Schubert Calculus with Puzzles
a dialogue between geometry and combinatorics

Portia Anderson

Department of Mathematics
Cornell University

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Definition

We define the **Grassmannian** of k-planes in $\mathbb{C}^n$ as

$$\mathrm{Gr}(k, \mathbb{C}^n) := \{V \leq \mathbb{C}^n \mid \dim(V) = k\}.$$

Facts

- Compact smooth complex manifold/projective variety with $\dim = k(n-k)$
- $G := \mathrm{GL}_n(\mathbb{C})$ acts transitively. $\mathrm{Gr}(k, \mathbb{C}^n) \cong G/P$.
- The B-orbits X_λ°, called **Schubert cells**, are indexed by binary strings $\lambda \in \binom{[n]}{k}$ and give a cell decomposition of $\mathrm{Gr}(k, \mathbb{C}^n)$.
- The orbit closures $X_\lambda := \overline{X_\lambda^\circ}$, called **Schubert varieties**, give a $\mathbb{Z}$-basis for $H^*(\mathrm{Gr}(k, \mathbb{C}^n))$ via Poincaré duality.
- The **opposite Schubert varieties** $X^\lambda := w_0 \cdot X_{w_0 \lambda}$ give a dual basis under the perfect pairing $\int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X^\mu] = \delta_{\lambda,\mu}$.

Definition

Given a string $\lambda = \lambda_1\lambda_2\cdots\lambda_n$ with content $0^{n-k}1^k$, the **Schubert variety** $X_\lambda \subseteq \mathrm{Gr}(k, \mathbb{C}^n)$ is defined by

$$X_\lambda = X_\lambda(F_\bullet) := \{V \in \mathrm{Gr}(k,n) \mid \dim(V \cap F_i) \geq \lambda_1 + \cdots + \lambda_i\},$$

where $F_\bullet$ is the standard complete flag $0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$ (i.e. $F_i = \mathrm{span}\{\vec{e}_1, \ldots, \vec{e}_i\}$ for each i).

Example

$$X_{0101} = \left\{ \text{2-planes } V \leq \mathbb{C}^4 \;\middle|\; \begin{array}{l} \dim(V \cap F_1) \geq 0, \\ \dim(V \cap F_2) \geq 0+1 = 1, \\ \dim(V \cap F_3) \geq 0+1+0 = 1, \\ \dim(V \cap \mathbb{C}^4) \geq 0+1+0+1 = 2 \end{array} \right\} \subseteq \mathrm{Gr}(2, \mathbb{C}^4)$$

$$= \left\{ \text{2-planes } V \leq \mathbb{C}^4 \mid \dim(V \cap F_2) \geq 1 \right\}$$

This is the set of all 2-planes V that intersect the 2-plane F_2.
If we projectivize, this becomes “the set of all lines that intersect a line” $\mathbb{P}(F_2)$.

A classic Schubert calculus problem

Question

Take four (projective) lines L_1, L_2, L_3, L_4 generically positioned in space. How many (projective) lines intersect all four?

Answer

Recall that $X_{0101} = \{$"Lines that intersect a certain line"$\}$.
Perturb with generic elements $g_i \in \mathrm{GL}_n(\mathbb{C})$ so that:

$$\begin{aligned} g_1 \cdot X_{0101} &= \{\text{"Lines that intersect } L_1\text{"}\} \\ g_2 \cdot X_{0101} &= \{\text{"Lines that intersect } L_2\text{"}\} \\ g_3 \cdot X_{0101} &= \{\text{"Lines that intersect } L_3\text{"}\} \\ g_4 \cdot X_{0101} &= \{\text{"Lines that intersect } L_4\text{"}\} \end{aligned}$$

A point in their intersection is a line that intersects all four.

$$\begin{aligned} &[(g_1 \cdot X_{0101}) \cap (g_2 \cdot X_{0101}) \cap (g_3 \cdot X_{0101}) \cap (g_4 \cdot X_{0101})] \\ &= [g_1 \cdot X_{0101}][g_2 \cdot X_{0101}][g_3 \cdot X_{0101}][g_4 \cdot X_{0101}] \\ &= [X_{0101}][X_{0101}][X_{0101}][X_{0101}] \\ &= 2[X_{1100}] \end{aligned}$$

Two points!

