Schubert calculus and edge-swapping symmetries of Knutson-Tao puzzles

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Table of Contents

Intro

Schubert calculus classically is about computing structure constants in the cohomology ring of the **Grassmannian** by taking transverse intersections of **Schubert varieties**.

Puzzles are combinatorial gadgets that perform these calculations.

Sometimes, a **symmetry** that is readily observed in puzzles can reveal a more obscure geometric phenomenon, or vice versa.

Today we'll examine some of this interplay with **edge-swapping symmetries of parallelogram and hexagon-shaped puzzles**.

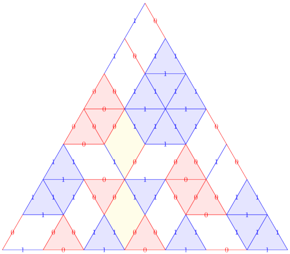


Table of Contents

The Grassmannian

Definition

We define the **Grassmannian** of k-planes in $\mathbb{C}^n$ as

$$\mathrm{Gr}(k, \mathbb{C}^n) := \{k\text{-dimensional subspaces } V \text{ of } \mathbb{C}^n\}.$$

Facts

- Smooth complex manifold/projective variety with $\dim = k(n-k)$
- $G := \mathrm{GL}_n(\mathbb{C})$ acts transitively. $\mathrm{Gr}(k, \mathbb{C}^n) \cong G/P$ a homogeneous space.
- The B-orbits X_λ° are called **Schubert cells**.
  - Indexed by binary strings $\lambda \in \binom{[n]}{k}$
  - Also indexed by partitions λ fitting in a $k \times (n-k)$ rectangle.
  - Give a cell decomposition of $\mathrm{Gr}(k, \mathbb{C}^n)$
- The cell closures $X_\lambda := \overline{X_\lambda^\circ}$ are called **Schubert varieties**.
  - The classes $[X_\lambda]$ are a $\mathbb{Z}$-basis for $H^*(\mathrm{Gr}(k, \mathbb{C}^n))$ via Poincaré duality
  - The structure constants are Littlewood-Richardson numbers $c_{\lambda,\mu}^{\nu}$.
  - The **opposite Schubert varieties** $X^\lambda := w_0 \cdot X_{w_0\lambda}$ give a dual basis under the perfect pairing $\int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X^\mu] = \delta_{\lambda,\mu}$.

We can represent an element $V \in \mathrm{Gr}(k, \mathbb{C}^n)$ as a $k \times n$ matrix of rank k. Choose a basis for V and put the basis vectors as the rows of a matrix. Example:

$$V = \text{rowspan} \begin{bmatrix} 7 & 0 & 1 & 0 & 11 & 8 & 0 \\ 0 & 6 & 0 & 9 & 3 & 4 & 0 \\ 4 & 4 & 9 & 10 & 8 & 2 & 0 \end{bmatrix} \in \mathrm{Gr}(3, \mathbb{C}^7)$$

Then row equivalent matrices all represent the same V. Choose a unique representative by using the RREF:

$$V = \text{rowspan} \begin{bmatrix} 7 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 1 & 0 & 0 \\ 2 & 2 & 0 & 5 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Gr}(3, \mathbb{C}^7)$$

We have bijections

$$\{M \in M_{k \times n}(\mathbb{C}) \mid \text{rank}(M) = k\}/\text{row equivalence} \xrightarrow{\text{rowspan}} \mathrm{Gr}(k, \mathbb{C}^n)$$

$$\{\text{RREFs in } M_{k \times n}(\mathbb{C}) \text{ with } k \text{ pivots}\} \xrightarrow{\text{rowspan}} \mathrm{Gr}(k, \mathbb{C}^n)$$

Definition

For a binary string $\lambda \in \binom{[n]}{k}$, we can define the **Schubert cell** in $\mathrm{Gr}(k, \mathbb{C}^n)$ as

$$X_\lambda^\circ := \text{rowspan}\{\text{RREFs in } M_{k\times n}(\mathbb{C}) \text{ with pivots in columns } \lambda\}.$$

$$\text{codim}(X_\lambda^\circ) = |\lambda| = \#\{\text{inversions in string } \lambda\} = \#\{\text{boxes in } \lambda \text{ as a partition}\}.$$

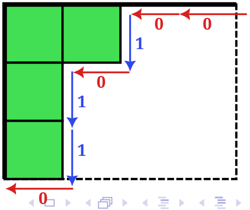
Example

$$X_{0010110}^\circ = \text{rowspan}\left\{\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 \end{bmatrix}\right\} \subseteq \mathrm{Gr}(3, \mathbb{C}^7)$$

Below highlights the correspondence between the string, RREF, and partition defining $X_{0010110}^\circ$.

