Non-classical Random Walks
Simple models, surprising results, and (embarrassing) open problems

Jonathon Peterson
Department of Mathematics
University of Wisconsin

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Outline

1. Classical Random Walks
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2. Random Walks in Random Environments
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3. Reinforced Random Walks
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4. Excited Random Walks
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4. Excited Random Walks
Simple Random Walks on $\mathbb{Z}^d$

A very simple model for random motion.

*Transition probabilities:* Probability measure $\mu$ on $\mathbb{Z}^d$.

**Example:**

$$\mu(x) = \begin{cases} 
1/4 & |x| = 1 \\
0 & |x| \neq 1
\end{cases}$$
A very simple model for random motion.

*Transition probabilities:* Probability measure $\mu$ on $\mathbb{Z}^d$.

**Example:**

$$\mu(x) = \begin{cases} 
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\end{cases}$$

**Simple random walk**

- Start at the origin
- Move from $x$ to $x + e$ with probability $\mu(e)$. 
Applications of Random Walks

- Modelling the Stock Market
- Discrete model for chemical diffusion
- Will an insurance company go bankrupt?
Question 1: Average Speed

\[ X_n = \text{location of RW after } n \text{ steps.} \]
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**Question:** What is \( \lim_{n \to \infty} \frac{X_n}{n} \)? (Does limit even exist?)
Question 1: Average Speed

\[ X_n = \text{location of RW after } n \text{ steps.} \]

**Question:** What is \( \lim_{n \to \infty} \frac{X_n}{n} \)? (Does limit even exist?)

\[ X_n = \xi_1 + \xi_2 + \xi_3 + \ldots + \xi_n, \text{ where } \xi_i \text{ are i.i.d. with distribution } \mu. \]

**Theorem (Law of Large Numbers)**

*If \( \xi_i \) is a sequence of i.i.d. random variables with finite expectation, then with probability 1,*

\[
\lim_{n \to \infty} \frac{\xi_1 + \xi_2 + \xi_3 + \ldots + \xi_n}{n} = E[\xi_1].
\]
Question 2: Recurrence/Transience

**Recurrent**: Infinitely many returns to the starting point.

**Transient**: Finitely many returns to the starting point (possibly zero).

If \( E[\xi_1] = \nu \neq 0 \), then LLN \( \Rightarrow \) Transient.
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**Theorem**

*A simple random walk in $\mathbb{Z}^d$ with $E[X_1] = 0$ is*

- Recurrent if $d \leq 2$.
- Transient if $d \geq 3$. 
**Question 2: Recurrence/Transience**

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If $E[\xi_1] = v \neq 0$, then LLN $\Rightarrow$ Transient.

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**Theorem**

A simple random walk in $\mathbb{Z}^d$ with $E[X_1] = 0$ is

- **Recurrent if** $d \leq 2$.
- **Transient if** $d \geq 3$.

**Math Joke:**
A drunk person will eventually get home.
A drunk bird might never get home.
Question 3: Limiting Distribution

Let $\nu = E[\xi_1]$. 
LLN $\Rightarrow X_n \approx n\nu$.

**Question:** How far away from $n\nu$ is $X_n$?
Question 3: Limiting Distribution

Let $v = E[\xi_1]$.  
LLN $\implies X_n \approx nv$. 

**Question:** How far away from $nv$ is $X_n$?

**Theorem (Central Limit Theorem - $d = 1$)**

Let $E[\xi_1] = v$ and $\text{Var}(\xi_1) = \sigma^2$. Then,

$$
\lim_{n \to \infty} P \left( \frac{X_n - nv}{\sigma \sqrt{n}} \leq t \right) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.
$$
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Bar chart of probability distributions looks like a normal bell curve centered at $nv$. 

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Non-classical Random Walks  
Classical Random Walks
Example: Probability distribution of simple random walk with $p = 0.75$ after 50 steps.
Brownian Motion

Random walks are discrete approximations of continuous models. Central Limit Theorem: Scale time by $n$ and space by $\sqrt{n}$.

$$B_n(t) = \frac{X_{nt} - ntv}{\sigma \sqrt{n}}.$$
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4. Excited Random Walks
RWRE on $\mathbb{Z}^d$

**Environment:** Transition probabilities $\omega_x$ assigned to each $x \in \mathbb{Z}^d$.

**Random Environment:** The $\omega_x$ are randomly assigned.
Non-classical Random Walks

Random Walks in Random Environments

**RWRE on** $\mathbb{Z}^d$

**Environment:** Transition probabilities $\omega_x$ assigned to each $x \in \mathbb{Z}^d$.

**Random Environment:** The $\omega_x$ are randomly assigned.

**Example:** (Dimension 1) Two types of environments:

- **Type 1**
  - Transition probabilities: $\frac{1}{4}$ and $\frac{3}{4}$

- **Type 2**
  - Transition probabilities: $\frac{2}{3}$ and $\frac{1}{3}$

Random environment: Choose type 1 with probability $p$. 
One-dimensional RWRE: Recurrence/Transience

\[ \omega_x = P(\xrightarrow{\downarrow} \xrightarrow{\uparrow} \xrightarrow{\downarrow} \xrightarrow{\uparrow}) \]
\[ 1 - \omega_x = P(\xrightarrow{\downarrow} \xrightarrow{\uparrow} \xrightarrow{\downarrow}) \]

