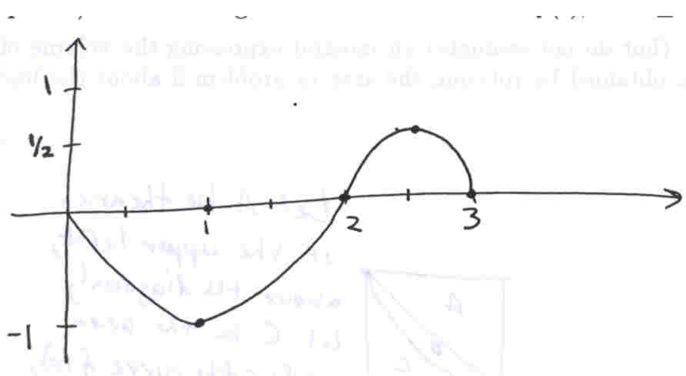
Projects Math 112 Prelim 1 October 2, 2001 Fall 2001

SOLUTIONS TO 2, 3, and 4a

2. (20 points) The following is a sketch of a function $f(t)$, for $0 \leq t \leq 3$.



Let $g(x) = \displaystyle\int_0^x f(t)\,dt$, for $0 \leq x \leq 3$.

There are two ways to do this problem. One is to think of $g(x)$ as the area under the curve $f(t)$ from 0 to x. The other is to use the FTC to realize $g'(x) = f(x)$.

a) Find the interval(s) where $g(x)$ is increasing.
Either way you think about the problem, you see $g(x)$ is increasing when $f(t)$ is positive, so on the interval $2 \leq x \leq 3$. Why? In the area interpretation, this is because the area from 0 to x is increasing as x moves from 2 to 3, since you are adding on positive area. If you use the FTC to think of $f(x) = g'(x)$, then you realize immediately that g is increasing when f is positive.

b) Find the interval(s) where $g(x)$ is decreasing.
Using the above reasoning, $g(x)$ is decreasing on $0 \leq x \leq 2$. For, $f(x) = g'(x)$ is negative there. Also, since f is negative, as x moves from 0 toward 2, you are adding on more negative area, so $g(x)$ is becoming more and more negative, thus decreasing.

c) Find the x-value(s) where $g(x)$ has a global minimum.
The candidates for extrema of g on the interval $0 \leq x \leq 3$ are critical points

(i.e. places where $f = g'$ is zero) or endpoints. These are $x = 0, 2$, and 3. Note $g(0) = 0$, since the integral from 0 to 0 of any function is 0. Then g decreases til $x = 2$, and there $g(2)$ is some large negative number. Then all the way from $x = 2$ to $x = 3$, g is increasing. Thus g had to have its global minimum at $x = 2$.

d) Find the x-value(s) where $g(x)$ has a global maximum.
This is trickier. Notice the positive area under the curve from 2 to 3 is much smaller than the negative area from 0 to 2. Thus $g(x)$ is negative for every $x > 0$. Thus the global max. occurs when $g(0) = 0$, at $x = 0$.

e) Find $g'(1)$.
Here we need the FTC. Since $g' = f$, we know $g'(1) = f(1)$, and we estimate that $f(1) = -1$. Thus, $g'(1) = -1$.

3. (15 points) a) Estimate $\int_0^4 x^2\,dx$ using the left and right endpoint approximations with $n = 4$.

Using $n = 4$ rectangles, we get endpoints $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$, and a rectangle width $\Delta x = 1$. So, the left hand endpoint approximation is

$$L_4 = (1) * (0^2 + 1^2 + 2^2 + 3^2) = 14,$$

and the right hand endpoint approximation is

$$R_4 = (1) * (1^2 + 2^2 + 3^2 + 4^2) = 30.$$

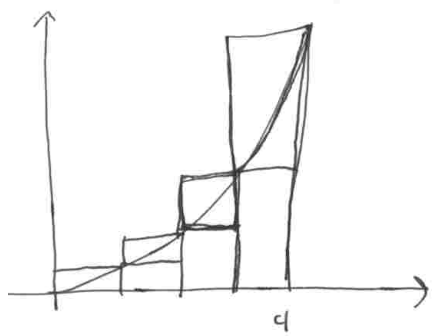
b) Compute $\int_0^4 x^2\,dx$.
Since the antiderivative of x^2 is $x^3/3$, we use the Evaluation Theorem to compute:

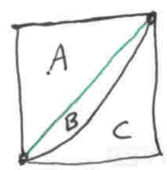
$$\int_0^4 x^2\ dx = \frac{x^3}{3}|_0^4 = \frac{4^3}{3} - \frac{0^3}{3} = \frac{64}{3},$$

which is about 21.33333333333.

c) Which choice of endpoints (left or right) in part a) gave a more accurate approximation? Explain why that same choice will give a more accurate approximation for every value of n. Give specific reasons. (Hint: Draw a picture.)

Left endpoints will always give a better approximation in this case. Basically, this is because the function is concave up and increasing, so the underestimate error from left-hand endpoints is less than the overestimate from right hand endpoints, because the function lies below the diagonal line connecting the endpoints. See the figure; it shows the rectangle which is the difference between left-hand and right-hand rectangles over one interval.





The above explanation, together with the picture, are a solution. But, if you want to see how one might make the argument really sound, read the following. Consider the picture of the box above. As marked in the figure, let A be the area of the upper left, above the diagonal, let C be the area under the curve, and let B be the area in between. Now $A+B$ is the error in using right-hand endpoints, and C is the error in using left-hand endpoints, so we want to determine which of the areas $A+B$ and C is bigger. It's pretty clear that $A+B$ is bigger. To make that precise, notice that $A = B+C$. Hence, $A+B = (B+C)+B$. Next, we know $B > 0$, because the function is concave up and increasing (look at the picture). Hence, $(B+C)+B = 2B+C > C$. This gives us what we desired, that $A+B$, the right-hand error, is bigger than C, the left-hand error. Thus left-hand endpoints give a better approximation.

4. (14 points) Determine by comparison whether the following improper integrals are convergent or divergent.

a) $\displaystyle\int_1^\infty \frac{dx}{x^2+\sqrt{x}}$

Here the first obvious choice of what to compare the integrand to is $1/x^2$. Notice that for $x \geq 1$ (which is true over the entire integral), we have $x^2 + \sqrt{(x)} > x^2$. Hence, $1/(x^2 + \sqrt{(x)}) < 1/x^2$. Thus by the comparison theorem, since we know that the larger integral $\int_1^\infty dx/(x^2)$ converges, then so does the smaller $\int_1^\infty dx/(x^2+\sqrt{(x)})$. (We are allowed to use the fact that $\int_1^\infty dx/(x^2)$ converges, without proof).