Schubert calculus is about computing the structure constants of the cohomology of the Grassmannian $H^*(\mathrm{Gr}(k, \mathbb{C}^n))$ in the Schubert variety basis.

These are the coefficients $c_{\lambda,\mu}^{\nu}$ (Littlewood-Richardson numbers) appearing in the product expansions

$$[X_\lambda][X_\mu] = \sum_{\nu} c_{\lambda,\mu}^{\nu} [X_\nu].$$

One way is to compute these is to count the points in triple intersections of Schubert varieties:

$$c_{\lambda,\mu}^{\nu} = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu].$$

Another way is to count puzzles.

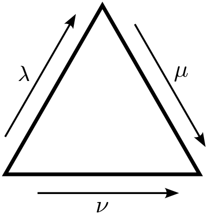
Theorem (Knutson-Tao)

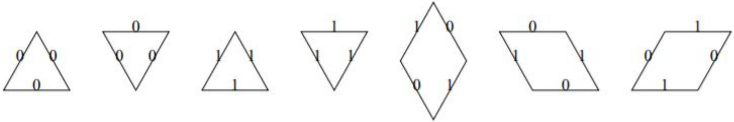
$$c_{\lambda,\mu}^{\nu} = \#\{\Delta_{\lambda,\mu}^{\nu}\text{-}\mathit{puzzles}\}.$$

Puzzles

Definition

Let $\Delta^{\nu}_{\lambda,\mu}$ denote the equilateral triangle whose edges are labeled with binary strings $\lambda, \mu, \nu \in \binom{[n]}{k}$ in the orientations shown at right. Then a $\Delta^{\nu}_{\lambda,\mu}$-**puzzle** is a filling of $\Delta^{\nu}_{\lambda,\mu}$ using the **puzzle pieces** below.



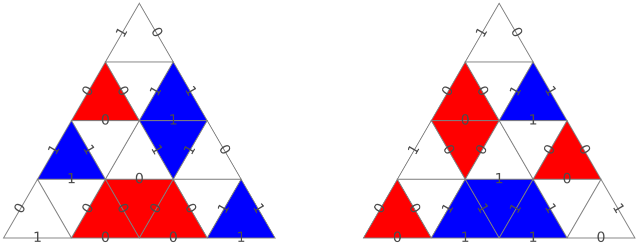


Examples

Puzzles

Example

$$[X_{0101}][X_{0101}] = 1 \cdot [X_{1001}] + 1 \cdot [X_{0110}]$$



These are the only two puzzles with 0101 and 0101 on the NW and NE edges.

$$c^{1001}_{0101,0101} = 1 \quad \text{and} \quad c^{0110}_{0101,0101} = 1$$

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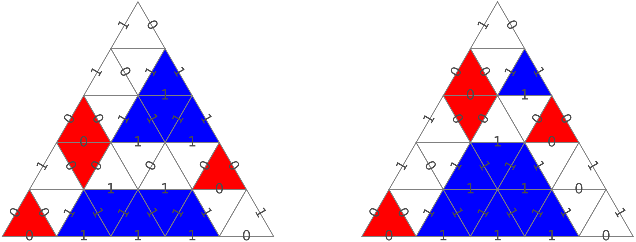
Commutativity

$$[X_\lambda][X_\mu] = [X_\mu][X_\lambda]$$

$$\implies \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\mu][X_\lambda][X^\nu]$$

$$\implies c^\nu_{\lambda,\mu} = c^\nu_{\mu,\lambda}$$

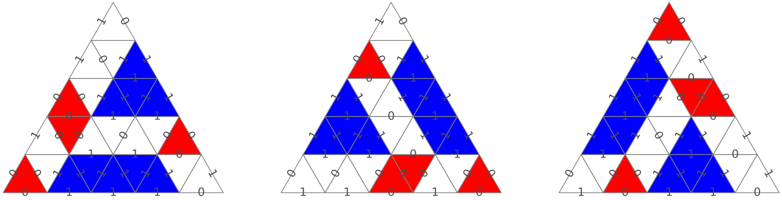
$$\implies \#\{\Delta^\nu_{\lambda,\mu}\text{-puzzles}\} = \#\{\Delta^\nu_{\mu,\lambda}\text{-puzzles}\}$$



Non-obvious from just looking at puzzles.