**Question 2:** When is the RWRE transient to \(+\infty\)?

Reasonable guess: \( E[\omega_0] = P(X_1 = 1) > \frac{1}{2} \).
One-dimensional RWRE: Recurrence/Transience

\[ \omega_x = P(\begin{array}{c} \circ \ x \ -1 \\ x \ x+1 \end{array}) \quad 1 - \omega_x = P(\begin{array}{c} \circ \ x \ x-1 \\ x+1 \end{array}) \]

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Reasonable guess: \( E[\omega_0] = P(X_1 = 1) > \frac{1}{2} \). \text{ WRONG}
Non-classical Random Walks

Random Walks in Random Environments

One-dimensional RWRE: Recurrence/Transience

$$\omega_x = P(\textcircled{\scriptsize x} \xrightarrow{\text{\scriptsize x+1}} \textcircled{\scriptsize x} \xrightarrow{\text{\scriptsize x-1}} \textcircled{\scriptsize x-1})$$

$$1 - \omega_x = P(\textcircled{\scriptsize x+1} \xrightarrow{\text{\scriptsize x}} \textcircled{\scriptsize x} \xrightarrow{\text{\scriptsize x-1}} \textcircled{\scriptsize x-1})$$

**Question 2:** When is the RWRE transient to $$+\infty$$?

Reasonable guess: $$E[\omega_0] = P(X_1 = 1) > \frac{1}{2}$$. **WRONG**

**Theorem (Solomon ’75)**

Let $$\rho_x = \frac{1 - \omega_x}{\omega_x}$$. Then

- $$X_n \to +\infty \iff E[\ln(\rho_0)] < 0$$
- $$X_n \to -\infty \iff E[\ln(\rho_0)] > 0$$
- $$X_n \text{ recurrent} \iff E[\ln(\rho_0)] = 0.$$
One-dimensional RWRE: Recurrence/Transience

\[ \omega_x = P( \xrightarrow{x-1} x \xrightarrow{x+1} ) \quad 1 - \omega_x = P( \xrightarrow{x-1} x \xrightarrow{x} x+1 ) \]

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- \( X_n \to -\infty \iff E[\ln(\rho_0)] > 0 \)
- \( X_n \text{ recurrent} \iff E[\ln(\rho_0)] = 0. \)

Follows from formula the probability of reaching \(-x\) before \(y\).
One-dimensional RWRE: LLN

Let $E[\ln(\rho_0)] < 0$ (that is $X_n \rightarrow +\infty$).

**Question 2:** What is a formula for $v = \lim_{n \rightarrow \infty} \frac{X_n}{n} \rightarrow \nu$?
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Let $E[\ln(\rho_0)] < 0$ (that is $X_n \to +\infty$).

Question 2: What is a formula for $v = \lim_{n \to \infty} \frac{X_n}{n} \to v$?

Reasonable guess: $v = 2E[\omega_0] - 1 = E[X_1]$. 
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**Question 3:** When is $\frac{X_n}{n} \longrightarrow \nu > 0$?

Reasonable guess: Whenever $X_n \longrightarrow +\infty$. 


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If $E_P \ln(\rho_0) < 0$, then $\lim_{n \to \infty} \frac{X_n}{n} = \begin{cases} \frac{1 - E[\rho_0]}{1 + E[\rho_0]} & E[\rho_0] < 1 \\ 0 & E[\rho_0] \geq 1. \end{cases}$
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Follows from computation of $E[\text{Time to go one unit right}]$. 
RWRE on Trees

Fix $\beta > 1$.

Node with $k$ “children”, RWRE moves
- up with probability $\frac{1}{1+k\beta}$
- down with probability $\frac{\beta}{1+k\beta}$.

Question 3: Does the RWRE move down with positive speed?

Reasonable guess: Yes. $\beta > 1$ gives bias downward.

Wrong

Theorem (Lyons, Pemantle, & Peres '96)
There exists a constant $\beta_c$ such that
- Positive speed for $\beta \in (1, \beta_c)$ (weak bias)
- Zero speed for $\beta \in \left[\beta_c, \infty\right)$ (strong bias).
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RWRE on $\mathbb{Z}^d$

RWRE on $\mathbb{Z}^d$ are very difficult when $d \geq 2$. 
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Embarrassing Open Problem

*How do you tell if the RWRE is recurrent/transient?*
RWRE on $\mathbb{Z}^d$

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**Embarrassing Open Problem**

*How do you tell if the RWRE is recurrent/transient?*

**Embarrassing Open Problem**

*For $0 \neq \ell \in \mathbb{R}^d$ is*

$$\mathbb{P}\left( \lim_{n \to \infty} X_n \cdot \ell = +\infty \right) \in \{0, 1\}?$$

That is, *is there a direction of transience?*
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That is, *is there a direction of transience?*

There are examples with two possible directions for transience.*
A Strange Example

Random up/right tree,
A Strange Example

Random up/right tree, and the dual down/left tree.
A Strange Example

Random up/right tree, and the dual down/left tree.