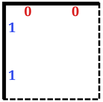
$\lambda = 0010110$

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 \end{bmatrix}$$
$$\begin{matrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{matrix}$$

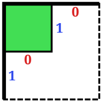


Schubert cell decomposition of $\mathrm{Gr}(2, \mathbb{C}^4)$

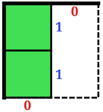
$$X^\circ_{0011} = \mathrm{rowspan}\left\{\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}\right\}$$



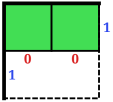
$$X^\circ_{0101} = \mathrm{rowspan}\left\{\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}\right\}$$



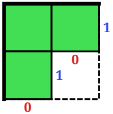
$$X^\circ_{0110} = \mathrm{rowspan}\left\{\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix}\right\}$$



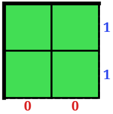
$$X^\circ_{1001} = \mathrm{rowspan}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}\right\}$$



$$X^\circ_{1010} = \mathrm{rowspan}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}\right\}$$



$$X^\circ_{1100} = \mathrm{rowspan}\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right\}$$



Schubert calculus is about computing structure constants in the Schubert variety basis $[X_\lambda]$ of $H^*(\mathrm{Gr}(k,\mathbb{C}^n))$. (Recall $X_\lambda := \overline{X_\lambda^\circ}$.)

These are the coefficients (Littlewood-Richardson numbers) $c_{\lambda,\mu}^\nu$ appearing in the product expansions

$$[X_\lambda][X_\mu] = \sum_\nu c_{\lambda,\mu}^\nu [X_\nu].$$

Geometric method

Count the points in triple intersections of Schubert varieties (perturbed to be transverse):

$$c_{\lambda,\mu}^\nu = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda \cap (g \cdot X_\mu) \cap X^\nu]$$

If the transverse intersection is finite, then the integral counts the intersection points. Otherwise, it equals 0.

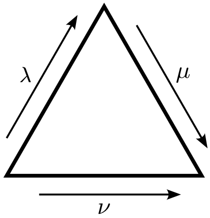
Puzzle method

Count Knutson-Tao puzzles:

$$c_{\lambda,\mu}^\nu = \#\{\Delta_{\lambda,\mu}^\nu\text{-puzzles}\}.$$

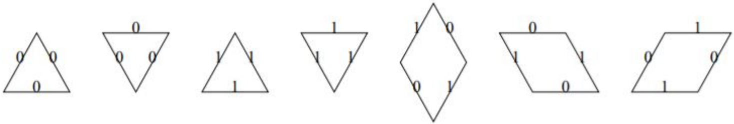
Table of Contents

Puzzles



Definition

Let $\Delta^{\nu}_{\lambda,\mu}$ denote the equilateral triangle whose edges are labeled with binary strings $\lambda, \mu, \nu \in \binom{[n]}{k}$ in the orientations shown at right. Then a $\Delta^{\nu}_{\lambda,\mu}$-**puzzle** is a filling of $\Delta^{\nu}_{\lambda,\mu}$ using the **puzzle pieces** below.



Examples

Example: $[X_{0101}][X_{0101}]$

Recall:

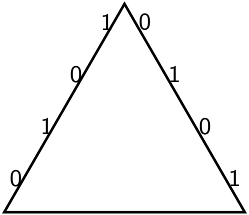
$$c^{\nu}_{\lambda,\mu} = \#\{\Delta^{\nu}_{\lambda,\mu}\text{-puzzles}\}.$$

Example

Compute the coefficients in the product expansion

$$[X_{0101}][X_{0101}] = \sum_{\nu} c^{\nu}_{0101,0101}[X_{\nu}] = \sum_{\nu} \#\{\Delta^{\nu}_{0101,0101}\text{-puzzles}\} \cdot [X_{\nu}]$$

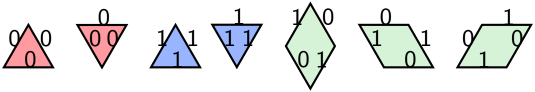
To answer this, we will look for all puzzles that fill this boundary:



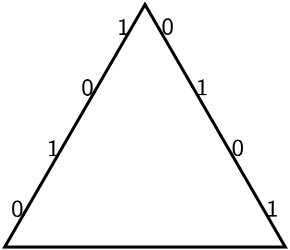
Which strings ν can appear on the bottom edge? And how many puzzles exist for each?

Example: $[X_{0101}][X_{0101}]$

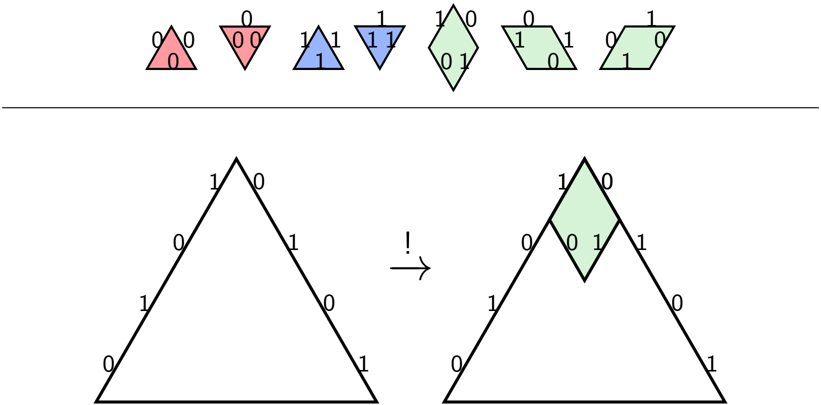
Allowed pieces:



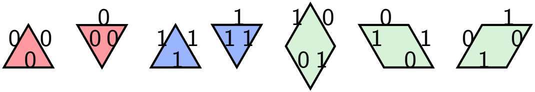
Starting at the top, let's try to fill in the boundary.