$\mathbb{Z}_3$ Symmetry

A 120° rotation of a puzzle is still a puzzle.

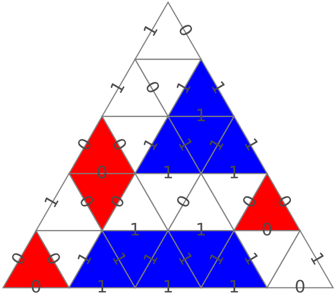


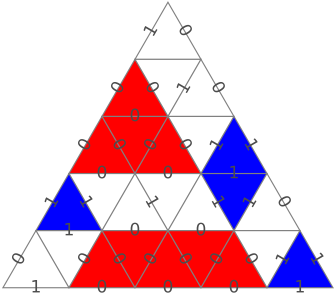
$$\implies c^{\nu}_{\lambda,\mu} = c^{\overleftarrow{\lambda}}_{\mu,\overleftarrow{\nu}} = c^{\overleftarrow{\mu}}_{\overleftarrow{\nu},\lambda}$$

$$\implies \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\mu][X_{\overleftarrow{\nu}}][X^{\overleftarrow{\lambda}}] = \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_{\overleftarrow{\nu}}][X_\lambda][X^{\overleftarrow{\mu}}]$$

Dual Grassmannian Symmetry

Flip over a puzzle and exchange the 1's and 0's. This gives you a puzzle.





$$\implies c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\mu^*,\lambda^*}$$

$$\implies \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu] = \int_{\mathrm{Gr}(n-k,(\mathbb{C}^n)^*)} [X_{\mu^*}][X_{\lambda^*}][X^{\nu^*}]$$

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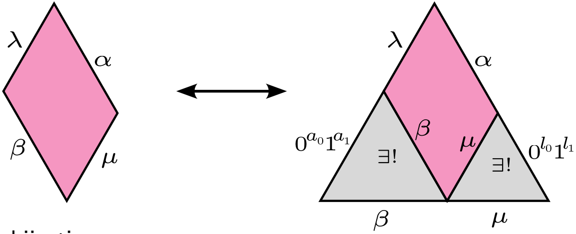
Schubert Calculus with Parallelogram-Shaped Puzzles

Label the edges of a parallelogram (clockwise from NW) with strings $\lambda, \alpha, \mu, \beta$ such that

$$\mathrm{sort}(\alpha) = \mathrm{sort}(\beta) = 0^{a_0}1^{a_1} \quad \text{and} \quad \mathrm{sort}(\lambda) = \mathrm{sort}(\mu) = 0^{\ell_0}1^{\ell_1}.$$

We can trivially complete any puzzle with boundary $\varparallelogram_{\lambda,\alpha,\mu,\beta}$ to a triangular puzzle with boundary $\Delta^{\overleftarrow{\beta}\,\bar{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)}$ like so:



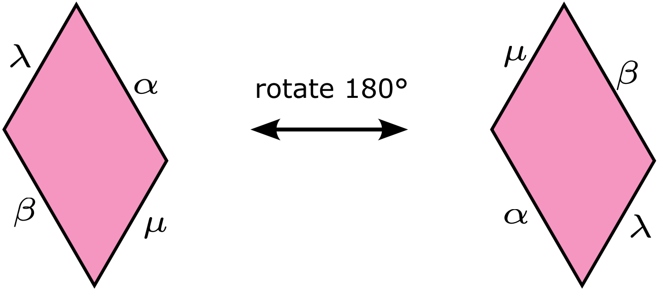
This gives a bijection

$$\{\varparallelogram_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} \leftrightarrow \left\{\Delta^{\overleftarrow{\beta}\,\bar{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)}\text{-puzzles}\right\}$$

and thus a geometric interpretation of parallelograms.

