi^{-1}(\rho)} \frac{1}{rb(\tau)} d\tau} d\rho. \end{aligned}$$

Since b is bounded below by m and bounded above by M , then

$$e^{\int_1^r \frac{1}{Mr} d\tau} \leq \chi(r) = e^{\int_1^r \frac{1}{rb(\tau)} d\tau} \leq e^{\int_1^r \frac{1}{mr} d\tau}$$

and after integrating, we have

$$r^{1/M} \leq \chi(r) \leq r^{1/m}.$$

This inequality says that χ is growing at least the $1/M$ -th power of r but at most $1/m$ -th power of r . Thus when we take the inverse of χ^{-1} , we have

$$\rho^m \leq \chi^{-1}(\rho) \leq \rho^M.$$

Using these bounds on χ^{-1} , we have

$$\hat{\phi}(\rho) \geq \int_0^\rho \frac{\rho^{2m}}{\rho} d\rho \geq \frac{1}{2m} \rho^{2m},$$

and

$$\int_1^{\chi^{-1}(\rho)} \frac{1}{rb(r)} dr \geq \int_1^{\chi^{-1}(\rho)} \frac{1}{Mr} dr = \frac{1}{M} \log \chi^{-1}(\rho).$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^2} |\zeta|^{2n} e^{-\phi/\hbar} \omega &\leq 2\pi \int_{\mathbb{R}^+} \rho^{2n} (\chi^{-1}(\rho))^2 e^{-\hat{\phi}(\rho)/\hbar} M (\chi^{-1}(\rho))^{-1/M} d\rho \\ &\leq 2M\pi \int_{\mathbb{R}^+} \rho^{2n} \rho^{M(2-1/M)} e^{-\rho^{2m}/2m\hbar} d\rho \\ &= 2M\pi \int_{\mathbb{R}^+} \rho^{2n+2M-1} e^{-\rho^{2m}/2m\hbar} d\rho < \infty \end{aligned}$$

for any n . Hence $\{\zeta^n, n = 0, 1, 2, \dots\}$ are L^2 -integrable with respect to $e^{-\phi/\hbar}\omega$.

Next, we would like to show that this set forms a basis in $\mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega)$. Recall that $\mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega)$ is the space of holomorphic functions which are L^2 integrable with respect to $e^{-\phi/\hbar}\omega$. It suffices to show that if $g \in \mathcal{HL}^2(\mathbb{R}^2, e^{-\phi/\hbar}\omega)$ and $\int_{\mathbb{R}^2} g \overline{\zeta^m} e^{-\phi/\hbar}\omega = 0$ for any m , then $g \equiv 0$. However, any holomorphic function $g \in \mathcal{H}(\mathbb{R}^2)$ can be written as a Taylor series $g = \sum_{n=0}^{\infty} a_n \zeta^n$ (pointwise).

Let $B(\sigma)$ be a disc centered at the origin with radius σ . In coordinates (r, t) , $\zeta(r, t) = f(r)e^{it}$ and hence

$$\int_{B(\sigma)} g \overline{\zeta^m} e^{-\phi/\hbar}\omega = \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} e^{i(n-m)t} dt \int_0^\sigma f^{n+m}(r) e^{-\phi(r)/\hbar} r dr.$$

Note that if $m \neq n$, $\int_{-\pi}^{\pi} e^{i(n-m)t} dt = 0$ and thus if we let

$$p_n := 2\pi \int_0^\sigma f^{2n}(r) e^{-\phi(r)/\hbar} r dr, \quad \gamma_n(\sigma) := \frac{2\pi}{p_n} \int_0^\sigma f^{2n}(r) e^{-\phi(r)/\hbar} r dr,$$

then

$$\int_{B(\sigma)} g \overline{\zeta^m} e^{-\phi/\hbar}\omega = a_m \gamma_m(\sigma) p_m.$$

Observe that $0 \leq \gamma_n(\sigma) \leq 1$ and $\lim_{\sigma \rightarrow \infty} \gamma_n(\sigma) = 1$. Thus