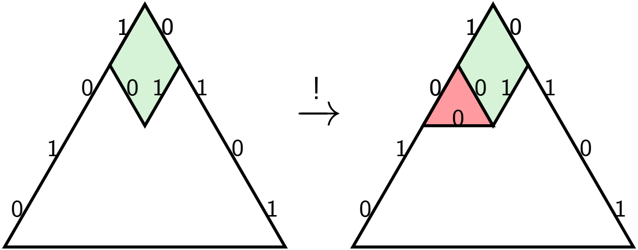


Example: $[X_{0101}][X_{0101}]$

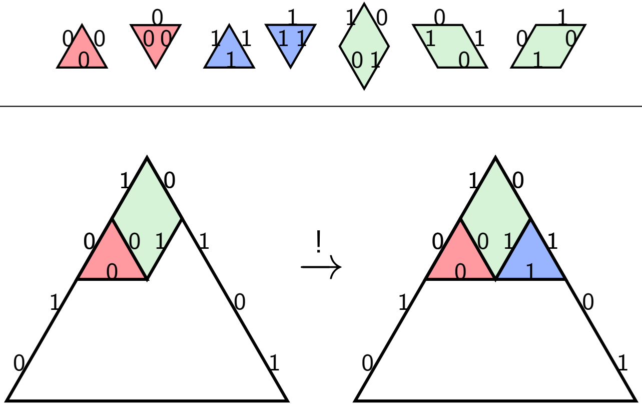


Example: $[X_{0101}][X_{0101}]$

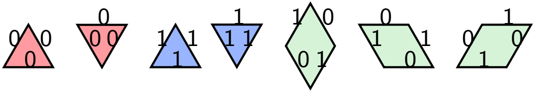


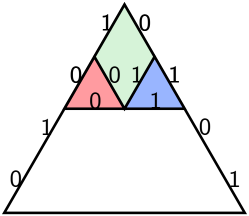


Example: $[X_{0101}][X_{0101}]$

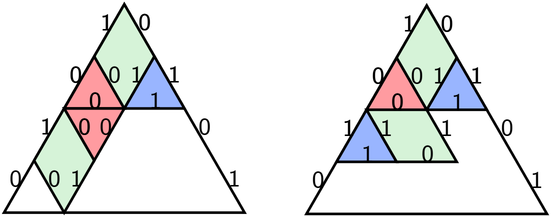


Example: $[X_{0101}][X_{0101}]$

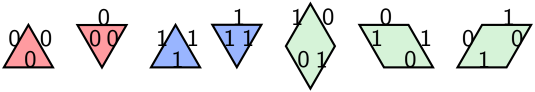


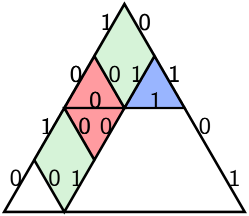


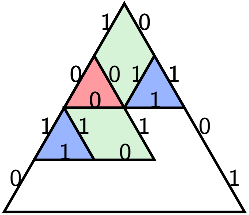
$\downarrow$ Two ways forward $\downarrow$



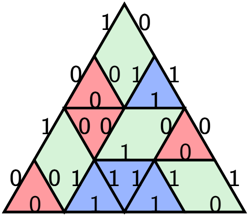
Example: $[X_{0101}][X_{0101}]$

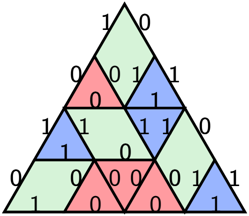






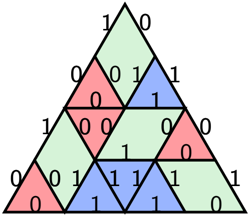
$\downarrow !$ Unique way to finish each $\downarrow !$



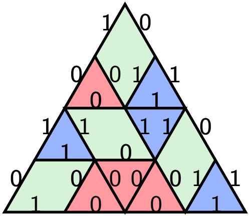


Example: $[X_{0101}][X_{0101}]$

$\therefore$ We've found **all two** puzzles with 0101 and 0101 on the NW and NE edges:



$\nu = 0110$



$\nu = 1001$

This tells us our answer:

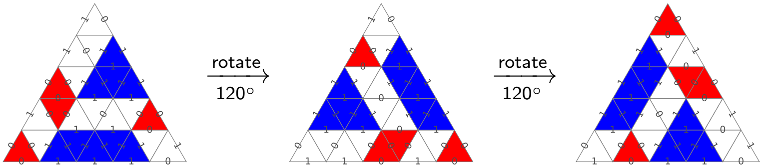
$$c^{\nu}_{0101,0101} = \begin{cases} 1, & \nu = 0110 \\ 1, & \nu = 1001 \\ 0, & \text{otherwise} \end{cases}$$

$$[X_{0101}][X_{0101}] = \sum_{\nu} c^{\nu}_{0101,0101}[X_{\nu}] = 1 \cdot [X_{0110}] + 1 \cdot [X_{1001}]$$

Table of Contents

$\mathbb{Z}_3$ Symmetry

A 120° rotation of a puzzle is still a puzzle.