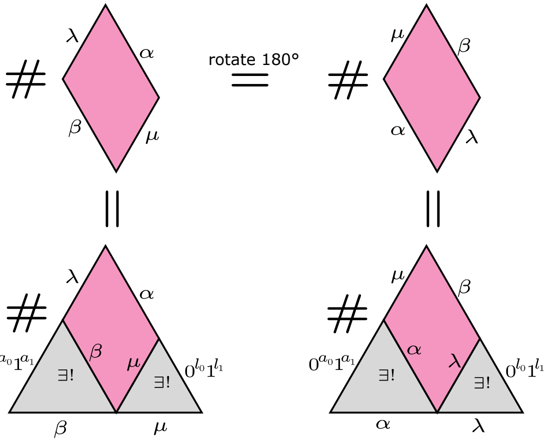
Also, 180° rotation yields a bijection:

$$\left\{\varnothing_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\right\} \leftrightarrow \left\{\varnothing_{\mu,\beta,\lambda,\alpha}\text{-puzzles}\right\}$$



Silly from puzzle standpoint, but what about the geometric meaning?

Rotational Symmetry of Parallelograms



$$\Longrightarrow \; c^{\overleftarrow{\beta}\overleftarrow{\mu}}_{\mathsf{sort}(\beta)\lambda,\alpha\,\mathsf{sort}(\mu)} = c^{\overleftarrow{\alpha}\overleftarrow{\lambda}}_{\mathsf{sort}(\alpha)\mu,\beta\,\mathsf{sort}(\lambda)}$$

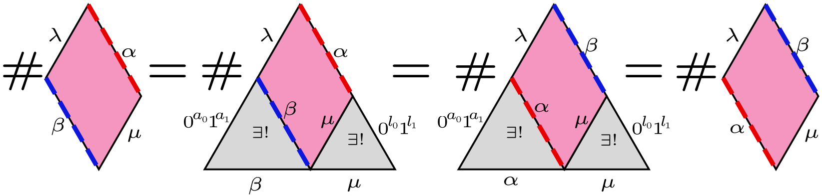
Original goal:

Understand this symmetry geometrically. This led to a stronger result.

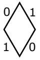
Theorem (1)

Swapping just two opposite edge labels, α and β, we have:

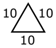
$$\#\{\text{▱}_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} = \#\{\text{▱}_{\lambda,\beta,\mu,\alpha}\text{-puzzles}\}$$



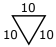
This also holds if we allow any one of the following additional pieces:



(equivariant)



(K-theory)



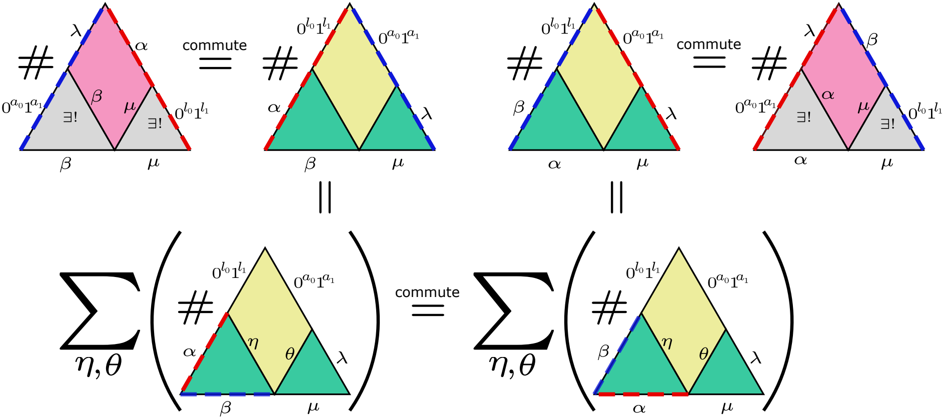
*(K-theory)**

Corollary

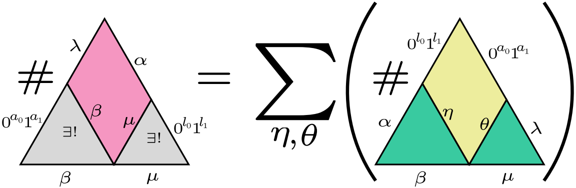
Can also swap λ and μ, i.e. $\#\{\text{▱}_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} = \#\{\text{▱}_{\mu,\alpha,\lambda,\beta}\text{-puzzles}\}$.

Puzzle-Based Proof

Here's a cartoon proof of Theorem 1 It relies on commutativity of (triangular) puzzle edges.



Puzzle-Based Proof



Turns out the yellow region has ≤ 1 filling. Shifts the counting problem into just the smaller green triangles.

