Transition phase for the speed of the biased random walk on a percolation cluster

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The biased random walk on a percolation cluster
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The mathematical model
Let us consider $\mathbb{Z}^d$ for $d \geq 2$. We perform an edge percolation of parameter $p$

- Each edge of $\mathbb{Z}^d$ is open with probability $p$ independently of all others.
Biased random walk

For simplicity in this talk assume the drift is along a direction $e_1$ of the grid. We take $\exp(\lambda)$ with $\lambda > 0$ to be its strength.

In the environment $\omega$ the transitions probabilities are given by

\[
\begin{array}{ccc}
\beta^{-1} & \beta & \beta^{-1} \\
1 & 1 & 1
\end{array}
\]

with $\beta = \exp(\lambda)$

$\rightarrow$ direction of the drift
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$$
\begin{align*}
\beta^{-1} & \quad 1 \\
1 & \quad \beta \\
1 & \quad \beta^{-1} \\
1 & \quad 1
\end{align*}
$$

with $\beta = \exp(\lambda)$ → direction of the drift

We can define a similar model with a drift in any direction.
Reasons for studying this model

This model was first considered in the physics literature (Barma, Dhar (83); Dhar (84); Dhar, Stauffer (98)).
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It is a challenging problem of RWRE, considered by Berger, Gantert, Peres (03); Sznitman (03). It is not uniformly elliptic and the environment is asymmetric.

\[ c(x, y) = \exp(\lambda(x + y) \cdot e_1) \mathbf{1}_{\{[x, y]\text{open}\}}. \]
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Representative of RWRE with directional transience and trapping.
Berger, Gantert, Peres and Sznitman 2003

Fix $p$ and $d$. The random walk is transient in the direction $e_1$, i.e.

$$\lim X_n \cdot e_1 = \infty, \quad P^\omega\text{-a.s. for } \omega - P_p\text{-a.s.}.$$

Moreover

$$\lim \frac{X_n}{n} = \nu(\lambda), \quad P^\omega\text{-a.s. for } \omega - P_p\text{-a.s.}.$$

There exists $\lambda_c^{(1)} \geq \lambda_c^{(2)} > 0$ such that

- if $\lambda < \lambda_c^{(2)}$, then $\nu(\lambda) \cdot e_1 > 0$,
- if $\lambda > \lambda_c^{(1)}$, then $\nu(\lambda) = 0$. 
Behavior of the walk
Transition phase for the speed of the biased random walk on a percolation cluster

Behavior of the walk

Local behavior

![Diagram showing local behavior of a biased random walk on a percolation cluster. The diagram features a grid with a marked starting point and a direction indicating the walk's path.]
Local behavior
Local behavior
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Local behavior
Local behavior
Local behavior
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Local behavior

There are traps in the environment.

To exit a trap the walk has to backtrack (go opposite to the drift) for \( n \) steps. It takes

\[
T_{\text{exit}} \approx \exp(2\lambda n),
\]

units of time to do so.
Global behavior

The trajectory looks uni-dimensional.
The phase transition
One important conjecture is that $\lambda_c^{(1)} = \lambda_c^{(2)}$ (phase transition).
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The phase transition

Backtracking function

The most effective type of traps we encounter (represented dually)

\[ d = 2 \]
\[ d \geq 3 \]

direction of the drift

a vertical slice
Let us introduce the number of steps you need to backtrack to exit the trap at 0

\[ \mathcal{BK} = \min_{(p(i))_{i \in \mathbb{N}} \in \mathcal{P}} \max_{i \geq 0} p(i) \cdot (-e_1), \]

where \( \mathcal{P} \) is the set of infinite open self-avoiding paths starting from 0.

We can prove that there exists \( \xi(p, d) \in (0, \infty) \) such that

\[ \mathbb{P}[\mathcal{BK} \geq n \mid 0 \text{ is in the infinite cluster}] \approx \exp(-\xi(p, d) n). \]
Transition phase for the speed

F. and Hammond 2011

In $\mathbb{Z}^d$, let us introduce $\gamma = \frac{\xi}{2\lambda}$. We have

$$\lim \frac{X_n}{n} = \nu, \quad P^\omega\text{-a.s. for } \omega - P_p\text{-a.s.},$$

where

if $\gamma > 1$, i.e. $\lambda < \xi/2$, then $\nu \cdot e_1 > 0$,

if $\gamma < 1$, i.e. $\lambda > \xi/2$, then $\nu = 0$.