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} g \overline{\zeta^m} e^{-\phi/\hbar} \omega = \lim_{\sigma \rightarrow \infty} \int_{B(\sigma)} g \overline{\zeta^m} e^{-\phi/\hbar} \omega \\ &= a_m p_m \lim_{\sigma \rightarrow \infty} \gamma_m(\sigma) = a_m p_m, \end{aligned}$$

for any m . Therefore, $a_m = 0$ for all m and hence $g \equiv 0$. This shows that $\{\zeta^n : n = 0, 1, 2, \dots\}$ forms a basis for $\mathcal{H}L^2(\mathbb{R}^2, e^{-\phi/\hbar} \omega)$. A similar argument will also show that ζ^n is orthogonal to ζ^m if $m \neq n$. By the definition of p_n , $\{\zeta^n/p_n\}_{n=0}^\infty$ hence forms an orthonormal basis and are normalized eigenfunctions of $\tilde{Q}(H)$.

Now let $\{q_n\}_{n=0}^\infty$ be a sequence of positive numbers such that z^n/q_n are normalized eigenfunctions of $\hat{Q}_{std}(H)$. Note that $\{z^n/q_n\}_{n=0}^\infty$ is an orthonormal basis for $\hat{\mathbb{H}}_{std}$. The linear map $\Psi : \mathcal{H}L^2(\mathbb{R}^2, e^{-\phi/\hbar} \omega) \rightarrow \hat{\mathbb{H}}_{std}$ sends for $n = 0, 1, 2, \dots$,

$$\Psi \left(\frac{1}{p_n} \zeta^n \right) = \frac{1}{q_n} z^n.$$

Then Ψ is a unitary map and $\hat{Q}_{std}(H)\Psi = \Psi\tilde{Q}(H)$. ■

Remark 4.7 *We can impose growth conditions on b in the statement of Theorem 4.6 instead of assuming boundedness to show that ζ^n is integrable.*

5 Discussion

It is indeed surprising that 2 different Kähler polarizations yield unitarily equivalent quantum spaces. Is there a reason why this is true for the harmonic oscillator?

I believe that the main reason is because of the rotational invariance of the classical Hamiltonian. Because of the rotational invariance, the Hamiltonian vector field X_H is just $\partial/\partial t$ where t is angular variable measured clockwise. Hence the flow of the Hamiltonian vector field is just the circles. Given $-iX_H$, each eigenfunction is of the form $g(r)e^{i\lambda t}$, where (r, t) are coordinates on \mathbb{R}^2 . For any $\lambda \in \mathbb{R}$, its multiplicity will be infinite.

We chose a non-standard complex structure J on \mathbb{R}^2 such that it is 'constant' along the flow. Kähler polarization involves restricting ourselves to an invariant subspace of the pre-quantum line bundle. The positivity of Kähler polarization somehow restricts the values of

the eigenvalues to be positive integers and the eigenspace corresponding to each non-negative integer has multiplicity one.

The eigenfunctions of $-iX_H$ with eigenvalue 1 are of the form $\zeta(r, t) = g(r)e^{it}$. There are many choices of g , which give this same eigenvalue. Each J will pick a particular choice of g and we chose such a ζ to be our uniformization map. As such, the quantum space for the harmonic oscillator is determined solely by ζ , since the quantum space is spanned by $\{\zeta^n\}_{n=0}^\infty$. In this sense, each J we pick determines ζ , which in turn determines the quantum space. Furthermore, because $\zeta(r, t) = g(r)e^{it}$, and given a rotational invariant measure, the set $\{\zeta^n\}_{n=0}^\infty$ forms an orthonormal basis (suitably normalized). By mapping ζ for a particular choice of J to $\tilde{\zeta}$ of another choice \tilde{J} gives us the unitarily equivalence of the 2 respective quantum spaces. Of course, we need to verify these statements by calculations but hopefully this gives some insight as to why the harmonic oscillator gives such special results.