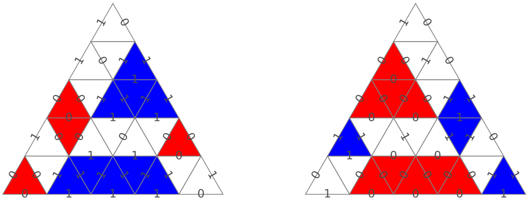
$$\implies c^{\nu}_{\lambda,\mu} = c^{\overleftarrow{\lambda}}_{\mu,\overleftarrow{\nu}} = c^{\overleftarrow{\mu}}_{\overleftarrow{\nu},\lambda}$$

$$\implies \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\mu][X_{\overleftarrow{\nu}}][X^{\overleftarrow{\lambda}}] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_{\overleftarrow{\nu}}][X_\lambda][X^{\overleftarrow{\mu}}]$$

where $\overleftarrow{\lambda} := \lambda$ reversed.

Dual Grassmannian Symmetry

Take the mirror image of a puzzle and exchange the 0s and 1s. It's again a puzzle!



Let * be the operation of reversing a string and exchanging the 0s and 1s. Then this mirroring trick gives a bijection

$$\{\Delta^{\nu}_{\lambda,\mu}\text{-puzzles}\} \leftrightarrow \{\Delta^{\nu^*}_{\mu^*,\lambda^*}\text{-puzzles}\}$$

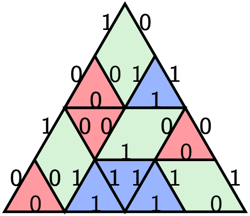
$$\implies c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\mu^*,\lambda^*}$$

Geometric version: The bijection $\mathrm{Gr}(k, \mathbb{C}^n) \to \mathrm{Gr}(n-k, (\mathbb{C}^n)^*)$ defined by $V \mapsto V^{\perp}$ sends $[X_\lambda]$ to $[X_{\lambda^*}]$ and intersection points to intersection points, so

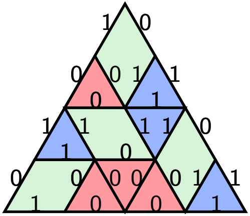
$$\int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(n-k,(\mathbb{C}^n)^*)} [X_{\lambda^*}][X_{\mu^*}][X^{\nu^*}].$$

Dual Grassmannian Symmetry

Our earlier example shows this symmetry as well.



$\nu = 0110$



$\nu = 1001$

$$c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\mu^*,\lambda^*}$$

$$c^{0110}_{0101,0101} = c^{(0110)^*}_{(0101)^*,(0101)^*} = c^{1001}_{0101,0101}$$

Commutativity

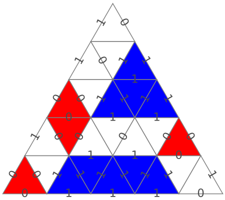
Commutativity of the cohomology classes implies commutativity of puzzles, i.e. we can swap the side edge labels and get the same number of puzzles.

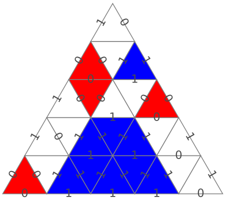
$$[X_\lambda][X_\mu] = [X_\mu][X_\lambda]$$

$$\implies \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\mu][X_\lambda][X^\nu]$$

$$\implies c^\nu_{\lambda,\mu} = c^\nu_{\mu,\lambda}$$

$$\implies \#\{\Delta^\nu_{\lambda,\mu}\text{-puzzles}\} = \#\{\Delta^\nu_{\mu,\lambda}\text{-puzzles}\}$$

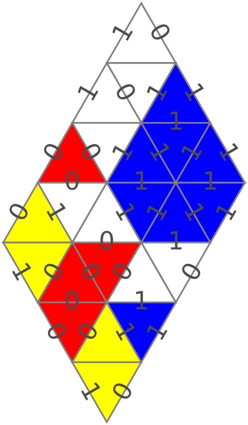


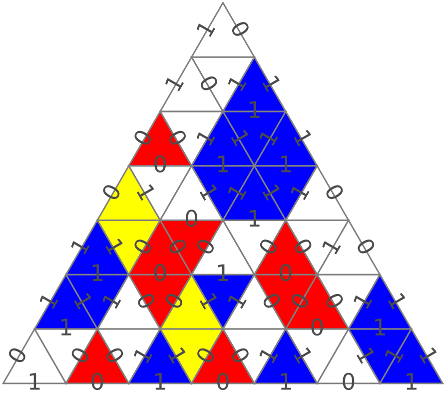


$c^{01110}_{01011,01101} = c^{01110}_{01101,01011} = 1$, but no visible relationship between the puzzles.

Table of Contents

Schubert Calculus with Parallelogram-Shaped Puzzles

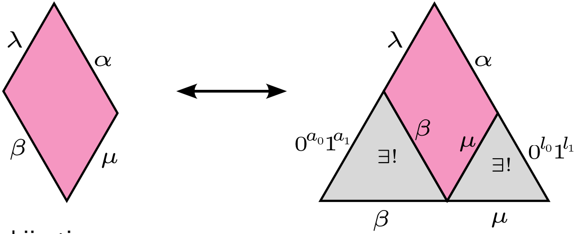




Label the edges of a parallelogram (clockwise from NW) with strings $\lambda, \alpha, \mu, \beta$ such that

$$\mathrm{sort}(\alpha) = \mathrm{sort}(\beta) = 0^{a_0}1^{a_1} \quad \text{and} \quad \mathrm{sort}(\lambda) = \mathrm{sort}(\mu) = 0^{\ell_0}1^{\ell_1}.$$