Moreover, if $\gamma \leq 1$ then

$$\lim \frac{\ln X_n \cdot e_1}{\ln n} = \gamma, \quad P^\omega\text{-a.s. for } \omega - P_p\text{-a.s.}.$$
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Sketch of proof
Hitting time of level $n$

From the drawing

we see that ideally the hitting time of the level $n$ is essentially the time spent in the $Cn$ first traps encountered.
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we see that ideally the hitting time of the level $n$ is essentially the time spent in the $Cn$ first traps encountered.

Hence the system is similar to a one-dimensional Bouchaud Trap Model (as conjectured by Sznitman 06).
Hitting time of level $n$

Thus the hitting time of the level $n$ is an i.i.d. sum

$$\Delta_n \approx \sum_{i=0}^{Cn} T_{\text{exit}}^{(i)},$$

is a sum of i.i.d. times to exit traps. Hence we need to know if the expectation of $T_{\text{exit}}^{(i)}$ is finite in an averaged sense.
We recall that

1. We need \( \exp(2\lambda n) \) units of time to backtrack \( n \) steps in a trap,

2. the number of steps we backtrack is typically \( \mathcal{B}_\mathcal{K} \), which is such that

\[
P[\mathcal{B}_\mathcal{K} \geq n \mid 0 \text{ is in the infinite cluster}] \approx \exp(-\xi(p, d)n).
\]
We recall that

1. We need $\exp(2\lambda n)$ units of time to backtrack $n$ steps in a trap,
2. the number of steps we backtrack is typically $B_K$, which is such that

$$\mathbb{P}[B_K \geq n \mid 0 \text{ is in the infinite cluster}] \approx \exp(-\xi(p, d)n).$$

From this we can see that averaged over the environment

$$\mathbb{P}\left[T_{exit}^{(i)} \geq t\right] = \mathbb{P}\left[\exp(2\lambda B_K) \geq t\right] \approx \mathbb{P}\left[B_K \geq \frac{1}{2\lambda} \ln t\right] \approx t^{-\gamma},$$

with $\gamma = \xi/2\lambda$. 
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Sketch of proof

Time spent in a trap

In the end

\[ \Delta_n \approx \sum_{i=0}^{Cn} T^{(i)}_{\text{exit}}, \]

with

\[ \mathbb{P}[T^{(i)}_{\text{exit}} \geq t] \approx t^{-\gamma}. \]
Time spent in a trap

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\[ \Delta_n \approx \sum_{i=0}^{Cn} T_{\text{exit}}^{(i)}, \]

with

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Then

1. if \( \gamma > 1 \), \( \mathbb{E}[T_{\text{exit}}^{(i)}] < \infty \) so \( \Delta_n \approx Cn \) and \( v = X_n/n \approx 1/C \),
2. if \( \gamma < 1 \), \( \mathbb{E}[T_{\text{exit}}^{(i)}] = \infty \) so \( \Delta_n \approx \infty n \) and \( v \approx 1/\infty = 0 \).
The key ingredients
1) **Condition implying** $(T)_\gamma$

**Lemma**

Denoting $B(L, L^\alpha) = [-L, L]_{e_1} \times [-L^\alpha, L^\alpha]_{e_{1^\perp}}^{d-1}$. For large $\alpha$,

\[\mathbb{P}[X_n \text{ does not exit } B(L, L^\alpha) \text{ in the positive direction}] \leq ce^{-cL}.\]
1) **Condition implying** \((T)_\gamma\)

**Lemma**

Denoting \(B(L, L^\alpha) = [-L, L]_{e_1} \times [-L^\alpha, L^\alpha]_{e_1}^{d-1}\). For large \(\alpha\)

\[\mathbb{P}[X_n \text{ does not exit } B(L, L^\alpha) \text{ in the positive direction}] \leq c e^{-cL}.\]

This lemma proves that regeneration boxes are small \((\approx \ln^C t)\). So the walk is truly one-dimensional.
2) Exit time of a box