Can assign probabilities so that

$$P(\text{up/right}) = P(\text{down/left}) = \frac{1}{2}$$
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4. Excited Random Walks
Reinforced Random Walks

Start with all edges having weight = 1.
Reinforced Random Walks

Start with all edges having weight = 1.

Transition probabilities proportional to edge weights.
Reinforced Random Walks

Start with all edges having weight $= 1$.

Transition probabilities proportional to edge weights.

When an edge is crossed, increase the weight by 1.
Start with all edges having weight $= 1$.

Transition probabilities proportional to edge weights.

When an edge is crossed, increase the weight by 1.

Move according to updated edge weights.
ERRW as a RWRE

RWRE: Transition probabilities fixed (randomly) at start.
ERRW: Transition probabilities change over time.
ERRW as a RWRE

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ERRW: Transition probabilities change over time.

Theorem (Diaconis ’88)

An ERRW has the same distribution as a certain RWRE.

- Simple random environment if graph is a tree (even infinite).
ERRW as a RWRE

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An ERRW has the same distribution as a certain RWRE.

- Simple random environment if graph is a tree (even infinite).
- Example: ERRW on \( \mathbb{Z} \)

\[
\omega_x \sim \begin{cases} 
  \text{Beta}(2, 1) & x \leq -1 \\
  \text{U}(0, 1) & x = 0 \\
  \text{Beta}(1, 2) & x \geq 1 
\end{cases}
\]
ERRW as a RWRE

RWRE: Transition probabilities fixed (randomly) at start.
ERRW: Transition probabilities change over time.

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*An ERRW has the same distribution as a certain RWRE.*

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U(0, 1) & x = 0 \\
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\end{cases}
$$

- Complicated random environment if graph has cycles.
Recurrence of ERRW

Reinforcement should make recurrence more likely.
Recurrence of ERRW

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Embarrassing Open Problem

Is an ERRW on $\mathbb{Z}^d$ with $d \geq 2$ recurrent?
Recurrence of ERRW

Reinforcement should make recurrence more likely.

Embarrassing Open Problem

Is an ERRW on $\mathbb{Z}^d$ with $d \geq 2$ recurrent?

Partial progress when $d = 2$ (Merkel & Rolles ’08).
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Excited (Cookie) Random Walks

\((M, p)\) Cookie Random Walk

Initially \(M\) cookies at each site.
Excited (Cookie) Random Walks

$(M, p)$ Cookie Random Walk
Initially $M$ cookies at each site.

- **Cookie available**: Eat cookie. Move right with probability $p > \frac{1}{2}$.
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\[
\begin{align*}
1 - p \quad & \quad p
\end{align*}
\]
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Initially \(M\) cookies at each site.

- **Cookie available:** Eat cookie. Move right with probability \(p > \frac{1}{2}\)
- **No cookies:** Move right/left with probability \(\frac{1}{2}\).
Excited (Cookie) Random Walks

\((M, p)\) Cookie Random Walk
Initially \(M\) cookies at each site.

- **Cookie available**: Eat cookie. Move right with probability \(p > \frac{1}{2}\).
- **No cookies**: Move right/left with probability \(\frac{1}{2}\).
Recurrence/Transience and LLN

\(M\) cookies with strength \(p_1, p_2, \ldots, p_M > \frac{1}{2}\).

\[\alpha = \sum_{i=1}^{M} (2p_i - 1) - 1.\]

**Theorem (Zerner '05)**

*The cookie RW is transient if and only if \(\alpha > 0\).*
Recurrence/Transience and LLN

$M$ cookies with strength $p_1, p_2, \ldots, p_M > \frac{1}{2}$.

$$\alpha = \sum_{i=1}^{M} (2p_i - 1) - 1.$$ 

**Theorem (Zerner ’05)**

*The cookie RW is transient if and only if $\alpha > 0$.*

**Theorem (Basdevant & Singh ’08)**

*Let $v = \lim_{n \to \infty} X_n/n = v$. Then, $v > 0 \iff \alpha > 1$.***
(M, p) cookie random walk

M cookies all of strength \( p > \frac{1}{2} \). \( v > 0 \iff p > \frac{1}{2} + \frac{1}{M} \).
\((M, p)\) cookie random walk

\(M\) cookies all of strength \(p > \frac{1}{2}\). \(v > 0 \iff p > \frac{1}{2} + \frac{1}{M} \).

Embarrassing Open Problem

Find a formula for \(v = v(M, p)\).

Figure: Simulations of speed for a \((3, p)\) cookie walk (Basdevant and Singh '08)
(M, p) cookie random walk

M cookies all of strength \( p > \frac{1}{2} \). \( v > 0 \iff p > \frac{1}{2} + \frac{1}{M} \).

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Figure: Simulations of speed for a (3, p) cookie walk (Basdevant and Singh ’08)