We can trivially complete any puzzle with boundary $\varnothing_{\lambda,\alpha,\mu,\beta}$ to a triangular puzzle with boundary $\Delta^{\overleftarrow{\beta}\bar{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)}$ like so:



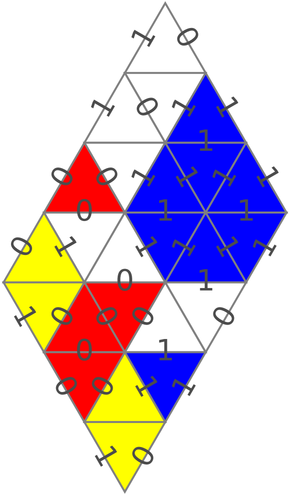
This gives a bijection

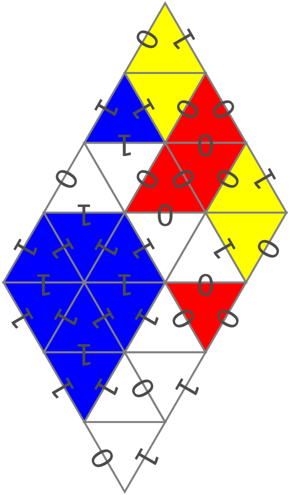
$$\{\varnothing_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} \leftrightarrow \left\{\Delta^{\overleftarrow{\beta}\bar{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)}\text{-puzzles}\right\}$$

and thus a geometric interpretation of parallelograms.

Rotational Symmetry of Parallelograms

Rotate 180°



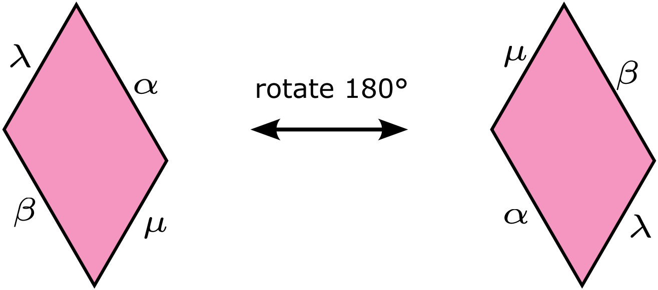


Another puzzle...obviously!

Rotational Symmetry of Parallelograms

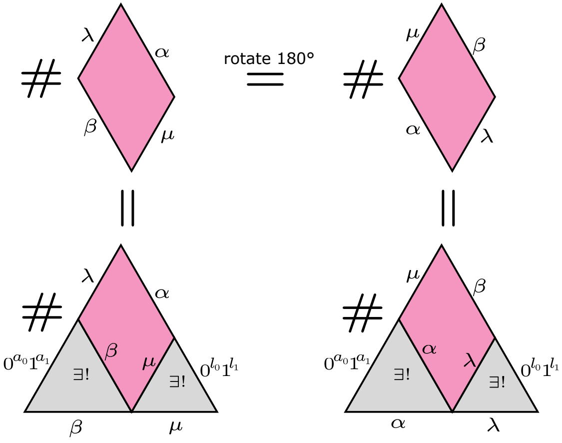
180° rotation yields a bijection:

$$\left\{\varGamma_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\right\} \leftrightarrow \left\{\varGamma_{\mu,\beta,\lambda,\alpha}\text{-puzzles}\right\}$$



Trivial from puzzle standpoint, but what about the geometric meaning?

Rotational Symmetry of Parallelograms



$$\Longrightarrow \quad c^{\overleftarrow{\beta}\,\overleftarrow{\mu}}_{\mathrm{sort}(\beta)\lambda,\,\alpha\,\mathrm{sort}(\mu)} = c^{\overleftarrow{\alpha}\,\overleftarrow{\lambda}}_{\mathrm{sort}(\alpha)\mu,\,\beta\,\mathrm{sort}(\lambda)}$$

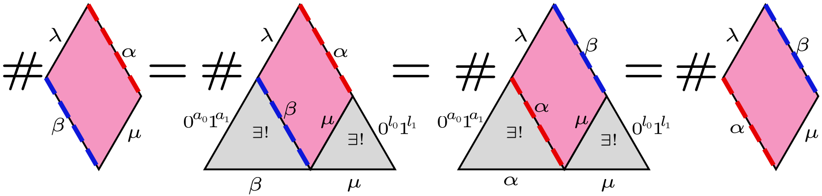
Original goal:

Understand this symmetry geometrically. This led to a stronger result.

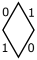
Theorem (A.)

Swapping just two opposite edge labels, α and β, we have:

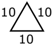
$$\#\{\varclubsuit_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} = \#\{\varclubsuit_{\lambda,\beta,\mu,\alpha}\text{-puzzles}\}$$



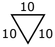
This also holds if we allow any one of the following additional pieces:



(T-equivariant cohomology)



(K-theory)