**Lemma**

*For any $\alpha$, we have for all $L$ not too big (e.g. $\approx \ln^C t$)*

$$\mathbb{P}[T_{B(L,L^\alpha)}^{\text{exit}} \geq t] \approx t^{-\gamma}.$$
Technical aspects

2) Exit time of a box

Lemma

For any \( \alpha \), we have for all \( L \) not too big (e.g. \( \approx \ln \alpha t \))

\[
P[T \text{exit}_{B(L,L^\alpha)} \geq t] \approx t^{-\gamma}.
\]

This lemma reflects that the time spent in a small box (e.g. a regeneration box) is mainly given by the time spent in traps.
Proof of \((T)_\gamma\)

- Exit times for biased reversible random walks can be efficiently approximated through spectral estimates (by Saloff-Coste).
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The Carne-Varopoulos estimate

\[ P_0[X_n = y] \leq 2 \left( \frac{\pi(y)}{\pi(0)} \right)^{1/2} \exp \left( - \frac{d(0, y)^2}{2n} \right), \]

tells us that in short time we can only be at places not too far from 0 and with high invariant measure.
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Proof of \((T)_\gamma\)

1. Exit times for biased reversible random walks can be efficiently approximated through spectral estimates (by Saloff-Coste).

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tells us that in short time we can only be at places not too far from 0 and with high invariant measure.

3. By replacing the original graph by one where the traps have been replaced by edges, we can
   - conserve exit probabilities
   - speed up the walk
Consider the biased random walk in positive conductances $c_*$ (Shen 02).

For $d \geq 2$, we have

$$\lim_{n \to \infty} \frac{X_n}{n} = \nu, \quad \mathbb{P} - \text{a.s.},$$

where

1. if $E_*[c_*] < \infty$, then $\nu \cdot \vec{\ell} > 0$,
2. if $E_*[c_*] = \infty$, then $\nu = 0$. 

Another model of biased random walks
Open problems

1. understand the scaling limits,
2. Einstein relation,
3. understand the behavior of the speed in the ballistic regime.
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Related models

Thanks!
Tail estimates on regeneration times

The idea is that if the time spent in a regeneration box is large then

1. either the regeneration box is large ($\geq \ln^C t$),
2. or the walk spends a lot of time in a small box.
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1. either the regeneration box is large ($\geq \ln^C t$),
2. or the walk spends a lot of time in a small box.

The first part is a consequence of $(T)_\gamma$.

The second one has probability $t^{-\gamma}$ from our estimates on exit time of boxes.
Small regeneration boxes

Condition \((T)_\gamma\) implies little backtracking

\[
P[(X_{\tau_2} - X_{\tau_1}) \cdot \vec{\ell} \geq t] \leq Ce^{-ct^{1/3d}},
\]

so we have small variations along \(\vec{\ell}\).
Small regeneration boxes

Condition \((T)_\gamma\) implies little backtracking

\[
\mathbb{P}[(X_{\tau_2} - X_{\tau_1}) \cdot \vec{\ell} \geq t] \leq Ce^{-ct^{1/3d}},
\]

so we have small variations along \(\vec{\ell}\).

Condition \((T)_\gamma\) then also implies little variations in orthogonal directions. So introducing the volume of a regeneration box

\[
\text{Vol}_\tau = \inf\{k, \ (X_i - X_{\tau_1})_{i \in [\tau_1, \tau_2]} \subset B(k, k^\alpha)\},
\]
Small regeneration boxes

Condition $\left( T \right)_\gamma$ implies little backtracking

$$\mathbb{P}\left[ (X_{\tau_2} - X_{\tau_1}) \cdot \vec{\ell} \geq t \right] \leq Ce^{-ct^{1/3d}},$$

so we have small variations along $\vec{\ell}$.

Condition $\left( T \right)_\gamma$ then also implies little variations in orthogonal directions. So introducing the volume of a regeneration box

$$\text{Vol}_\tau = \inf\{k, (X_i - X_{\tau_1})_{i \in [\tau_1, \tau_2]} \subset B(k, k^\alpha)\},$$

we have tails on the size of boxes.

$$\mathbb{P}[\text{Vol}_\tau \geq k] \leq ce^{-ck^{1/3d}} \quad \text{or} \quad \text{Vol}_\tau \leq \ln^C t, \text{ w.h.p.}$$