Theorem (2)

For a fixed pair η, θ, the number of H^* puzzles filling the yellow region is either 0 or 1. It is 1 iff $\overleftarrow{\eta}$ and $\overleftarrow{\theta}$ (as partitions) are complements in an $\ell_0 \times a_1$ rectangle. In this case write $\theta = \eta'$, as it is unique to η. Then we have

$$\#\{\varnothing_{\lambda,\alpha,\mu,\beta}\text{-}puzzles\} = \sum_{\overleftarrow{\eta} \subseteq \ell_0 \times a_1} c^{\eta}_{\alpha,\beta} \cdot c^{\eta'}_{\lambda,\mu}. \qquad (*)$$

We can prove this directly, going piece by piece. But we can also obtain (*) as

$$\int_{\mathrm{Gr}(a_1, a_0+a_1) \times \mathrm{Gr}(\ell_1, \ell_0+\ell_1)} ([X_\alpha][X_\beta] \otimes [X_\lambda][X_\mu]) \left(\sum_{\overleftarrow{\eta} \subseteq a_1 \times \ell_0} [X^\eta] \otimes [X^{\eta'}] \right).$$

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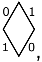
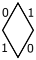
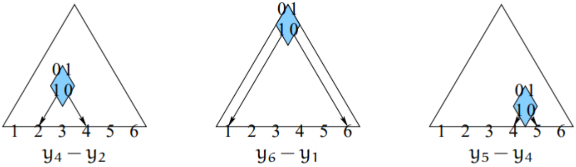
Equivariant Puzzles

If we allow the additional puzzle piece 

, then puzzles compute the structure constants $c^{\nu}_{\lambda,\mu} \in \mathbb{Z}[y_1, \ldots, y_n]$ of the T-**equivariant cohomology** of the Grassmannian, $H^*_T(\mathrm{Gr}(k, \mathbb{C}^n))$.

To do this, we give each equivariant piece 

a **weight** $\mathrm{wt}(p) = y_j - y_i$, where (i, j) corresponds to its position.



Each puzzle contributes the product of the weights of its equivariant pieces.

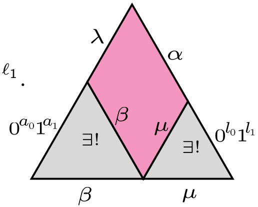
Theorem (Knutson-Tao)

$$c^{\nu}_{\lambda,\mu} = \sum_{\substack{\text{Puzzles } P: \\ \partial P = \Delta^{\nu}_{\lambda,\mu}}} wt(P) = \sum_{\substack{\text{Puzzles } P: \\ \partial P = \Delta^{\nu}_{\lambda,\mu}}} \left(\prod_{\substack{\text{equivariant} \\ \text{pieces } p \text{ in } P}} wt(p) \right)$$

Equivariant Edge Swapping

As before, let $\alpha, \beta, \lambda, \mu$ be strings with

$$\text{sort}(\alpha) = \text{sort}(\beta) = 0^{a_0}1^{a_1} \quad \text{and} \quad \text{sort}(\lambda) = \text{sort}(\mu) = 0^{\ell_0}1^{\ell_1}.$$



Now, **allowing equivariant pieces**, consider the puzzle to the left. What happens if we swap edges?

Let $a := a_0 + a_1$ and $\ell := \ell_0 + \ell_1$, and define

$$\Phi_a := \left[\begin{array}{c|c} J_a & \mathbf{0} \\ \hline \mathbf{0} & I_\ell \end{array}\right] \quad \text{and} \quad \Phi_\ell := \left[\begin{array}{c|c} I_a & \mathbf{0} \\ \hline \mathbf{0} & J_\ell \end{array}\right],$$

where $I_a, J_a \in S_a$ are the identity and anti-diagonal permutations respectively, and similarly for $I_\ell, J_\ell \in S_\ell$.

Theorem (3)

In $H_T^*(Gr(k, \mathbb{C}^n))$, where $n = a_0 + a_1 + \ell_0 + \ell_1$ and $k = a_1 + \ell_1$, we have

$$c^{\overleftarrow{\beta}\overleftarrow{\mu}}_{\text{sort}(\beta)\lambda, \alpha\, \text{sort}(\mu)} = \Phi_a \cdot c^{\overleftarrow{\alpha}\overleftarrow{\mu}}_{\text{sort}(\alpha)\lambda, \beta\, \text{sort}(\mu)} = \Phi_\ell \cdot c^{\overleftarrow{\beta}\overleftarrow{\lambda}}_{\text{sort}(\beta)\mu, \alpha\, \text{sort}(\lambda)}.$$

In other words swapping $\alpha \leftrightarrow \beta$ reverses the $y_1, \ldots, y_a$, and swapping $\lambda \leftrightarrow \mu$ reverses the $y_{a+1}, \ldots, y_n$.