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A Almost Complex Structure

Let M be a complex manifold with complex dimension n and $z = x + iy$ be a holomorphic coordinate chart. Then

$$\partial_{z^k} = \frac{1}{2}(\partial_{x^k} - i\partial_{y^k}), \quad \overline{\partial_{z^k}} = \frac{1}{2}(\partial_{x^k} + i\partial_{y^k})$$

and thus

$$\partial_{x^k} = \partial_{z^k} + \overline{\partial_{z^k}}, \quad \partial_{y^k} = -i(\partial_{z^k} - \overline{\partial_{z^k}}).$$

Let TM be the tangent space to M as a real manifold. Then there exists a unique complex structure J on TM with

$$J\partial_{z^k} = i\partial_{z^k}, \quad J\overline{\partial_{z^k}} = -i\overline{\partial_{z^k}}.$$

Thus

$$J\partial_{x^k} = -\partial_{y^k}, \quad J\partial_{y^k} = \partial_{x^k}.$$

Definition A.1 Let M be a complex manifold. A smooth section $Z \in \Gamma^\infty(TM^\mathbb{C})$ is said to be a holomorphic vector field if $Z \in \Gamma^\infty(TM^{(1,0)})$ and Zf is holomorphic on any open set $U \subset M$ where f is holomorphic.

Theorem A.2 Assume that M is a complex manifold with the induced almost complex structure J . Then the following are equivalent.

1. $L_X J \equiv 0$.
2. $[X, JY] = J[X, Y]$ for all smooth sections $Y \in \Gamma^\infty(TM)$.
3. $(X - iJX)$ is a holomorphic vector field.

Proof. Let $Y \in \Gamma^\infty(TM)$, then

$$[X, JY] = L_X(JY) = (L_X J)Y + J L_X Y = (L_X J)Y + J[X, Y],$$

and hence

$$(L_X J)Y = J[X, Y] - [X, JY]. \tag{A.1}$$

This implies the equivalence of (1) and (2).

Write $X = a\partial_x + b\partial_y$ and $Y = u\partial_x + v\partial_y$, so that $JY = -v\partial_x + u\partial_y$. Any $W \in TM^{(1,0)}$ can be written as $W = \frac{1}{2}(X - iJX)$. If f is holomorphic, then $W\bar{f} = 0$, i.e. $iJX\bar{f} = X\bar{f}$ or $-iJXf = Xf$. Thus

$$Wz = \frac{1}{2}(Xz - iJXz) = \frac{1}{2}(Xz + Xz) = Xz = a + ib.$$

Hence

$$\frac{1}{2}(X - iJX) = F\partial_z$$

where $F := a + ib$.

Let us compute $(L_X J)Y$ using Equation (A.1).

$$\begin{aligned}
(L_X J)Y &= J\{Xu\partial_x + Xv\partial_y - (Ya\partial_x + Yb\partial_y)\} - \{-Xv\partial_x + Xu\partial_y - (JYa\partial_x + JYb\partial_y)\} \\
&= \{-Xv\partial_x + Xu\partial_y - (-Yb\partial_x + Ya\partial_y)\} - \{-Xv\partial_x + Xu\partial_y - (JYa\partial_x + JYb\partial_y)\} \\
&= (JYa + Yb)\partial_x + (-Ya + JYb)\partial_y \\
&= \{(JYa + Yb) + i(-Ya + JYb)\}\partial_z + \{(JYa + Yb) - i(-Ya + JYb)\}\partial_{\bar{z}} \\
&= (JY - iY)F\partial_z + (JY + iY)\bar{F}\partial_{\bar{z}} \\
&= (JY - iY)F\partial_z + \overline{(JY - iY)F}\partial_{\bar{z}}.
\end{aligned}$$

Thus, $L_X J = 0$ iff $(JY - iY)F = 0$ iff F is holomorphic. This shows the equivalence of (2) and (3). ■

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