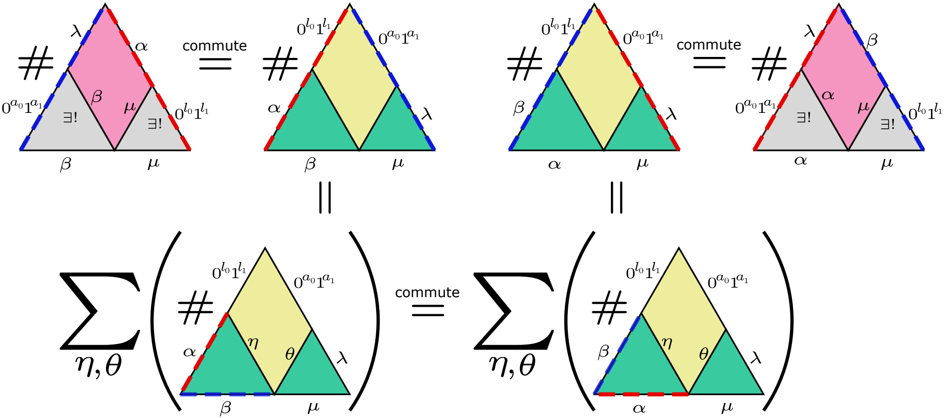
*(K-theory)**

Corollary

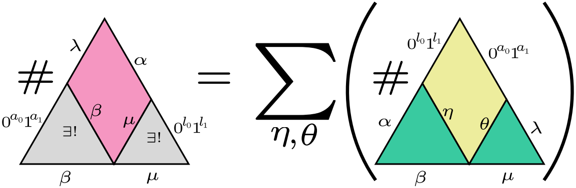
Can also swap λ and μ, i.e. $\#\{\varclubsuit_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} = \#\{\varclubsuit_{\mu,\alpha,\lambda,\beta}\text{-puzzles}\}$.

Puzzle-Based Proof

Here's a cartoon proof of the theorem that works for H^* and K-theory
It relies on commutativity and rotational symmetry of (triangular) puzzles.



Note: This doesn't work for equivariant puzzles because we can't rotate the equivariant piece .



It turns out the yellow region has ≤ 1 filling. This shifts the counting problem into just the smaller green triangles.

Theorem (A.)

For a fixed pair η, θ, the number of H^* puzzles filling the yellow region is either 0 or 1. It is 1 iff $\overleftarrow{\eta}$ and $\overleftarrow{\theta}$ (as partitions) are complements in an $\ell_0 \times a_1$ rectangle. In this case write $\theta = \eta'$, as it is unique to η. Then we have

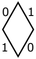
$$\#\left\{\varnothing_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\right\} = \sum_{\overleftarrow{\eta} \subseteq \ell_0 \times a_1} c_{\alpha,\beta}^{\eta} \cdot c_{\lambda,\mu}^{\eta'}. \qquad (*)$$

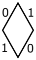
We can prove this directly, going piece by piece. But we can also obtain (*) as

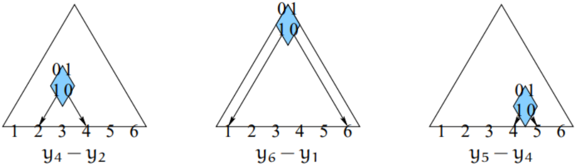
$$\int_{\mathrm{Gr}(a_1, a_0+a_1) \times \mathrm{Gr}(\ell_1, \ell_0+\ell_1)} ([X_\alpha][X_\beta] \otimes [X_\lambda][X_\mu]) \left(\sum_{\overleftarrow{\eta} \subseteq a_1 \times \ell_0} [X^\eta] \otimes [X^{\eta'}] \right).$$

Table of Contents

Equivariant Puzzles

If we allow the additional piece , then puzzles compute the structure constants $c^{\nu}_{\lambda,\mu} \in \mathbb{Z}[y_1, \ldots, y_n]$ in the T-**equivariant cohomology** of the Grassmannian, $H^*_T(\mathrm{Gr}(k, \mathbb{C}^n))$ (which extends the ordinary cohomology).

To do this, we give each equivariant piece  a **weight** $\mathrm{wt}(p) = y_j - y_i$, where (i, j) are coordinates for its position in the puzzle.



The weight $\mathrm{wt}(P)$ of a puzzle P is the product of the weights of its pieces.

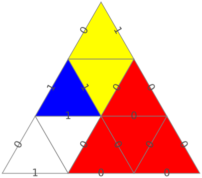
Theorem (Knutson-Tao)

$$c^{\nu}_{\lambda,\mu} = \sum_{\substack{\text{Puzzles } P: \\ \partial P = \Delta^{\nu}_{\lambda,\mu}}} wt(P) = \sum_{\substack{\text{Puzzles } P: \\ \partial P = \Delta^{\nu}_{\lambda,\mu}}} \left(\prod_{\substack{\text{equivariant} \\ \text{pieces } p \text{ in } P}} wt(p) \right)$$

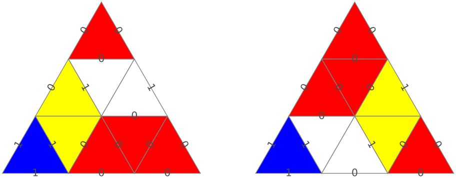
Equivariant puzzles have commutativity and dual symmetry, but not rotational symmetry.

Example

Here is an example showing commutativity.
(The yellow equivariant pieces contribute weights $y_j - y_i$.)



versus

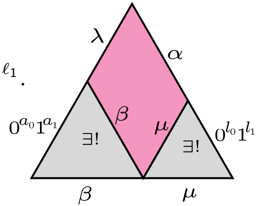


$c_{010,100}^{100} = y_3 - y_1$

$c_{100,010}^{100} = (y_2 - y_1) + (y_3 - y_2) = y_3 - y_1$

Warning: The structure constant is the same, but the *number* of puzzles may be different!