Idea of proof

$F_\bullet := F_0 \subset \cdots \subset F_n$ standard flag, $\tilde{F}_\bullet := \tilde{F}_0 \subset \cdots \subset \tilde{F}_n$ anti-standard flag, $\mathbb{C}^n = F_a \oplus \tilde{F}_\ell$.
Have a T-invariant closed immersion:

$$\begin{aligned} \delta : \mathrm{Gr}(a_1, F_a) \times \mathrm{Gr}(\ell_1, \tilde{F}_\ell) &\hookrightarrow \mathrm{Gr}(k, \mathbb{C}^n) \\ (V_a, V_\ell) &\mapsto V_a \oplus V_\ell \end{aligned}$$

$$\begin{aligned} c^{\overleftarrow{\beta}\overleftarrow{\mu}}_{\mathrm{sort}(\beta)\lambda,\alpha\,\mathrm{sort}(\mu)} &= \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_{\mathrm{sort}(\beta)\lambda}][X_{\alpha\,\mathrm{sort}(\mu)}][X^{\overleftarrow{\beta}\overleftarrow{\mu}}] \\ &= \int_{\mathrm{Gr}(k,\mathbb{C}^n)} [X_{\mathrm{sort}(\alpha)\lambda}]\delta_*[(X_\alpha \cap X^{\overleftarrow{\beta}}) \times X^{\overleftarrow{\mu}}] \\ &= \int_{\mathrm{Gr}(a_1,F_a)\times\mathrm{Gr}(\ell_1,\tilde{F}_\ell)} \delta^*[X_{\mathrm{sort}(\beta)\lambda}] \left([X_\alpha][X^{\overleftarrow{\beta}}] \otimes [X^{\overleftarrow{\mu}}]\right) \\ &= \int_{\mathrm{Gr}(a_1,F_a)\times\mathrm{Gr}(\ell_1,\tilde{F}_\ell)} \delta^*[\Phi_a \cdot X_{\mathrm{sort}(\alpha)\lambda}] \left([J_a \cdot X_\beta][J_a \cdot X^{\overleftarrow{\alpha}}] \otimes [I_\ell \cdot X^{\overleftarrow{\mu}}]\right) \\ &= \Phi_a \cdot \int_{\mathrm{Gr}(a_1,F_a)\times\mathrm{Gr}(\ell_1,\tilde{F}_\ell)} \delta^*[X_{\mathrm{sort}(\alpha)\lambda}] \left([X_\beta][X^{\overleftarrow{\alpha}}] \otimes [X^{\overleftarrow{\mu}}]\right) \\ &= \Phi_a \cdot c^{\overleftarrow{\alpha}\overleftarrow{\mu}}_{\mathrm{sort}(\alpha)\lambda,\beta\,\mathrm{sort}(\mu)} \end{aligned}$$

Equivariant Edge Swapping

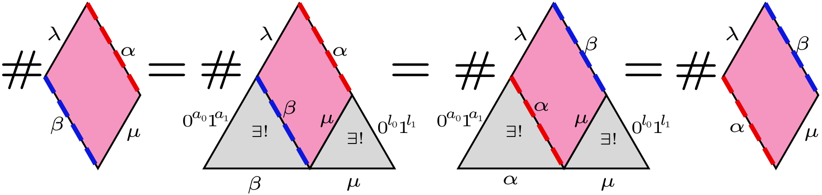
Corollary

The **number** *of equivariant parallelogram puzzles is invariant under edge swapping of* $\alpha \leftrightarrow \beta$ *or* $\lambda \leftrightarrow \mu$*.*

$$\#\{\varlozenge_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} = \#\{\varlozenge_{\lambda,\beta,\mu,\alpha}\text{-puzzles}\}$$

and

$$\#\{\varlozenge_{\lambda,\alpha,\mu,\beta}\text{-puzzles}\} = \#\{\varlozenge_{\mu,\alpha,\lambda,\beta}\text{-puzzles}\}$$



(This doesn't automatically follow. Requires a further simple proof.)