Equivariant Edge Swapping



As before, let $\alpha, \beta, \lambda, \mu$ be strings with

$$\mathrm{sort}(\alpha) = \mathrm{sort}(\beta) = 0^{a_0}1^{a_1} \quad \text{and} \quad \mathrm{sort}(\lambda) = \mathrm{sort}(\mu) = 0^{\ell_0}1^{\ell_1}.$$

Now, **allowing equivariant pieces**, consider the puzzle to the left. What happens if we swap edges?

Let $a := a_0 + a_1$ and $\ell := \ell_0 + \ell_1$, and define permutation matrices

$$\Phi_a := \left[\begin{array}{c|c} J_a & \mathbf{0} \\ \hline \mathbf{0} & I_\ell \end{array}\right] \quad \text{and} \quad \Phi_\ell := \left[\begin{array}{c|c} I_a & \mathbf{0} \\ \hline \mathbf{0} & J_\ell \end{array}\right],$$

where $I_a, J_a \in S_a$ are the identity and anti-diagonal permutations respectively, and similarly for $I_\ell, J_\ell \in S_\ell$.

Theorem (A.)

In $H_T^*(Gr(k, \mathbb{C}^n))$, where $n = a_0 + a_1 + \ell_0 + \ell_1$ and $k = a_1 + \ell_1$, we have

$$c^{\overleftarrow{\beta}\overleftarrow{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)} = \Phi_a \cdot c^{\overleftarrow{\alpha}\overleftarrow{\mu}}_{\mathrm{sort}(\alpha)\lambda,\beta\,\mathrm{sort}(\mu)} = \Phi_\ell \cdot c^{\overleftarrow{\beta}\overleftarrow{\lambda}}_{\mathrm{sort}(\beta)\mu,\alpha\,\mathrm{sort}(\lambda)}.$$

In other words swapping $\alpha \leftrightarrow \beta$ reverses the $y_1, \ldots, y_a$, and swapping $\lambda \leftrightarrow \mu$ reverses the $y_{a+1}, \ldots, y_n$.

A glimpse of the geometric proof

$F_\bullet := F_0 \subset \cdots \subset F_n$ standard flag, $\tilde{F}_\bullet := \tilde{F}_0 \subset \cdots \subset \tilde{F}_n$ anti-standard flag,
$\mathbb{C}^n = F_a \oplus \tilde{F}_\ell$.
We have a T-invariant closed immersion:

$$\delta : \mathrm{Gr}(a_1, F_a) \times \mathrm{Gr}(\ell_1, \tilde{F}_\ell) \hookrightarrow \mathrm{Gr}(k, \mathbb{C}^n)$$
$$(V_a, V_\ell) \mapsto V_a \oplus V_\ell$$

$$\begin{aligned}
c^{\overleftarrow{\beta}\overleftarrow{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)} &= \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_{\mathrm{sort}(\beta)\lambda}][X_{\alpha\,\mathrm{sort}(\mu)}][X^{\overleftarrow{\beta}\overleftarrow{\mu}}] \\
&= \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_{\mathrm{sort}(\alpha)\lambda}]\delta_*[(X_\alpha \cap X^{\overleftarrow{\beta}}) \times X^{\overleftarrow{\mu}}] \\
&= \int_{\mathrm{Gr}(a_1,F_a)\times\mathrm{Gr}(\ell_1,\tilde{F}_\ell)} \delta^*[X_{\mathrm{sort}(\beta)\lambda}] \left([X_\alpha][X^{\overleftarrow{\beta}}] \otimes [X^{\overleftarrow{\mu}}]\right) \\
&= \int_{\mathrm{Gr}(a_1,F_a)\times\mathrm{Gr}(\ell_1,\tilde{F}_\ell)} \delta^*[\Phi_a \cdot X_{\mathrm{sort}(\alpha)\lambda}] \left([J_a \cdot X_\beta][J_a \cdot X^{\overleftarrow{\alpha}}] \otimes [I_\ell \cdot X^{\overleftarrow{\mu}}]\right) \\
&= \Phi_a \cdot \int_{\mathrm{Gr}(a_1,F_a)\times\mathrm{Gr}(\ell_1,\tilde{F}_\ell)} \delta^*[X_{\mathrm{sort}(\alpha)\lambda}] \left([X_\beta][X^{\overleftarrow{\alpha}}] \otimes [X^{\overleftarrow{\mu}}]\right) \\
&= \Phi_a \cdot c^{\overleftarrow{\alpha}\overleftarrow{\mu}}_{\mathrm{sort}(\alpha)\lambda,\beta\,\mathrm{sort}(\mu)}
\end{aligned}$$

Equivariant Edge Swapping

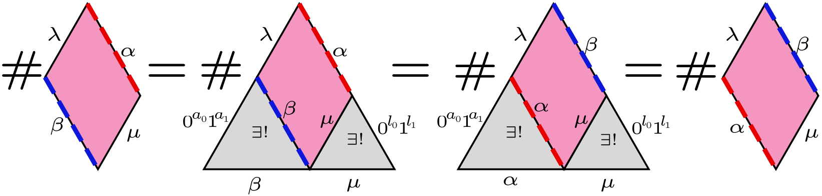
Corollary

The **number** *of equivariant parallelogram puzzles is invariant under edge swapping of* $\alpha \leftrightarrow \beta$ *or* $\lambda \leftrightarrow \mu$.