Remark

This bijection is combinatorially mysterious!

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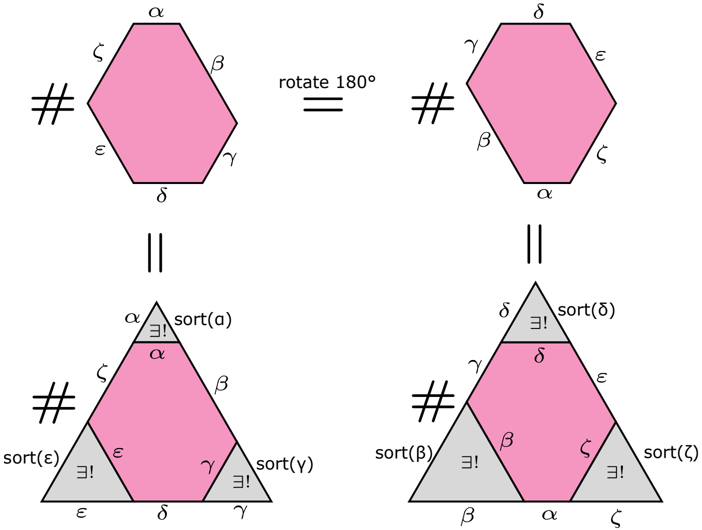
Hexagons

The *original* original question was an analogous one about hexagons.



For two nice cases where the edge labels have symmetric content, namely

$$\text{sort}(\alpha) = \text{sort}(\delta), \qquad \text{sort}(\beta) = \text{sort}(\varepsilon), \qquad \text{sort}(\gamma) = \text{sort}(\zeta) \tag{1}$$

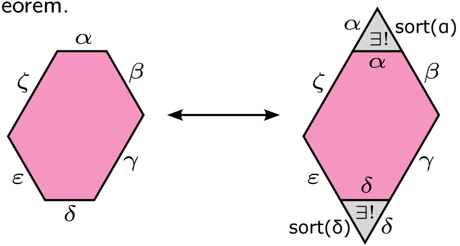
$$\text{sort}(\alpha) = \text{sort}(\gamma) = \text{sort}(\varepsilon), \qquad \text{sort}(\beta) = \text{sort}(\delta) = \text{sort}(\zeta) \tag{2}$$

we found that (at least in H^*) we can swap

$$\alpha \leftrightarrow \delta \qquad \beta \leftrightarrow \varepsilon \qquad \gamma \leftrightarrow \zeta \tag{1}$$

$$\alpha \leftrightarrow \gamma \leftrightarrow \varepsilon \qquad \beta \leftrightarrow \delta \leftrightarrow \zeta \tag{2}$$

(1) can be seen by completing the hexagon to a parallelogram and applying our edge-swapping theorem.



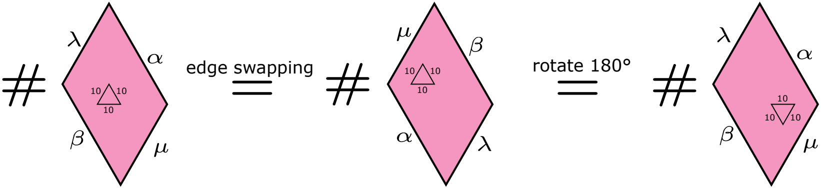
Further Questions

Are there interesting ways to extend these parallelogram/hexagon puzzle symmetries to other cohomology theories? What would be the interpretation, and what could we learn?

- K-theory: Puzzles with 

and puzzles with 

compute the structure constants in the dual $[\mathcal{O}_{X_\lambda}]$ and $[\mathcal{I}_{X_\lambda}]$ bases for $K(\mathrm{Gr}(k, \mathbb{C}^n))$ respectively. What is the geometric proof? Also this is weird:



(Implies structure constant is the same in both $[\mathcal{O}_{X_\lambda}]$ and $[\mathcal{I}_{X_\lambda}]$ bases)

- SMM (Segre-Schwartz-MacPherson): Puzzles containing *both* 

and 

compute the structure constants for the SSM classes of Schubert varieties.