$$\#\{\varnothing_{\lambda,\alpha,\mu,\beta}\text{-equivariant puzzles}\} = \#\{\varnothing_{\lambda,\beta,\mu,\alpha}\text{-equivariant puzzles}\}$$

and

$$\#\{\varnothing_{\lambda,\alpha,\mu,\beta}\text{-equivariant puzzles}\} = \#\{\varnothing_{\mu,\alpha,\lambda,\beta}\text{-equivariant puzzles}\}$$

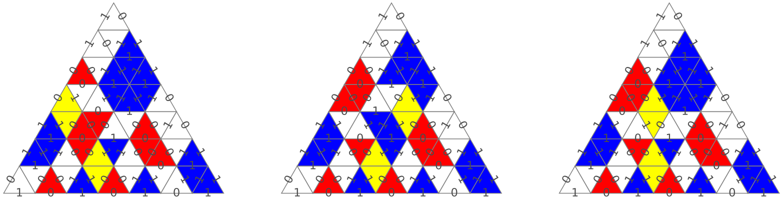


(This is not automatic from the theorem. Requires a further simple proof.)

Remark

This bijection remains combinatorially mysterious!

$$\{\square_{0011,011,1010,101}\text{-equivariant puzzles}\} =$$



After swapping λ and μ:

$$\{\square_{1010,011,0011,101}\text{-equivariant puzzles}\} =$$

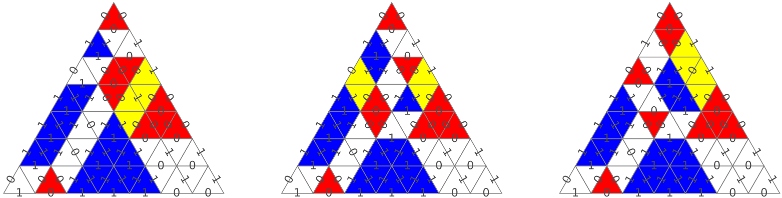
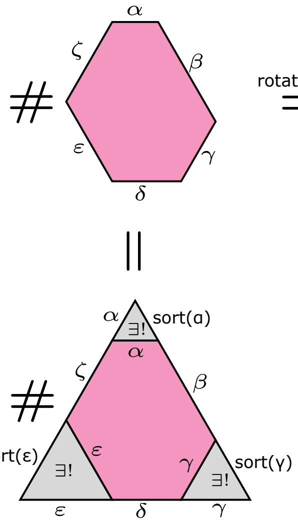


Table of Contents

We can generalize some of our questions and results by considering hexagonal puzzles. (Note: A parallelogram is a degenerate hexagon.)



Edge swapping results on Hexagons

For two nice cases where the edge labels have symmetric content, namely

$$\text{sort}(\alpha) = \text{sort}(\delta), \qquad \text{sort}(\beta) = \text{sort}(\varepsilon), \qquad \text{sort}(\gamma) = \text{sort}(\zeta) \tag{1}$$

$$\text{sort}(\alpha) = \text{sort}(\gamma) = \text{sort}(\varepsilon), \qquad \text{sort}(\beta) = \text{sort}(\delta) = \text{sort}(\zeta) \tag{2}$$

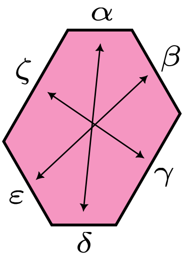
we found that (at least in H^* and K-theory) we can swap

$$\alpha \leftrightarrow \delta \qquad \beta \leftrightarrow \varepsilon \qquad \gamma \leftrightarrow \zeta \tag{1}$$

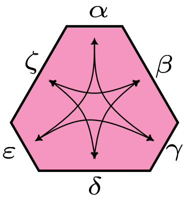
$$\alpha \leftrightarrow \gamma \leftrightarrow \varepsilon \qquad \beta \leftrightarrow \delta \leftrightarrow \zeta \tag{2}$$

and get the same number of puzzles.

Case (1)



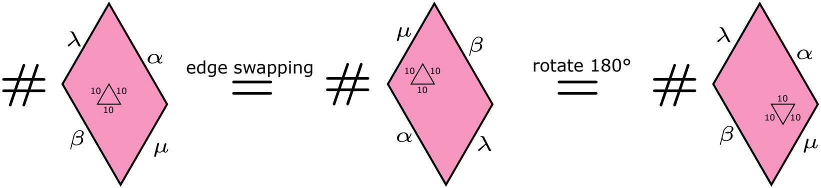
Case (2)



Further Questions

Are there interesting ways to extend these parallelogram/hexagon puzzle symmetries to other cohomology theories? What is the interpretation, and what could we learn?

- K-theory: Puzzles with  and puzzles with  compute the structure constants in the dual $[\mathcal{O}_{X_\lambda}]$ and $[\mathcal{I}_{X_\lambda}]$ bases for $K(\mathrm{Gr}(k, \mathbb{C}^n))$ respectively. Here is a strange observation:



  (Implies structure constant is the same in both $[\mathcal{O}_{X_\lambda}]$ and $[\mathcal{I}_{X_\lambda}]$ bases)

- What is a satisfying statement we can make about hexagons for equivariant cohomology?

- SMM (Segre-Schwartz-MacPherson): Puzzles containing *both*  and  compute the structure constants for the SSM classes of Schubert varieties. Can we swap edges and what would it mean?

Thank you!